퍼지 및 지능시스템학회 논문지 2003, Vol. 13, No. 1, pp. 97-101

A note on fuzzy knowledge spaces

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요 약

최근 수학구조 및 교수-학습과 관련된 연구에 지식공간 이론을 응용하고자하는 논문들이 많이 나오고 있다. 실제로 유의미 학습과 관련된 수행평가와 수학문제를 푸는 능력에 관한 평가를 연구하는데 지식구조가 응용되고 있지만 이를 활용하는데는 많은 애로사항이 있으며 이를 보완하기 위한 여러 가지 방법이 연구되어오고 있다. 특히, Schrepp교수는 스피드문제의 경우로 제한하여 지식공간론을 응용한 일반화된 수학구조의 연구방법을 제시하였다. 본 논문에서는 주관적 지식의 평가를 하게되는 수학구조 및 공간에 관한 연구를 하는데 효과적으로 응용될 수 있는 퍼지지식공간론에 관한 전반적인 기초 이론을 정의하고 그 성질들을 연구하고자한다.

Key words : Knowledge spaces, fuzzy knowledge spaces, fuzzy sets, fuzzy surmise relations, fuzzy surmise functions.

1. Introduction

Knowledge space theory was initiated by Doignon and Falmagne [2]. A knowledge domain is represented by a finite set Q of problems and a knowledge state is the set of problems a person is capable of solving. A family $\Im$ of knowledge states is called a knowledge structure if the empty set and Q are elements of $\Im$. When $\Im$ is closed under union, the structure is called a knowledge space (see [1,2,3,4,5]).

We consider two main applications of knowledge space theory in M. Schrepp [4,5]. First, knowledge spaces can be used for an efficient assessment of knowledge. Second, knowledge spaces can be used to test psychological models of problem solving processes. Psychological models of problem solving processes describe the basic cognitive abilities subjects must possess in order to solve problems from the underlying knowledge domain. Such a detailed analysis of the cognitive processes allows one often not only to predict whether or not a subject with specific abilities will solve a problem, but also to predict how far the subject will come in his/her effort to solve problem. Therefore, the assumption that every problem is solved either correctly or incorrectly by a subject is too restrictive. M. Schrepp [4] obtained that the assumption were generalized to problem domains in which solutions were evaluated on a linear scale concerning their equality.

In references [7, 8, 9], we have studied some new concepts and their applications in fuzzy set theory. Fuzzy set theory is very useful tool to the issue which concerns the effects of vagueness. So, we will use fuzzy set theory to generalize knowledge space theory. In particular, we define new concepts of fuzzy knowledge structures, fuzzy knowledge spaces, quasi-ordinal fuzzy knowledge spaces, fuzzy surmise relations, fuzzy surmise functions and investigate some properties of them.

2. New concepts and basic properties.

In this section, we introduce new concepts of fuzzy knowledge space theory. Using this concept of a fuzzy set in [10], we define a fuzzy knowledge structure and a fuzzy knowledge space. Let $\wp(Q)$ be the power set of a finite set Q of problems. A fuzzy set Ψ of a set $\wp(Q)$ is defined by

$$\Psi=\{(A, m_{\Psi}(A)) \mid A\in \wp(Q)\}$$

where $m_{\Psi}: \wp(Q) \to [0,1]$ is a function and it is called the membership function of a fuzzy set $\wp(Q)$.

Definition 2.1 A fuzzy set of $\wp(Q)$ is called a fuzzy knowledge structure Ψ on Q if $m_{\Psi}(\emptyset)=1$ and $m_{\Psi}(Q)=1$.

Definition 2.2 A fuzzy knowledge structure Ψ is called a fuzzy knowledge space on Q if for each $F_1, F_2\in \wp(Q)\backslash\{\emptyset\}$,

$$m_{\Psi}(F_1\cup F_2)=\max\{m_{\Psi}(F_1), m_{\Psi}(F_2)\}.$$

Definition 2.3 A fuzzy knowledge space Ψ is called a quasi-ordinal fuzzy knowledge space on Q if for each $F_1, F_2\in \wp(Q)\backslash\{\emptyset\}$,

접수일자 : 2002년 11월 9일
완료일자 : 2003년 2월 3일

$$m_\Psi(F_1 \cap F_2) = \min\{m_\Psi(F_1), m_\Psi(F_2)\}.$$

Let $0 \le \alpha \le 1$. We note that

$$\Psi^\alpha = \{F \in \wp(Q) \mid m_\Psi(F) \ge \alpha\}$$

is called the α-level knowledge space of a fuzzy knowledge space Ψ. Clearly, we then have the following theorem.

Theorem 2.4 (1) If Ψ is a fuzzy knowledge space on Q and $0 < \alpha \le 1$, then the α-level knowledge space Ψ^α on Q is a knowledge space.

