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TIGHTNESS OF LEVEL-WISE CONTINUOUS FUZZY RANDOM VARIABLES†

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ABSTRACT

In this paper, we first obtain some characterizations of compact subsets of the space of level-wise continuous fuzzy numbers in R by the modulus of continuity. Using this, we establish the tightness for a sequence of level-wise continuous fuzzy random variables.

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1. INTRODUCTION

The notion of tightness of random variables plays an important role in limit theorems of stochastic processes and its applications. Prohorov (1956) gave the relationships between tightness and weak convergence of probability measures on complete separable metric spaces which include $C[0,1]$ and $D[0,1]$. These results can be also found in Billingsley (1968).

Since Puri and Ralescu (1986) introduced the concept of a fuzzy random variable, there has been increasing interests in limit theorems for fuzzy random variables because of its usefulness in several applied fields. Thus it seems to be important that we ask how to characterize the tightness of fuzzy random variables. Related to this subject, Joo (2003) obtained some results.

In this paper, we restrict ourselves to $CF(R)$-valued fuzzy random variables, where $CF(R)$ is the family of level-wise continuous fuzzy numbers in R. We first present some characterizations of compact subsets of $CF(R)$ and establish the tightness of random elements of $CF(R)$. Section 2 is devoted to describe some preliminary results, and the main results are given in Section 3.

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2. PRELIMINARY RESULTS

In this section, we describe some preliminary results for fuzzy numbers. Let R denote the real line. A fuzzy number in R is a fuzzy set $\tilde{u} : R \to [0,1]$ with the following properties:

(1) $\tilde{u}$ is normal, *i.e.*, there exists $x \in R$ such that $\tilde{u}(x) = 1$;

(2) $\tilde{u}$ is upper semicontinuous;

(3) $\tilde{u}$ is a convex fuzzy set, *i.e.*, $\tilde{u}\big(\lambda x + (1-\lambda)y\big) \geq \min\big(\tilde{u}(x), \tilde{u}(y)\big)$ for $x, y \in R$ and $\lambda \in [0,1]$;

(4) $\operatorname{supp} \tilde{u} = \operatorname{cl}\{x \in R : \tilde{u}(x) > 0\}$ is compact,

where $\operatorname{cl}(A)$ denote the closure of A.

We denote the family of all fuzzy numbers by $F(R)$. For a fuzzy set $\tilde{u}$, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1, \\ \operatorname{supp} \tilde{u}, & \text{if } \alpha = 0. \end{cases}$$

Then it follows that $\tilde{u}$ is a fuzzy number in R if and only if $L_1\tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the end points of the intervals $L_\alpha \tilde{u} = [u_\alpha^1, u_\alpha^2]$.

THEOREM 2.1. (a) *For $\tilde{u} \in F(R)$, let us consider u_α^1 and u_α^2 as functions of $\alpha \in [0,1]$. Then the following properties hold:*

(1) *u^1 is a bounded increasing function on $[0,1]$;*

(2) *u^2 is a bounded decreasing function on $[0,1]$;*

(3) $u_1^1 \leq u_1^2$;

(4) *u^1 and u^2 are left continuous on $(0,1]$ and right continuous at 0.*

(b) *If v^1 and v^2 satisfy the above (1)–(4), then there exists a unique $\tilde{v} \in F(R)$ such that $L_\alpha \tilde{v} = [v_\alpha^1, v_\alpha^2]$ for all $\alpha \in [0,1]$.*

PROOF. See Goetschel and Voxman (1986). □

By the above theorem, we can identify a fuzzy number $\tilde{u}$ in R with the parameterized representation $\{(u_\alpha^1, u_\alpha^2) \mid 0 \le \alpha \le 1\}$, where u^1 and u^2 satisfy (1)–(4) of Theorem 2.1.

