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A NOTE ON LATTICE DISTRIBUTIONS ON THE TORUS †

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ABSTRACT

In the recent papers by Harris and Park (1994) and by Hui and Park (2000), a family of lattice distributions derived from a sum of independent identically distributed random variables is examined. In this paper we generalize a result of Hui and Park (2000) on lattice distributions on the torus using the Poisson summation formula.

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1. INTRODUCTION AND THE MAIN THEOREMS

In a recent paper by Hui and Park (2000), they examine a family of lattice distribution derived from a sum of independent identically distributed random variables having a probability density function of the form

$$f(x) = \sum_{j=-\infty}^{\infty} a_j \chi_{I_j}(x),$$

where $\{I_j\}_{j=-\infty}^{\infty}$ is a uniform partition of $\mathbf{R}$ with $|I_j| = |I|$, $a_j \geq 0$ for each j, and $\sum_{j=-\infty}^{\infty} a_j = |I|^{-1}$.

This paper extends the result of Hui and Park (2000) to a family of lattice distribution on the torus. More specifically, let T_{m+1} denotes the sum of $m+1$ independent random vectors having a probability density function of the form

$$g(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^n} h_{\mathbf{k}} \chi_{Q_{\mathbf{k}}}(\mathbf{x}), \quad \mathbf{x} \in \mathbf{R}^n, \tag{1.1}$$

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where

$$\sum_{\mathbf{k}\in\mathbf{Z}^n} h_{\mathbf{k}} = |H|^{-n}, \quad Q_{\mathbf{k}} = \prod_{j=1}^{n}[\alpha + k_j|H|, \alpha + (k_j+1)|H|),$$
$$\text{for } \mathbf{k} = (k_1, k_2, ..., k_n) \in \mathbf{Z}^n, \quad |Q_{\mathbf{k}}| = |H|^n.$$

Define a lattice distribution, g_{m+1} on the torus derived from the probability density function $g_{T_{m+1}}$ by

$$\begin{aligned} g_{m+1}(\mathbf{x}) &= g_{T_{m+1}}(\mathbf{x}), \quad \text{for } \mathbf{x} = (\alpha+\delta_1+k_1|H|, ..., \alpha+\delta_n+k_n|H|), \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{1.2}$$

THEOREM 1.1. *For* $0 \le \delta_i < |H|$, $i = 1, 2, ..., n$,

$$\sum_{\mathbf{k}\in\mathbf{Z}^n} g_{m+1}(\alpha+\delta_1+k_1|H|, ..., \alpha+\delta_n+k_n|H|) = |H|^{-n}.$$

THEOREM 1.2. *Let* $W_{m+1,\delta}$ *be a random vector having the lattice distribution* g_{m+1} *on the torus. For* $0 \le \delta_i < |H|$, $i = 1, 2, ..., n$ *and* $l_1, l_2, ..., l_n$ *such that* $\sum_{j=1}^{n} l_j = L \le m$,

$$\mathbf{E}\left[W_{m+1,\delta}^{L}\right] = \mathbf{E}\left[T_{m+1}^{L}\right],$$

where

$$W_{m+1,\delta}^{L} = W_1^{l_1} W_2^{l_2} \cdots W_n^{l_n} \quad \textit{and} \quad T_{m+1}^{L} = T_1^{l_1} T_2^{l_2} \cdots T_n^{l_n}.$$

2. PROOFS OF THE THEOREMS

Let

$$\hat{g}_{T_{m+1}}(\mathbf{s}) = \int_{\mathbf{R}^n} g_{T_{m+1}}(\mathbf{t}) e^{-2\pi i \mathbf{s}\cdot\mathbf{t}} d\mathbf{t} \tag{2.1}$$

be the Fourier transform of $g_{T_{m+1}}$, where $\mathbf{s}\cdot\mathbf{t} = \sum_{j=1}^{n} s_j t_j$.

