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THE OPTIMAL CAPACITY OF THE FINITE DAM WITH COMPOUND POISSON INPUTS

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ABSTRACT

We consider the finite dam with compound Poisson inputs which is called $M/G/1$ finite dam. We assign some costs related to operating the dam and calculate the long-run average cost per unit time. Then, we find the optimal dam capacity under which the average costs is minimized.

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1. INTRODUCTION

In this paper, the finite dam with compound Poisson inputs is considered. Input of water occurs according to a Poisson process of rate $\nu > 0$. The amounts of inputs are independent and identically distributed with common distribution function G. However, at each moment of input, the level of water in the reservoir is bounded by $k > 0$. That is, as soon as the level exceeds k, the amount of overflow is lost instantaneously and the level becomes k. The release rate of the dam is 1 as long as the dam is not empty. This model is also called $M/G/1$ finite dam or $M/G/1$ queue with uniformly bounded virtual waiting times.

The optimization of dam has been studied by many authors. Faddy (1974) introduced P_λ^M-policy as a release policy in the finite dam with a Wiener process input and showed the optimality of the policy. In Yeh (1985), $P_{\lambda,\tau}^M$-policy, the generalization of P_λ^M-policy, is applied to the finite dam with a Wiener process input and the long-run average cost per unit time is determined. Lee and Lee (1997) applied P_λ^M-policy to the infinite dam with compound Poisson inputs and found the optimal M and λ.

However, the studies on optimization have been concentrated mainly on the release policy of dam. We are interested in the capacity of dam instead of the

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releasing policy. We optimize the capacity of the finite dam with compound Poisson inputs. To do this, we assign three costs related to operating the dam:

(i) $f(k)$ is the cost for construction and maintenance of the dam per unit time when the capacity is k. $f(\cdot)$ is assumed to be a nonconstant, increasing, and convex function;

(ii) $a \geq 0$ is the cost for unit amount of overflow of the dam;

(iii) $b \geq 0$ is the cost per unit time for emptiness of the dam.

Then, we calculate the long-run average cost per unit time in operating the dam and find the optimal k which minimizes the average costs. Throughout this paper, we say f is *increasing* if $f(x_1) \leq f(x_2)$ for $x_1 < x_2$, and *strictly increasing* if $f(x_1) < f(x_2)$ for $x_1 < x_2$. Besides, by *convex* function we mean the function whose derivative is increasing.

2. THE LONG-RUN AVERAGE COST PER UNIT TIME

Note that the process of the level of water in the reservoir is a regeneration process and the time epochs when the first input of water is.occurred after the dam has been empty are regeneration points. Therefore, by the renewal reward theorem (see Ross, 1983, p. 78), the long-run average cost per unit time is given by

$$f(k) + \frac{E\big[\text{cost for overflow and emptiness during a regeneration cycle}\big]}{E\big[\text{the length of a regeneration cycle}\big]}.$$

By Kim *et al.* (2001), the expected length of wet period is $(H(k)-1)/\nu$, where $H(x) = \sum_{n=0}^{\infty} \rho^n G_e^{n*}(x)$, $\rho = \nu m$ is the traffic intensity of the dam, $m = \int_0^{\infty}(1-G(s))ds$, $G_e(x) = (1/m)\int_0^x (1-G(s))ds$ is the equilibrium distribution function of G, and $n*$ is the n-fold recursive Stieltjes convolution with G_e^{0*} being Heaviside function. Hence, the expected length of a regeneration cycle, the sum of the expected wet period and dry (or idle) period, is $H(k)/\nu$.

On the other hand, the expected amount of overflow during a cycle is calculated as follows:

$$\begin{aligned}
&E\big[\text{the amount of overflow during a cycle}\big] \\
&\quad = E\big[\text{the amount of input during a cycle}\big] \\
&\qquad -E\big[\text{the amount of output during a cycle}\big] \\
&\quad = \nu m E\big[\text{the length of a cycle}\big] - E\big[\text{the length of wet period}\big] \\
&\quad = \frac{1-(1-\rho)H(k)}{\nu}.
\end{aligned}$$

In the second equality of the above equations, we use the property of Poisson process and the renewal reward theorem, by which

$$\frac{E\big[\text{the amount of input during a cycle}\big]}{E\big[\text{the length of a cycle}\big]}$$

is the long-run average amount of input per unit time, that is, νm. Now, the long-run average cost per unit time is given by

$$\begin{aligned}
C(k) &= f(k) + \frac{a\cdot\{1-(1-\rho)H(k)\}/\nu + b\cdot(1/\nu)}{H(k)/\nu} \\
&= f(k) + \frac{a+b}{H(k)} - a(1-\rho).
\end{aligned}$$

3. Optimization

In this section, we show that there exists a unique k which minimizes $C(k)$ by proving that $C(k)$ is a convex function of k. To prove this, we need to know some properties of $H(k)$.

LEMMA 3.1. *The expected number of overflows during a cycle is $H'(k)/\nu$.*

PROOF. In Bae *et al.* (2001), the $M/G/1$ queue with impatient customers where the customers wait only for k is considered. We observe that the number of workload's crossings over k before reaching 0 in the $M/G/1$ queue with impatient customers is equal, in distribution, to the number of overflows during a cycle in the finite dam. Hence, if we denote, by $N(x)$, the number of overflows during a cycle given that the initial level of water is x, then from the result in Bae *et al.* (2001), we see that

$$P\{N(x)\geq n\} = \left(1-\frac{H(k-x)}{H(k)}\right)\left(1-\frac{1}{H(k)}\right)^{n-1}, \quad n=1,2,3,\ldots$$

and that

$$E\big[N(x)\big] = H(k) - H(k-x), \quad 0 \le x \le k.$$

Therefore, it follows, from conditioning on the amount of the first input after the dam has been empty, that

$$\begin{aligned} &E\big[\text{number of overflows during a cycle}\big] \\ &= \int_0^k E\big[N(x)\big]dG(x) + \int_k^\infty \Big\{1 + E\big[N(k)\big]\Big\}dG(x) \\ &= H(k) - \int_0^k H(k-x)dG(x) \\ &= H'(k)/\nu, \end{aligned}$$

where the last equality is justified in Bae *et al.* (2001).