Now, we define the metric d on $F(R)$ by

$$d(\tilde{u}, \tilde{v}) = \sup_{0 \le \alpha \le 1} \max\left(|u_\alpha^1 - v_\alpha^1|, |u_\alpha^2 - v_\alpha^2|\right). \tag{2.1}$$

Also, the norm $\|\tilde{u}\|$ of fuzzy number $\tilde{u}$ is defined as

$$\|\tilde{u}\| = d(\tilde{u}, \tilde{0}) = \max\left(|u_0^1|, |u_0^2|\right),$$

where $\tilde{0} = I_{\{0\}}$ is the indicator function of $\{0\}$. Then it is well-known that $F(R)$ is complete, but is not separable with respect to the metric d (see Klement *et al.*, 1986). Let us denote

$$CF(R) = \{\tilde{u} \in F(R) : u^1 \text{ and } u^2 \text{ are continuous on } [0,1]\}.$$

A fuzzy number $\tilde{u}$ is called level-wise continuous if $\tilde{u} \in CF(R)$. Then it is known that $\tilde{u} \in CF(R)$ if and only if for each $\beta \in (0,1)$, there exist at most two different $x_1, x_2 \in R$ such that $\tilde{u}(x_1) = \tilde{u}(x_2) = \beta$ (see Theorem 5.1 of Congxin and Ming, 1992).

THEOREM 2.2. *$(CF(R), d)$ is complete and separable.*

PROOF. If $\{\tilde{u}_n\}$ is a sequence of $CF(R)$ such that $d(\tilde{u}_n, \tilde{u})$ for some $\tilde{u} \in F(R)$, then u_n^1 and u_n^2 converge to u^1 and u^2 uniformly on $[0,1]$, respectively. This implies that u^1 and u^2 are continuous on $[0,1]$ and $CF(R)$ is a closed subspace of $F(R)$. Thus the completeness of $CF(R)$ is trivial.

Now to prove the separability of $CF(R)$, let $F_0(R)$ be the family of fuzzy numbers $\tilde{v}$ which for some positive integer k, there exist rational points $a_0 < a_1 < \cdots < a_k \le b_k < \cdots < b_1 < b_0$ such that $\tilde{v}(x) = j/k$ at $x = a_j$ or b_j for some j and linear in between. Then $F_0(R)$ is exactly the same as the family of fuzzy numbers $\tilde{v}$ such that for some k, v^1 and v^2 have the rational values at $\alpha = j/k$, $j = 0, 1, \ldots, k$ and linear in between. Then it is trivial that $F_0(R)$ is a countable subset of $CF(R)$. Now it suffices to prove that $F_0(R)$ is dense in $CF(R)$ *w.r.t.* the metric d.

Let $\tilde{u} \in CF(R)$ and $\epsilon > 0$ be arbitrary fixed. Since u^1 and u^2 are uniformly continuous on [0,1], there exists $\delta > 0$ such that

$$|\alpha - \beta| < \delta \text{ implies } |u_\alpha^i - u_\beta^i| < \frac{\epsilon}{2},\ i = 1, 2.$$

Now we choose a positive integer r such that $1/r < \delta$ and take rational points $a_0 < a_1 < \cdots < a_r \le b_r < \cdots < b_1 < b_0$ so that $|u^1_{j/r} - a_j| < \epsilon/2$ and $|u^2_{j/r} - b_j| < \epsilon/2$.

Now if we define $\tilde{v}(x) = j/r$ at $x = a_j$ or b_j for some j and linear in between, then $\tilde{v} \in F_0(R)$. By the construction of $\tilde{v}$, we have for $i = 1, 2$,

$$|u^i_{j/r} - v^i_{j/r}| < \frac{\epsilon}{2}.$$

If $(j-1)/r \le \alpha \le j/r$, then

$$|u^i_\alpha - v^i_{j/r}| \le |u^i_\alpha - u^i_{j/r}| + |u^i_{j/r} - v^i_{j/r}| < \epsilon.$$

Similarly, $|u^i_\alpha - v^i_{(j-1)/r}| < \epsilon$. Since v^i_α is a convex combination of $v^i_{(j-1)/r}$ and $v^i_{j/r}$, we obtain $|u^i_\alpha - v^i_\alpha| < \epsilon$. This implies $d(\tilde{u}, \tilde{v}) < \epsilon$. This completes the proof. □

