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THE CENTRAL LIMIT THEOREMS FOR THE MULTIVARIATE LINEAR PROCESS GENERATED BY WEAKLY ASSOCIATED RANDOM VECTORS†

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ABSTRACT

Let $\{\mathbb{X}_t\}$ be an m-dimensional linear process of the form $\mathbb{X}_t = \sum_{j=0}^{\infty} A_j \times \mathbb{Z}_{t-j}$, where $\{\mathbb{Z}_t\}$ is a sequence of stationary m-dimensional weakly associated random vectors with $E\mathbb{Z}_t = \mathbb{O}$ and $E||\mathbb{Z}_t||^2 < \infty$. We prove central limit theorems for multivariate linear processes generated by weakly associated random vectors. Our results also imply a functional central limit theorem.

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1. INTRODUCTION

Notions of positive dependence for collections of random variables have been much studied in recent years. The most prevalent positive dependence notion is that of association. A finite collection $\{Y_i,\ 1 \le i \le m\}$ of random variables is said to be associated if for all coordinatewise nondecreasing functions $f, g : \mathbb{R}^m \to \mathbb{R}$, $\mathrm{Cov}(f(Y_1, \dots, Y_m), g(Y_1, \dots, Y_m)) \ge 0$, where the covariance is defined. An infinite collection of random variables is associated if every finite subcollection is associated. This positive dependence notion was first defined by Esary *et al.* (1967). Associated sequences are widely encountered in applications: *e.g.* in reliability theory, in mathematical physics and in percolation theory (*cf.* Barlow

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and Proschan, 1975; Newman, 1980; Cox and Grimmett, 1984). Under some covariance restrictions a number of limit theorems have been proved for associated random variables. Newman (1980) proved the central limit theorem, and Newman and Wright (1981) extended this to a functional central limit theorem. Burton *et al.* (1986) defined weakly associated random vectors and proved a functional central limit theorem for such sequences. This was achieved by an extension of the Cramer-Wold device to suit the special needs of weakly associated vectors. And these results were extended to random vectors with values in a separable Hilbert space (see Burton *et al.*, 1986).

Let $\mathbb{X}_t,\ t = 0, \pm 1, \ldots$, be an m-dimensional linear process of the form

$$\mathbb{X}_t = \sum_{j=0}^{\infty} A_j \mathbb{Z}_{t-j} \tag{1.1}$$

defined on a probability space (Ω, A, P), where $\mathbb{X}_t,\ t = 0, \pm 1, \ldots$, is a sequence of strictly stationary m-dimensional random vectors with mean $\mathbb{O} : m \times 1$ and positive definite covariance matrix $\Gamma : m \times m$. The class of linear processes defined in (1.1) contains stationary multivariate autoregressive moving average processes(MARMA) and a popular approach to the development of asymptotics for time series has been the use of limit theorem for dependent random variables (see Brockwell and Davis, 1990). Fakhre-Zakeri and Lee (1993) established a central limit theorem for multivariate linear process generated by *iid* random vectors and they also derived a functional central limit theorem for multivariate linear process generated by martingale difference random vectors in 2000.

In this paper we prove the central limit theorem for stationary multivariate linear processes generated by weakly associated random vectors. We also extend this to a functional central limit theorem, which generalizes, to the stationary multivariate linear process, the functional central limit theorem of Burton *et al.* (1986) on the sum of weakly associated random vectors.

2. RESULTS

DEFINITION 2.1 (Burton *et al.*, 1986). *A sequence $\{\mathbb{Z}_t,\ t \geq 1\}$ of m-dimensional random vectors is said to be weakly associated if for all coordinatewise non-decreasing functions $f : \mathbb{R}^{nm} \to \mathbb{R}$ and $g : \mathbb{R}^{km} \to \mathbb{R}$ and for any permutation π of the positive integers we have*

$$\mathrm{Cov}(f(\mathbb{Z}_{\pi(1)}, \ldots, \mathbb{Z}_{\pi(n)}), g(\mathbb{Z}_{\pi(n+1)}, \ldots, \mathbb{Z}_{\pi(n+k)})) \geq 0, \tag{2.1}$$

whenever this covariance is defined.