PROOF OF THEOREM 1.1. We apply the Poisson summation formula (see for example Folland, 1984) to $g_{T_{m+1}}$ and obtain the following result:

$$\sum_{\mathbf{k}\in\mathbf{Z}^n} g_{m+1}(\delta + \mathbf{k}|H|) = \frac{1}{|H|^n} \sum_{\mathbf{k}\in\mathbf{Z}^n} \hat{g}_{T_{m+1}}\left(\frac{\mathbf{k}}{|H|}\right) e^{2\pi i \mathbf{k}\cdot\delta/|H|}$$

where $\delta = (\delta_1, ..., \delta_n)$ and $\mathbf{k} = (k_1, ..., k_n)$. Since $g_{T_{m+1}}$ is $(m+1)$-fold convolution of g given in (1.1), we have $\hat{g}_{T_{m+1}} = \hat{g}^{m+1}$. The Fourier transform of g can be written as

$$\hat{g}(\mathbf{s}) = \sum_{\mathbf{k}\in\mathbf{Z}^n} h_{\mathbf{k}} \prod_{j=1}^{n} e^{2\pi i s_j \delta_j} \left[\frac{e^{-2\pi i k_j |H| s_j} - e^{-2\pi i (k_j+1)|H| s_j}}{2\pi i s_j} \right] . \tag{2.2}$$

for $\mathbf{s} \neq \mathbf{0}$ and $\hat{g}(\mathbf{0}) = 1$. We observe that

$$\hat{g}_{T_{m+1}}\left(\frac{\mathbf{k}}{|H|}\right) = \hat{g}^{m+1}\left(\frac{\mathbf{k}}{|H|}\right) = 0, \quad \mathbf{k} \neq \mathbf{0}$$

and $\hat{g}_{T_{m+1}}(\mathbf{0}) = 1$, because $e^{-2\pi i j} = 1$ for all integer j. The proof of Theorem 1.1 is complete. □

PROOF OF THEOREM 1.2. We apply the Poisson summation formula to $g_{T_{m+1}}(\mathbf{t})e^{-2\pi i \mathbf{s}\cdot\mathbf{t}}$ to obtain

$$\begin{aligned} &\sum_{\mathbf{k}\in\mathbf{Z}^n} g_{m+1}(\delta + \mathbf{k}|H|)e^{-2\pi i \mathbf{s}\cdot(\delta+\mathbf{k}|H|)} \\ &= \frac{1}{|H|^n} \sum_{\mathbf{k}\in\mathbf{Z}^n} \hat{g}_{T_{m+1}}\left(\mathbf{s} + \frac{\mathbf{k}}{|H|}\right) e^{2\pi i \mathbf{k}\cdot\delta/|H|}. \end{aligned} \tag{2.3}$$

Let $\hat{g}^{(L)}_{T_{m+1}}(\mathbf{s}) = \partial^L \hat{g}_{T_{m+1}}(\mathbf{s})/(\partial s_1^{l_1} \cdots \partial s_n^{l_n})$, where $l_1 + \cdots + l_n = L$ and $l_i \geq 0$ for $i = 1, ..., n$, be the L^{th} derivative of $\hat{g}_{T_{m+1}}(\mathbf{s})$. Since $\hat{g}_{T_{m+1}} = \hat{g}^{m+1}$ and $\hat{g}(\mathbf{k}|H|^{-1}) = 0$ for $\mathbf{k} \neq \mathbf{0}$ by (2.2), it follows that $\hat{g}^{(L)}_{T_{m+1}}(\mathbf{k}|H|^{-1}) = 0$ for $\mathbf{k} \neq \mathbf{0}$ and $L \leq m$. Now differentiate (2.3) with respect to $\mathbf{s}$ and evaluate it at $\mathbf{s} = \mathbf{0}$ to obtain

$$(-2\pi i)^L \sum_{\mathbf{k}\in\mathbf{Z}^n} \prod_{j=1}^{n} (\delta_j + k_j|H|)^{l_j} g_{m+1}(\delta + \mathbf{k}|H|) = \frac{1}{|H|^n} \hat{g}^{(L)}_{T_{m+1}}(\mathbf{0}),$$

which is a constant independent of δ. Furthermore, for $l_1, ..., l_n$ such that $l_i \geq 0$ and $\sum_{j=1}^{n} l_j = L \leq m$, we have

$$\begin{aligned} \frac{1}{(-2\pi i)^L} \hat{g}^{(L)}_{T_{m+1}}(\mathbf{0}) &= \mathbf{E}\left(T^L_{m+1}\right) \\ &= \sum_{\mathbf{k}\in\mathbf{Z}^n} (\delta + \mathbf{k}|H|)^L \cdot g_{m+1}(\delta + \mathbf{k}|H|) \cdot |H|^n \\ &= \mathbf{E}\left(W^L_{m+1,\delta}\right). \end{aligned}$$

This completes the proof of Theorem 1.2. □

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