For $\tilde{u} \in CF(R)$ and $0 < \delta \le 1$, we define

$$\begin{aligned}\tau_{\tilde{u}}(\delta) = \tau(\tilde{u}, \delta) &= \sup_{|\alpha-\beta|\le\delta} \max\left(|u^1_\alpha - u^1_\beta|, |u^2_\alpha - u^2_\beta|\right) \\ &= \Big\{ \sup_{0\le\alpha\le 1-\delta} \max\left(u^1_{\alpha+\delta} - u^1_\alpha, u^2_\alpha - u^2_{\alpha+\delta}\right)\Big\} \\ &\quad \vee \Big\{ \max\left(u^1_1 - u^1_{1-\delta}, u^2_{1-\delta} - u^2_1\right)\Big\}.\end{aligned}$$

Since u^1 and u^2 are uniformly continuous on $[0, 1]$, we can obtain the following lemma.

LEMMA 2.1. *$\lim_{\delta\to 0} \tau_{\tilde{u}}(\delta) = 0$ for each $\tilde{u} \in CF(R)$.*

The following lemma implies that $\tau_{\tilde{u}}(\delta)$ is continuous in $\tilde{u}$ for each fixed $\delta \in (0, 1]$.

LEMMA 2.2. *$|\tau_{\tilde{u}}(\delta) - \tau_{\tilde{v}}(\delta)| \le 2d(\tilde{u}, \tilde{v})$.*

PROOF. If $\alpha, \beta \in [0, 1]$, then for $i = 1, 2$,

$$\begin{aligned}|u^i_\alpha - u^i_\beta| &\le |u^i_\alpha - v^i_\alpha| + |u^i_\beta - v^i_\beta| + |v^i_\alpha - v^i_\beta| \\ &\le 2d(\tilde{u}, \tilde{v}) + |v^i_\alpha - v^i_\beta|,\end{aligned}$$

which implies $\tau_{\tilde{u}}(\delta) \le 2d(\tilde{u}, \tilde{v}) + \tau_{\tilde{v}}(\delta)$. Thus we have $\tau_{\tilde{u}}(\delta) - \tau_{\tilde{v}}(\delta) \le 2d(\tilde{u}, \tilde{v})$. By similar arguments, we can obtain $\tau_{\tilde{v}}(\delta) - \tau_{\tilde{u}}(\delta) \le 2d(\tilde{u}, \tilde{v})$. This completes the proof. □

Note that a continuous function $\phi : [0,\infty) \to [0,\infty)$ is called a modulus of continuity if $\phi(0) = 0$ and $\phi(s) \le \phi(s+t) \le \phi(s) + \phi(t)$. Thus if we define $\tau_{\tilde{u}}(0) = 0$ and $\tau_{\tilde{u}}(\delta) = \tau_{\tilde{u}}(1)$ for $\delta > 1$, then $\tau_{\tilde{u}}(\delta)$ as a function of δ is a modulus of continuity.

3. Main Results

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A fuzzy number valued function $\tilde{X} : \Omega \to F(R)$ can be identified with the parameterized representation $\{(X_\alpha^1, X_\alpha^2) \mid 0 \le \alpha \le 1\}$. A fuzzy number valued function $\tilde{X}$ is called a fuzzy random variable if for each $\alpha \in [0,1]$, X_α^1 and X_α^2 are random variables as real-valued functions. If $\tilde{X}$ is a random element of the metric space $(F(R), d)$, then it is a fuzzy random variable. But the converse is not true. If $\tilde{X}$ is $CF(R)$-valued, Theorem 3.1 of Joo and Kim (2000), and Corollary 3.3 of Kim (2002) imply that $\tilde{X}$ is a fuzzy random variable if and only if it is a random element of $(CF(R), d)$. Here we give a direct proof.

Theorem 3.1. *Let $\tilde{X} : \Omega \to CF(R)$. Then $\tilde{X}$ is a fuzzy random variable if and only if it is a random element of the metric space $(CF(R), d)$.*

Proof. (Sufficiency): Let $\tilde{X}$ be a random element of the metric space $(CF(R), d)$. For each $\alpha \in [0,1]$, let us define

$$f_\alpha^i : CF(R) \to R,\ f_\alpha^i(\tilde{u}) = u_\alpha^i,\ i = 1,2.$$

Then f_α^i is continuous. Since $X_\alpha^i = f_\alpha^i(\tilde{X})$, the sufficiency is proved.