LEMMA 2.2 (Newman and Wright, 1981). *Let $\{Y_1, \dots, Y_n\}$ be weakly associated random variables with $EY_i = 0$, $EY_i^2 < \infty$. Then*

$$E\left(\max_{1\le k\le n} |Y_1 + \cdots + Y_k|^2\right) \le E(|Y_1 + \cdots + Y_n|^2) \tag{2.2}$$

PROOF. See the proof of Theorem 2 of Newman and Wright (1981). □

LEMMA 2.3. *Let $\{\mathbb{Z}_t,\ t \ge 1\}$ be a strictly stationary sequence of weakly associated m-dimensional random vectors with $E(\mathbb{Z}_t) = 0$, $E||\mathbb{Z}_t||^2 < \infty$. Let $\mathbb{X}_t = \sum_{j=1}^{\infty} A_j \mathbb{Z}_{t-j}$, $\mathbb{S}_k = \sum_{t=1}^{k} \mathbb{X}_t$, $\tilde{\mathbb{X}}_t = (\sum_{j=1}^{\infty} A_j)\mathbb{Z}_t$ and $\tilde{\mathbb{S}}_k = \sum_{t=1}^{k} \tilde{\mathbb{X}}_t$. Assume*

$$\sum_{j=1}^{\infty} ||A_j|| < \infty \quad and \quad \sum_{j=1}^{\infty} A_j \ne \mathbb{O}_{m\times m}\ , \tag{2.3}$$

where for any $m \times m$, $m \ge 1$, matrix $A = (a_{ij})$, $||A|| = \sum_{i=1}^{m}\sum_{j=1}^{m} |a_{ij}|$ and $\mathbb{O}_{m\times m}$ denotes the $m \times m$ zero matrix and assume

$$E||\mathbb{Z}_1||^2 + 2\sum_{t=2}^{\infty}\sum_{i=1}^{m} E(\mathbb{Z}_{1i}\mathbb{Z}_{ti}) = \sigma^2 < \infty. \tag{2.4}$$

Then

$$n^{-1/2} \max_{1\le k\le n} ||\tilde{\mathbb{S}}_k - \mathbb{S}_k|| = o_p(1).$$

PROOF. First observe that

$$\begin{aligned}\tilde{\mathbb{S}}_k &= \sum_{t=1}^{k}\Big(\sum_{j=0}^{k-t} A_j\Big)\mathbb{Z}_t + \sum_{t=1}^{k}\Big(\sum_{j=k-t+1}^{\infty} A_j\Big)\mathbb{Z}_t \\ &= \sum_{t=1}^{k}\Big(\sum_{j=0}^{t-1} A_j\mathbb{Z}_{t-j}\Big) + \sum_{t=1}^{k}\Big(\sum_{j=k-t+1}^{\infty} A_j\Big)\mathbb{Z}_t,\end{aligned}$$

and thus

$$\begin{aligned}\tilde{\mathbb{S}}_k - \mathbb{S}_k &= -\sum_{t=1}^{k}\Big(\sum_{j=t}^{\infty} A_j\mathbb{Z}_{t-j}\Big) + \sum_{t=1}^{k}\Big(\sum_{j=k-t+1}^{\infty} A_j\Big)\mathbb{Z}_t \\ &= I_1 \ + \ I_2 \text{ (say)}.\end{aligned}$$

To prove

$$n^{-1/2} \max_{1\le k\le n} ||I_1|| = o_p(1), \tag{2.5}$$

note that

$$
\begin{aligned}
n^{-1} E \max_{1\le k\le n} & \left\| \sum_{t=1}^{k} \sum_{j=t}^{\infty} A_j \mathbb{Z}_{t-j} \right\|^2 \\
&= n^{-1}\ E \max_{1\le k\le n} \left\| \sum_{j=1}^{\infty} \sum_{t=1}^{j\wedge k} A_j\ \mathbb{Z}_{t-j} \right\|^2 \\
&\le n^{-1} \left\{ \sum_{j=1}^{\infty} \|A_j\| \left(E \max_{1\le k\le n} \left\| \sum_{t=1}^{j\wedge k} \mathbb{Z}_{t-j} \right\|^2 \right)^{1/2} \right\}^2 \\
&\qquad\qquad\qquad \text{by Minkowski inequality} \\
&\le \sigma^2 \left\{ \sum_{j=1}^{\infty} \|A_j\| \left(\frac{j\wedge k}{n} \right)^{1/2} \right\}^2
\end{aligned}
$$

by (2.2), (2.3) and (2.4). By the dominated convergence theorem the last term above tends to zero as $n \to \infty$. Thus (2.5) is proved by Markov inequality.