An alternative proof of the lemma is given in Kim and Lee (2002). □

LEMMA 3.2. *$H'(k)/H(k)$ is strictly decreasing in k.*

PROOF. Note that

$$\frac{H'(k)}{H(k)} = \frac{H'(k)/\nu}{H(k)/\nu} = \frac{E\big[\text{number of overflows during a cycle}\big]}{E\big[\text{length of a cycle}\big]},$$

which is equal to the long-run average number of overflows per unit time, and clearly decreases strictly as the capacity k of the dam increases. □

LEMMA 3.3. *$1/H(k)$ is strictly decreasing in k and the derivative is strictly increasing in k.*

PROOF. At first, by the definition of $H(k)$, we see that $H(k)$ is strictly increasing in k since each G_e^{n*} is the distribution function of a certain random variable. The derivative of $1/H(k)$ is given by

$$\frac{d}{dk}\left(\frac{1}{H(k)}\right) = -\frac{1}{H(k)}\frac{H'(k)}{H(k)}.$$

Here, $H(k)$ and $H'(k)$ are nonnegative, and $1/H(k)$ and $H'(k)/H(k)$ are strictly decreasing in k. Hence, the derivative of $1/H(k)$ is strictly increasing in k. □

LEMMA 3.4.

$$\text{(i)} \lim_{k \to 0} H'(k) = \nu; \quad \text{(ii)} \lim_{k \to \infty} \frac{d}{dk}\left(\frac{1}{H(k)}\right) = 0$$

PROOF.

(i) As k goes to 0, the length of wet period goes to 0 and the expected number of overflows during a cycle goes to 1. That is,

$$\lim_{k \to 0} \frac{H'(k)}{\nu} = 1.$$

(ii) Recall that $1/H(k)$ is positive, strictly decreasing and convex in k, and observe that $1/H(k)$ goes to $1-\rho$ if $\rho < 1$ and to 0 if $\rho \geq 1$. Hence, the derivative of $1/H(k)$ goes to 0. □

THEOREM 3.1. *If $f'(0+) \geq (a+b)\nu$, then $C(k)$ is strictly increasing in k, and if $f'(0+) < (a+b)\nu$, then there exists a unique k^* in $(0, \infty)$ such that $C(k)$ is minimized at $k = k^*$, and k^* is the unique solution of*

$$f'(k) = (a+b)\frac{H'(k)}{\{H(k)\}^2}.$$

PROOF. The derivative of $C(k)$ is given by

$$C'(k) = f'(k) + (a+b)\frac{d}{dk}\left(\frac{1}{H(k)}\right).$$

By the assumption on $f(k)$ and Lemma 3.3, $C'(k)$ is strictly increasing in k. Now, by Lemma 3.4, we have

$$\lim_{k \to 0} C'(k) = f'(0+) - (a+b)\nu$$

and

$$\lim_{k \to \infty} C'(k) = f'(\infty) > 0.$$

Thus, in conclusion, if $f'(0+) \geq (a+b)\nu$, then $C'(k) > 0$ for all $k > 0$, and if $f'(0+) < (a+b)\nu$, then $C(k)$ is minimized when $k = k^*$, and k^* is the unique solution of

$$f'(k) = (a+b)\frac{H'(k)}{\{H(k)\}^2}. \tag{3.1}$$

□

As for calculation and approximation of $H(k)$, we refer to Bae *et al.* (2002). In practice, to obtain k which minimizes $C(k)$, some numerical methods should be used to solve Eq. (3.1).

4. NUMERICAL EXAMPLES

In this section, we find k which minimizes $C(k)$ by numerical method in case that the input amounts of water are exponentially distributed. We put the mean of input 1. Then,

$$H(k) = \frac{1 - \rho e^{-(1/m-\nu)k}}{1-\rho} = \frac{1 - \rho e^{-(1-\rho)k}}{1-\rho}.$$

TABLE 4.1 $a = 100,\ b = 50,\ f(k) = k + 1$

$\nu = \rho$	0.2	0.4	0.6	0.8	1.2
k^*	3.72	5.18	6.86	9.20	9.41

TABLE 4.2 $\nu = 0.5,\ f(k) = k + 1$

k^*		a		
		200	300	400
b	0	6.52	7.30	7.86
	30	6.79	7.49	8.01
	60	7.02	7.66	8.14

TABLE 4.3 $\nu = 0.5,\ a = 200,\ b = 50$

$f(k)$	$1 + k/10$	$1 + k$	$1 + 150k$	$1 + k^2/10$	$e^{k/10}$
k^*	11.50	6.95	0.00	6.45	9.59

With ν varying and the other parameters fixed, we investigate the behavior of $C(k)$ and find k^*. The result is given in Table 4.1. In Table 4.2, we do for various a and b and fixed ν and $f(\cdot)$. Finally, Table 4.3 shows the relation of $f(\cdot)$ and k^*.

From the examples, we confirm that the larger capacity is required when the input occurs more frequently or the costs for overflow or emptiness are higher, and that the capacity should be small when the cost for construction and maintenance of dam is high. If $f'(0)$ is sufficiently large, *i.e.*, the minimal cost for construction of dam is very expensive, then the optimal capacity is 0, that is, it is better not to construct dam.

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