(Necessity): Suppose that X_α^1 and X_α^2 are random variables for each $\alpha \in [0,1]$. For $\tilde{v} \in CF(R)$, let $B_\epsilon(\tilde{v}) = \{\tilde{u} \in CF(R) : d(\tilde{u}, \tilde{v}) \le \epsilon\}$. Then

$$\begin{aligned} &\{\omega \in \Omega : \tilde{X}(\omega) \in B_\epsilon(\tilde{v})\} \\ &= \bigcap_{n=1}^{\infty} \{\omega : \max\left(|X_{\alpha_n}^1(\omega) - v_{\alpha_n}^1|,\ |X_{\alpha_n}^2(\omega) - v_{\alpha_n}^2|\right) \le \epsilon\}, \end{aligned}$$

where $\{\alpha_n : n = 1, 2, \ldots\}$ is an enumeration of all rational points in [0,1]. Hence $\{\omega \in \Omega : \tilde{X}(\omega) \in B_\epsilon(\tilde{v})\} \in \mathcal{A}$. Since $(CF(R), d)$ is separable, every open subsets of $CF(R)$ can be represented by a countable union of closed balls of $CF(R)$. Thus, for each open subset O of $(CF(R), d)$,

$$\{\omega \in \Omega : \tilde{X}(\omega) \in O\} \in \mathcal{A}.$$

This completes the proof. □

A fuzzy random variable $\tilde{X}$ is called level-wise continuous if it is $CF(R)$-valued. By Theorem 3.1, a level-wise continuous fuzzy random variable is a random element of the metric space $(CF(R), d)$. Thus we can apply the notion of tightness for random elements in a complete separable metric space to the case of fuzzy random variables.

DEFINITION 3.1. *$\{\tilde{X}_n\}$ be a sequence of level-wise continuous fuzzy random variables. Then $\{\tilde{X}_n\}$ is said to be tight if for each $\epsilon > 0$, there exists a compact subset K of $(CF(R), d)$ such that*

$$P(\tilde{X}_n \notin K) < \epsilon \quad \textit{for all } n.$$

Now we wish to characterize the tightness for a sequence of level-wise continuous fuzzy random variables. To this end, we first need to present a characterization of compact subsets of $(CF(R), d)$. A characterization of compact subsets of $(F(R), d)$ was obtained by Diamond and Kloeden (1989) by using support functions of fuzzy numbers, and that of $F(R)$ endowed with another metric was done by Ghil *et al.* (2001). Here we restrict ourselves to the space $(CF(R), d)$ and give another characterizations by using the modulus of continuity $\tau_{\tilde{u}}(\delta)$.

THEOREM 3.2. *Let A be a subset of $CF(R)$. Then A is relatively compact if and only if*

$$\sup_{\tilde{u} \in A} \|\tilde{u}\| < \infty \tag{3.1}$$

and

$$\lim_{\delta \to 0} \sup_{\tilde{u} \in A} \tau(\tilde{u}, \delta) = 0. \tag{3.2}$$

PROOF. The proof will be proceeded by similar arguments in Ghil *et al.* (2001). Let A be relatively compact. Then (3.1) is trivial. Since $\tau(\tilde{u}, \delta)$ is continuous in $\tilde{u}$ and non-increasing as $\delta \downarrow 0$, (3.2) follows from Dini's theorem (see Lemma 3.3 of Ghil *et al.*, 2001).