Next, we show that

$$n^{-1/2} \max_{1\le k\le n} \|I_2\| = o_p(1). \tag{2.6}$$

Write $I_2 = II_1 + II_2$, where

$$II_1 = A_1\mathbb{Z}_k + A_2(\mathbb{Z}_k + \mathbb{Z}_{k-1}) + \cdots + A_k(\mathbb{Z}_k + \cdots + \mathbb{Z}_1)$$

and

$$II_2 = (A_{k+1} + A_{k+2} + \cdots)\ (\mathbb{Z}_k + \cdots + \mathbb{Z}_1).$$

Let p_n be a sequence of positive integers such that

$$p_n \to \infty \ \text{ and } \ p_n/n \to 0 \ \text{ as } \ n \to \infty. \tag{2.7}$$

Then,

$$
\begin{aligned}
n^{-1/2} \max_{1\le k\le n} \|II_2\| &\le \Big(\sum_{i=0}^{\infty} \|A_i\| \Big) n^{-1/2} \max_{1\le k\le p_n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| \\
&\quad + \Big(\sum_{i>p_n} \|A_i\| \Big)\ n^{-1/2} \max_{1\le k\le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| \\
&= O_p\Big(\frac{p_n}{n}\Big) + O_p\Big(\sum_{i>p_n} \|A_i\| \Big) \\
&= o_p(1)
\end{aligned}
$$

by (2.3), (2.4) and (2.7).

It remains to prove that

$$Y_n \triangleq n^{-1/2} \max_{1\le k\le n} \|\Pi_1\| = o_p(1).$$

To this end, define for each $l \ge 1$

$$\Pi_{1,l} = B_1\mathbb{Z}_k + B_2(\mathbb{Z}_k + \mathbb{Z}_{k-1}) + \cdots + B_k(\mathbb{Z}_k + \cdots + \mathbb{Z}_1),$$

where

$$B_k = \begin{cases} A_k, & k \le l, \\ \mathbb{O}_{m\times m}, & k > l. \end{cases}$$

Let $Y_{n,l} = n^{-1/2}\max_{1\le k\le n} \|\Pi_{1,l}\|$. Clearly, for each $l \ge 1$,

$$Y_{n,l} = o_p(1). \tag{2.8}$$

On the other hand,

$$\begin{aligned} n(Y_{n,l} - Y_n)^2 &\le \max_{1\le k\le n} \left\| \sum_{i=1}^{k} (A_i - B_i)(\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}) \right\|^2 \\ &\le \max_{l<k\le n} \Big(\sum_{i=l+1}^{k} \|A_i\| \cdot \|\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}\| \Big)^2 \\ &\le \Big(\sum_{i>l} \|A_i\| \Big)^2 \max_{l<k\le n} \max_{l<i\le k} \|\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}\|^2 \\ &\le 4\Big(\sum_{i>l} \|A_i\| \Big)^2 \max_{l\le j\le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_j\|^2. \end{aligned}$$

From this result, (2.3) and (2.4), for any $\delta > 0$,

$$\begin{aligned} &\lim_{l\to\infty} \lim_{n\to\infty} \sup P(|Y_{n,l} - Y_n|^2 > \delta) \\ &\quad\le \lim_{l\to\infty} \lim_{n\to\infty} \sup\ 4\delta^{-1}\Big(\sum_{i>l} \|A_i\|^2 \Big)\, n^{-1} E \max_{1\le j\le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_j\|^2 \\ &\quad\le 4\delta^{-1}\sigma^2 \lim_{l\to\infty} \Big(\sum_{i>l} \|A_i\|^2 \Big) = 0. \end{aligned} \tag{2.9}$$

In view of (2.8) and (2.9), it follows from Theorem 4.2 of Billingsley (1968, p. 25) that $Y_n = o_p(1)$. This completes the proof of Lemma 2.3. □

LEMMA 2.4 (Burton *et al.*, 1986). *Let* $\{\mathbb{Z}_t,\ t \geq 1\}$ *be a strictly stationary weakly associated sequence m-dimensional real random vectors with* $E(\mathbb{X}_1) = \mathbb{O}$, $E||\mathbb{X}_1||^2 < \infty$. *Define, for* $u \in [0,1], n \geq 1$

$$W_n(u) = n^{-1/2} \sum_{t=1}^{[nu]} \mathbb{Z}_t \tag{2.10}$$

If (2.4) holds then, as $n \to \infty$

$$W_n \xrightarrow{w} B^m \tag{2.11}$$

where $\xrightarrow{w}$ *indicates weak convergence, and* B^m *is an m-dimensional Wiener process with covariance matrix* $\Gamma = [\sigma_{kj}]$,

$$\sigma_{kj} = E(Z_{1k}Z_{1j}) + \sum_{t=2}^{\infty} \{E(Z_{1k}Z_{tj}) + E(Z_{1j}Z_{tk})\}. \tag{2.12}$$

PROOF. See the proof of Theorem 2 of Burton *et al.* (1986). □

REMARK 1. In Lemma 2.4 by letting $u = 1$ we have

$$n^{-1/2} \sum_{t=1}^{n} \mathbb{Z}_t \xrightarrow{w} N(\mathbb{O}, \Gamma), \tag{2.13}$$

that is, $\{\mathbb{Z}_t\}$ satisfies the central limit theorem.

THEOREM 2.5. *Let* $\{\mathbb{Z}_t,\ t \geq 1\}$ *be a strictly stationary weakly associated sequence of m-dimensional random vectors with* $E(\mathbb{X}_t) = \mathbb{O}$, $E||\mathbb{X}_t||^2 < \infty$ *and* $\{\mathbb{X}_t\}$ *an m-dimensional linear process defined in (1.1). Set* $\mathbb{S}_n = \sum_{t=1}^n \mathbb{X}_t (\mathbb{S}_0 = \mathbb{O})$, $\tilde{\mathbb{S}}_n = \sum_{t=1}^n \tilde{\mathbb{X}}_t$ *as in Lemma 2.3. If (2.3) and (2.4) holds then*

$$n^{-1/2}\mathbb{S}_n \xrightarrow{\mathcal{D}} N(\mathbb{O}, T) \quad \text{as } n \to \infty, \tag{2.14}$$

where $T = (\sum_{j=1}^{\infty} A_j)\Gamma(\sum_{j=1}^{\infty} A_j)'$.

PROOF. First note that $n^{-1/2}\tilde{\mathbb{S}}_n = n^{-1/2}(\sum_{j=1}^{\infty} A_j)\sum_{t=1}^n \mathbb{Z}_t$ and that $n^{-1/2}\tilde{\mathbb{S}}_n \xrightarrow{\mathcal{D}} N(\mathbb{O}, T)$ according to Remark 1. Hence, $n^{-1/2}S_n \xrightarrow{\mathcal{D}} N(\mathbb{O}, T)$ follows by applying Lemma 2.3 and Theorem 4.1 of Billingsley (1968). □

We now introduce another central limit theorem.

THEOREM 2.6. *Let $\{\mathbb{Z}_t,\ t \geq 1\}$ be a strictly stationary weakly associated sequence of m-dimensional random vectors with $E(\mathbb{Z}_t) = \mathbb{O}$, $E||\mathbb{Z}_t||^2 < \infty$ and $\{\mathbb{X}_t\}$ an m-dimensional linear process defined in (1.1). If (2.4) and*

$$\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}||A_j|| < \infty \tag{2.15}$$

hold, then $n^{-1/2}\tilde{\mathbb{S}}_n \xrightarrow{\mathcal{D}} N(\mathbb{O}, T)$ as $n \to \infty$, where $T = (\sum_{j=1}^{\infty} A_j)\Gamma(\sum_{j=1}^{\infty} A_j)'$.