To prove the converse, suppose that (3.1) and (3.2) hold. Since $CF(R)$ is complete, it suffices to prove that A is totally bounded. For a given $\epsilon > 0$, we choose a positive integer k such that

$$\tau(\tilde{u}, 1/k) < \epsilon \quad \text{for all } \tilde{u} \in A.$$

Let $\sup_{\tilde{u} \in A} \|\tilde{u}\| = M$ and take a partition $-M = x_0 < x_1 < \cdots < x_n = M$ of $[-M, M]$ satisfying $x_j - x_{j-1} < \epsilon$ for all j. Let B be the family of fuzzy numbers

$\tilde{v}$ which there exist points $a_0 < a_1 < \cdots < a_k \leq b_k < \cdots < b_1 < b_0$ from $\{x_0, x_1, \ldots, x_n\}$ such that $\tilde{v}(x) = i/k$ at $x = a_i$ or b_i for some i and linear in between. Then B is exactly same as the family of fuzzy numbers $\tilde{v}$ such that v^1 and v^2 have the values in $\{x_0, x_1, \ldots, x_n\}$ at i/k, $i = 0, 1, \ldots, k$ and linear in between. Then it is trivial that B is a finite subset of $CF(R)$.

Now we shall show that B is a 2ϵ-net for A. Let $\tilde{u} \in A$. For $m = 0, 1, \ldots, k$, we put

$$a_m = \max\left\{x_j \,|\, x_j \leq u^1_{m/k}\right\},$$
$$b_m = \min\left\{x_j \,|\, x_j \geq u^2_{m/k}\right\},$$

and let $\tilde{v}$ is an element of B defined as above. Then

$$\max\left(|u^1_{m/k} - v^1_{m/k}|, |u^2_{m/k} - v^2_{m/k}|\right) < \epsilon.$$

If $(m-1)/k \leq \alpha \leq m/k$, since $\tau(\tilde{u}, 1/k) < \epsilon$, we have

$$|u^1_\alpha - v^1_{m/k}| \leq |u^1_\alpha - u^1_{m/k}| + |u^1_{m/k} - v^1_{m/k}| < 2\epsilon,$$

and similarly for $v^1_{(m-1)/k}$. Since u^1_α is a convex combination of $u^1_{(m-1)/k}$ and $u^1_{m/k}$, we obtain $|u^1_\alpha - v^1_\alpha| < 2\epsilon$.

By the same argument, $|u^2_\alpha - v^2_\alpha| < 2\epsilon$. Therefore, $d(\tilde{u}, \tilde{v}) < 2\epsilon$, which completes the proof. □

Suppose now that (3.2) holds and for some $\alpha \in [0, 1]$,

$$\sup_{\tilde{u} \in A} \max\left(|u^1_\alpha|, |u^2_\alpha|\right) < \infty.$$

Then we can choose a positive integer k large enough that $\sup_{\tilde{u} \in A} \tau(\tilde{u}, 1/k)$ is finite. Then for $i = 1, 2$, since

$$\begin{aligned} |u^i_0| &\leq |u^i_1| + \sum_{j=1}^{k} |u^i_{j\alpha/k} - u^i_{(j-1)\alpha/k}| \\ &\leq |u^i_\alpha| + k\tau(\tilde{u}, \delta), \end{aligned}$$

we have that $\sup_{\tilde{u} \in A} \|\tilde{u}\| < \infty$. In fact, under the condition that (3.2) holds, $\sup_{\tilde{u} \in A} \|\tilde{u}\| < \infty$ if and only if

$$\sup_{\tilde{u} \in A} \max\left(|u^1_1|, |u^2_1|\right) < \infty.$$

Therefore we conclude the following corollary.

COROLLARY 3.1. *Let A be a subset of $CF(R)$. Then A is relatively compact if and only if*

$$\sup_{\tilde{u}\in A} \max\left(|u_1^1|,\, |u_1^2|\right) < \infty$$

and

$$\lim_{\delta\to 0}\ \sup_{\tilde{u}\in A} \tau(\tilde{u},\delta) = 0.$$

Now we are in a position to characterize the tightness of level-wise continuous fuzzy random variables.