PROOF. Letting $\tilde{A}_i = \sum_{j=i+1}^{\infty} A_j$ and $\mathbb{Y}_t = \sum_{i=0}^{\infty} \tilde{A}_i\mathbb{Z}_{t-i}$, which is well defined since $\sum_{i=0}^{\infty}||\tilde{A}_i|| < \infty$ by (2.15), we have

$$\begin{aligned}\mathbb{X}_t &= \Big(\sum_{i=0}^{\infty} A_i\Big)\mathbb{Z}_t - \tilde{A}_0\mathbb{Z}_t + \sum_{i=1}^{\infty}(\tilde{A}_i - \tilde{A}_{i-1})\mathbb{Z}_{t-i}\\ &= \Big(\sum_{i=0}^{\infty} A_i\Big)\mathbb{Z}_t + \mathbb{Y}_{t-1} - \mathbb{Y}_t\end{aligned}$$

which implies that

$$\mathbb{S}_n = \Big(\sum_{i=0}^{\infty} A_i\Big)\sum_{t=1}^{n}\mathbb{Z}_t + \mathbb{Y}_0 - \mathbb{Y}_n.$$

According to Remark 1 we have $n^{-1/2}\sum_{t=1}^{n}\mathbb{Z}_t \to N(\mathbb{O}, \Gamma)$ as $n \to \infty$. Hence using this result on $(\sum_{i=0}^{\infty} A_i)\sum_{t=1}^{n}\mathbb{Z}_t$, this theorem is proved if

$$\frac{\mathbb{Y}_n}{\sqrt{n}} \xrightarrow{P} \mathbb{O} \quad \text{as } n \to \infty. \tag{2.16}$$

To prove (2.16) it is sufficient to show that

$$\frac{\mathbb{Y}_n}{\sqrt{n}} \to \mathbb{O} \ \ a.s. \ \ as\ n \to \infty. \tag{2.17}$$

But (2.17) follows from the fact that for any $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\left(\frac{|\mathbb{Y}_{n,j}|}{\sqrt{n}} > \epsilon\right) = \sum_{n=1}^{\infty} P(|\mathbb{Y}_{0,j}| > \sqrt{n}\epsilon) < \infty,$$

for all j, where $\mathbb{Y}_{n,j}$ denotes the j^{th} component of $\mathbb{Y}_n$. □

REMARK 2. If $\sum_{i=0}^{\infty}(i||A_i||)^2 < \infty$ the condition (2.15) is satisfied since

$$\sum_{i=0}^{\infty}||\tilde{A}_i||^2 \leq \sum_{i=0}^{\infty}\Big(\sum_{j=i+1}^{\infty}||A_j||\Big)^2 \leq 4\sum_{i=0}^{\infty}(i||A_i||)^2 < \infty.$$

See Theorem 3.31 of Hardy *et al.* (1952).

Finally, we derive the following functional central limit theorem.

THEOREM 2.7. *Let $\{\mathbb{Z}_t,\ t \geq 1\}$ be a strictly stationary weakly associated sequence of m-dimensional random vectors with $E(\mathbb{X}_1) = \mathbb{O}$, $E\|\mathbb{X}_1\|^2 < \infty$ and let $\{\mathbb{X}_t\}$ be an m-dimensional linear process defined in (1.1). Set $\mathbb{S}_n = \sum_{t=1}^{n} \mathbb{X}_t (\mathbb{S}_0 = \mathbb{O})$, and define for $u \in [0,1]$, $n \geq 1$, the stochastic process ξ_n by*

$$\xi_n(u) = n^{-1/2}\mathbb{S}_{[nu]} = n^{-1/2}\sum_{t=1}^{[nu]} \mathbb{X}_t. \tag{2.18}$$

If (2.3) and (2.4) holds then

$$\xi_n \xrightarrow{w} W^m$$

where $\xrightarrow{w}$ indicates weak convergence, and W^m is an m-dimensional Wiener process with covariance matrix $T = (\sum_{j=1}^{\infty} A_j)\Gamma(\sum_{j=1}^{\infty} A_j)'$ and $\Gamma = [\sigma_{kj}]$ as in (2.12).