THEOREM 3.3. *Let $\{\tilde{X}_n\}$ be a sequence of level-wise continuous fuzzy random variables. Then $\{\tilde{X}_n\}$ is tight if and only if*

(1) *For each $\eta > 0$, there exists a $\lambda > 0$ such that for all n,*

$$P\big(\{\omega : \|\tilde{X}_n(\omega)\| > \lambda\}\big) \le \eta; \tag{3.3}$$

(2) *For each $\epsilon > 0$ and $\eta > 0$, there exists a $\delta \in (0,1)$ such that for all n,*

$$P\big(\{\omega : \tau(\tilde{X}_n(\omega),\delta) \ge \epsilon\}\big) \le \eta. \tag{3.4}$$

PROOF. We first note that $\tau(\tilde{X}_n(\omega),\delta)$ is a real-valued random variable since $\tau(\cdot,\delta)$ is continuous. Suppose that $\{\tilde{X}_n\}$ is tight. For a given $\eta > 0$, we choose a compact subset K of $CF(R)$ such that

$$P(\tilde{X}_n \notin K) < \eta \ \text{ for all } n.$$

By (3.1), there exists a $\lambda > 0$ such that

$$K \subset \{\tilde{u} : \|\tilde{u}\| \le \lambda\}.$$

Thus (1) holds. Now for (2), let $\epsilon > 0$ be given. Then by (3.2), there exists a $\delta \in (0,1)$ such that

$$K \subset \{\tilde{u} :\ \tau(\tilde{u},\delta) < \epsilon\}.$$

Therefore, (2) follows.

To prove the converse, suppose that (1) and (2) hold. For given $\eta > 0$, we choose $\lambda > 0$ so that

$$P\big(\{\omega : \|\tilde{X}_n(\omega)\| > \lambda\}\big) \le \frac{\eta}{2} \ \text{ for all } n.$$

Then for each positive integer k, we choose δ_k so that for all n,

$$P\big(\{\omega : \tau(\tilde{X}_n(\omega), \delta_k) \geq 1/k\}\big) \leq \frac{\eta}{2^{k+1}}.$$

Let $A_0 = \{\tilde{u} : \|\tilde{u}\| \leq \lambda\}$ and $A_k = \{\tilde{u} : \tau(\tilde{u}, \delta_k) < 1/k\}$, $k = 1, 2 \ldots$ If K is the closure of $A = \bigcap_{k=0}^{\infty} A_k$, then for all n,

$$P(\tilde{X}_n \notin K) \leq \sum_{k=0}^{\infty} P(\tilde{X}_n \notin A_k) < \eta.$$

Since A satisfies (3.1) and (3.2), K is compact. Therefore, $\{\tilde{X}_n\}$ is tight. □

If we replace Theorem 3.2 by Corollary 3.1 in the proof of the above theorem, we can obtain the following corollary.

COROLLARY 3.2. *Let $\{\tilde{X}_n\}$ be a sequence of level-wise continuous fuzzy random variables and denote $\tilde{X}_n = \{(X^1_{n\alpha}, X^2_{n\alpha}) \mid 0 \leq \alpha \leq 1\}$. Then $\{\tilde{X}_n\}$ is tight if and only if*

(1) *For each $\eta > 0$, there exists a $\lambda > 0$ such that for all n,*

$$P\big(\{\omega : \max\big(|X^1_{n1}(\omega)|, |X^2_{n1}(\omega)|\big) > \lambda\}\big) \leq \eta; \tag{3.5}$$

(2) *For each $\epsilon > 0$ and $\eta > 0$, there exists a $\delta \in (0,1)$ such that for all n,*

$$P\big(\{\omega : \tau(\tilde{X}_n(\omega), \delta) \geq \epsilon\}\big) \leq \eta.$$

Since $CF(R)$ is separable and complete, a single level-wise continuous fuzzy random variable is tight. Thus if (3.3), (3.4) and (3.5) hold except for finitely many n, we may ensure that (3.3), (3.4) and (3.5) hold for all n by increasing λ and decreasing δ. Therefore we have the modified forms of Theorem 3.3 and Corollary 3.2.

THEOREM 3.4. *Let $\{\tilde{X}_n\}$ be a sequence of level-wise continuous fuzzy random variables. Then $\{\tilde{X}_n\}$ is tight if and only if*

(1) $\lim_{\lambda\to\infty} \limsup_{n\to\infty} P\big(\{\omega : \|\tilde{X}_n(\omega)\| > \lambda\}\big) = 0;$

(2) *For each $\epsilon > 0$,* $\lim_{\delta\to 0} \limsup_{n\to\infty} P\big(\{\omega : \tau(\tilde{X}_n(\omega), \delta) \geq \epsilon\}\big) = 0.$

COROLLARY 3.3. *Let $\{\tilde{X}_n\}$ be a sequence of level-wise continuous fuzzy random variables. Then $\{\tilde{X}_n\}$ is tight if and only if*

(1) $\lim_{\lambda\to\infty} \limsup_{n\to\infty} P\big(\{\omega : \max\big(|X^1_{n1}(\omega)|, |X^1_{n1}(\omega)|\big) > \lambda\}\big) = 0;$

(2) *For each* $\epsilon > 0$, $\lim_{\delta\to 0} \limsup_{n\to\infty} P\big(\{\omega : \tau(\tilde{X}_n(\omega), \delta) \geq \epsilon\}\big) = 0.$

As a final result, we give a sufficient condition for (2) of Theorem 3.3.

THEOREM 3.5. *Let $\{\tilde{X}_n\}$ be a sequence of level-wise continuous fuzzy random variables. Suppose that for each $\epsilon > 0$ and $\eta > 0$, there exists a $\delta \in (0,1)$ such that for each $\alpha \in [0,1]$,*

$$P\big(\{\omega : \max\big[X^1_{n(\alpha+\delta)}(\omega) - X^1_{n\alpha}(\omega),\ X^2_{n\alpha}(\omega) - X^2_{n(\alpha+\delta)}(\omega)\big] \geq \epsilon\}\big) \leq \delta\eta,$$

for all n. Then condition (2) of Theorem 3.3 holds.

To prove the above theorem, we need the following lemma.

LEMMA 3.1. *If $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ and $\min_{2\leq j\leq r-1}(\alpha_j - \alpha_{j-1}) \geq \delta$, then for each $\tilde{u} \in CF(R)$, we have*

$$\tau(\tilde{u}, \delta) \leq 2 \max_{1\leq j\leq r} \max\big\{u^1_{\alpha_j} - u^1_{\alpha_{j-1}}, u^2_{\alpha_{j-1}} - u^2_{\alpha_j}\big\}.$$

PROOF. Let $M = \max_{1\leq j\leq r} \max\big\{u^1_{\alpha_j} - u^1_{\alpha_{j-1}}, u^2_{\alpha_{j-1}} - u^2_{\alpha_j}\big\}$. If $\alpha_{j-1} \leq \alpha < \alpha + \delta \leq \alpha_j$, then

$$u^1_{\alpha+\delta} - u^1_\alpha \leq u^1_{\alpha_j} - u^1_{\alpha_{j-1}} \leq M.$$

If $\alpha_{j-1} \leq \alpha < \alpha_j < \alpha + \delta \leq \alpha_{j+1}$, then

$$u^1_{\alpha+\delta} - u^1_\alpha = \big(u^1_{\alpha_{j+1}} - u^1_{\alpha_j}\big) + \big(u^1_{\alpha_j} - u^1_{\alpha_{j-1}}\big) \leq 2M.$$

In any case, $u^1_{\alpha+\delta} - u^1_\alpha \leq 2M$. Similarly, $u^2_\alpha - u^2_{\alpha+\delta} \leq 2M$, and so we have $\tau(\tilde{u}, \delta) \leq 2M$. □

PROOF OF THEOREM 3.5. Let us denote by $\lfloor\lambda\rfloor$ the largest integer not greater than λ. If we denote $\lfloor 1/\delta\rfloor = r$ and take $\alpha_j = j\delta$ for $j = 0, 1, \ldots, r-1$ and $\alpha_r = 1$. Then by Lemma 3.1,

$$\begin{aligned} &P\big(\{\omega : \tau(\tilde{X}_n(\omega), \delta) \geq 2\epsilon\}\big) \\ &\leq \sum_{j=1}^{r} P\big(\{\omega : \max_{1\leq j\leq r} \max\big[X^1_{n\alpha_j}(\omega) - X^1_{n\alpha_{j-1}}(\omega), X^2_{n\alpha_{j-1}}(\omega) - X^2_{n\alpha_j}(\omega)\big] \geq \epsilon\}\big) \\ &\leq r\delta\eta \leq \eta, \end{aligned}$$

which completes the proof. □

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