PROOF. Let $\tilde{\xi}_n$ be the same as ξ_n defined in (2.18) with $\tilde{S}_{[nu]}$ in place of $S_{[nu]}$, (*i.e.*, $\tilde{\mathbb{X}}_t$ in place of $\mathbb{X}_t$)

$$\tilde{\xi}_n(u) = \Big(\sum_{j=0}^{\infty} A_j\Big) n^{-1/2}\Big(\sum_{t=1}^{[nu]} \mathbb{Z}_t\Big).$$

Then $\tilde{\xi}_n \xrightarrow{w} W^m$ follows from Lemma 2.4. Applying Lemma 2.3 and Theorem 4.1 of Billingsley (1968) we obtain $\xi_n \xrightarrow{w} W^m$, so Theorem 2.7 is proved. □

DEFINITION 2.8 (Burton *et al.*, 1986). *Let $\{\mathbb{Z}_t\}$ be a sequence of random variables taking values in separable Hilbert space $(H, <,>)$. $\{\mathbb{Z}_t\}$ is called weakly associated if for some orthonormal basis $(e_k,\ k \geq 1)$ of H and for any $m \geq 1$ the m-dimensional sequence*

$$(<\mathbb{Z}_t, e_1>, \ldots, <\mathbb{Z}_t, e_m>), \quad t \geq 1,$$

is weakly associated.

LEMMA 2.9 (Burton *et al.*, 1986). *Let $\{\mathbb{Z}_t\}$ be a strictly stationary sequence of H-valued random variables which are weakly associated. Define for $u \in [0,1]$,*

$$S_n(u) = n^{-1/2}\sum_{t=1}^{[nu]} \mathbb{Z}_t.$$

If $E(\mathbb{Z}_t) = \mathbb{O}$ *and* $\sigma^2 = E||\mathbb{Z}_1||^2 + 2\sum_{t=2}^{\infty} E(<\mathbb{Z}_1, \mathbb{Z}_t>) < \infty$, *then* $S_n \xrightarrow{w} W$, *where* W *is a Wiener process on* H *with covariance structure*

$$\Gamma(f,g) = E(<f,\mathbb{Z}_1><g,\mathbb{Z}_1>) + \sum_{t=2}^{\infty}\{E(<f,\mathbb{Z}_1><g,\mathbb{Z}_t>) + E(<g,\mathbb{Z}_1><f,\mathbb{Z}_t>)\}.$$

PROOF. See the proof of Theorem 3 of Burton *et al.* (1986). □

THEOREM 2.10. *Let* $\{\mathbb{Z}_t\}$ *be a strictly stationary sequence of* H*-valued random variables which are weakly associated and let* $\mathbb{X}_t = \sum_{j=0}^{\infty} A_i\mathbb{Z}_{t-j}$. *Define, for* $u \in [0,1]$, $n \geq 1$,

$$W_n(u) = n^{-1/2}\sum_{t=1}^{[nu]} \mathbb{X}_t. \tag{2.19}$$

If $E\mathbb{Z}_1 = \mathbb{O}$ *and* $\sigma^2 = E||\mathbb{Z}_1||^2 + 2\sum_{t=2}^{\infty} E(<\mathbb{Z}_1, \mathbb{Z}_t>) < \infty$, *then* $W_n \xrightarrow{w} W$ *where* W *is a Wiener process on* H *with covariance structure* $(\sum_{j=1}^{\infty} A_j)\Gamma(\sum_{j=1}^{\infty} A_j)'$, *and*

$$\Gamma(f,g) = E(<f,\mathbb{Z}_1><g,\mathbb{Z}_1>) + \sum_{t=2}^{\infty}\{E(<f,\mathbb{Z}_1><g,\mathbb{Z}_t>) + E(<g,\mathbb{Z}_1><f,\mathbb{Z}_t>)\}.$$

PROOF. Let $\tilde{W}_n$ be the same as W_n defined in (2.19) with $\tilde{\mathbb{X}}_t$ in place of $\mathbb{X}_t$

$$\tilde{W}_n(u) = n^{-1/2}\sum_{t=1}^{[nu]} \tilde{\mathbb{X}}_t = \Big(\sum_{j=0}^{\infty} A_j\Big) n^{-1/2}\Big(\sum_{t=1}^{[nu]} \mathbb{Z}_t\Big), \quad u \in [0,1]. \tag{2.20}$$

Then $\tilde{W}_n \xrightarrow{w} W$ follows from Lemma 2.9 (Theorem 3 of Burton *et al.*, 1986). Applying Lemma 2.3 and Theorem 4.1 of Billingsley (1968) we obtain $W_n \xrightarrow{w} W$, so Theorem 2.9 is proved. □

REMARK 3. In Theorem 2.9 letting $u = 1$ we obtain a central limit theorem for a multivariate linear process generated by a strictly stationary sequence of H-valued random variables which are weakly associated random vectors.

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