(2) If Ψ is a quasi-ordinal fuzzy knowledge space on Q and $0 < \alpha \le 1$, then the α-level quasi-ordinal knowledge space Ψ^α on Q is a quasi-ordinal knowledge space.

Proof. (1) If $F_1, F_2 \in \Psi^\alpha$, then we have that $m_\Psi(F_1) \ge \alpha$ and $m_\Psi(F_2) \ge \alpha$. Thus

$$m_\Psi(F_1 \cup F_2) = \max\{m_\Psi(F_1), m_\Psi(F_2)\} \ge \alpha.$$

That is, $F_1 \cup F_2 \in \Psi^\alpha$.

(2) By (1), Ψ^α is a knowledge space. Let $F_1, F_2 \in \Psi^\alpha$. We then have that $m_\Psi(F_1) \ge \alpha$ and $m_\Psi(F_2) \ge \alpha$. Thus

$$m_\Psi(F_1 \cap F_2) = \min\{m_\Psi(F_1), m_\Psi(F_2)\} \ge \alpha.$$

That is, $F_1 \cap F_2 \in \Psi^\alpha$.

3. Fuzzy surmise relations and fuzzy surmise functions.

Let $(Q, \mathfrak{I})$ be a knowledge structure and $\sqsubseteq$ a surmise relation on Q defined by

$$p \sqsubseteq q \Leftrightarrow p \in \cap \mathfrak{I}_q$$

where $p, q \in Q$ and $\mathfrak{I}_q = \{F \in \mathfrak{I} \mid q \in F\}$. In this section, we will define a fuzzy surmise relation and discuss some their properties.

Definition 3.1 Let (Q, Ψ) be a fuzzy knowledge structure and $0 \le \alpha \le 1$.

(1) A fuzzy surmise relation $\sqsubseteq_f$ on Q is defined by

$$p \sqsubseteq_f q \Leftrightarrow p \in \cap \Psi_q$$

where $\Psi_q = \{F \in \wp(Q) \mid q \in F \text{ and } m_\Psi(F) > 0\}$.

(2) The α-level surmise relation $\sqsubseteq_f^\alpha$ on Q of $\sqsubseteq_f$ is defined by

$$p \sqsubseteq_f^\alpha q \Leftrightarrow p \in \cap \Psi_q^\alpha .$$

Since Ψ^α is a knowledge space, it is easily to show that $\sqsubseteq_f^\alpha$ is a surmise relation on Q for each $\alpha \in (0,1]$.

Theorem 3.2 If Ψ is a fuzzy knowledge space on Q and $q \in Q$, then we have that

(1) Ψ_q is a finite set.

(2) $\inf\{m_\Psi(F) \mid F \in \Psi_q\} > 0$, and

(3) there exists $\alpha_0 \in (0,1]$ such that $\Psi_q = \Psi_q^{\alpha_0}$.

Proof. (1) Since Q is a finite set, the power set $\wp(Q)$ of Q is finite.

(2) By (1), Ψ_q is a finite set and hence

$$\{m_\Psi(F) \in (0,1] \mid F \in \Psi_q\}$$

is a finite set. That is, there exists $F_0 \in \Psi_q$ such that

$$\inf\{m_\Psi(F) \in (0,1] \mid F \in \Psi_q\} = m_\Psi(F_0) > 0.$$

(3) Clearly, $\Psi_q^{\alpha_0} \subset \Psi_q$. Let $F \in \Psi_q$. By (2), there exists

$$\alpha_0 = \inf\{m_\Psi(F) \mid F \in \Psi_q\} > 0$$

and hence $m_\Psi(F) \ge \alpha_0$. So $F \in \Psi_q^{\alpha_0}$. That is, $\Psi_q \subset \Psi_q^{\alpha_0}$.

We remark that since Ψ_q is a finite set, there exists a set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ such that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n \le 1$ and

$$\emptyset \ne \Psi_q^{\alpha_n} \subset \Psi_q^{\alpha_{n-1}} \subset \cdots \subset \Psi_q^{\alpha_1} \subset \Psi_q .$$

Theorem 3.3 If Ψ is a fuzzy knowledge space on Q and if any $p, q \in Q$ and $p \sqsubseteq_f q$ and α_0 is as in Theorem 3.2 (3), then there exists $\alpha_1 \in [\alpha_0, 1]$ such that $p \sqsubseteq_f^{\alpha_1} q$.

Proof. By Theorem 3.2 (3), there exists $\alpha_0 \in (0,1]$ such that $\Psi_q = \Psi_q^{\alpha_0}$. We can choose α_1 with

$$\alpha_1 = \inf\{\alpha \in [\alpha_0, 1] \mid p \in \cap \Psi_q^\alpha\}.$$

Since $\{\alpha \in [\alpha_0, 1] \mid p \in \cap \Psi_q^\alpha\}$ is finite,

$$\alpha_1 \in \{\alpha \in [\alpha_0, 1] \mid p \in \cap \Psi_q^\alpha\}.$$

We then have $p \in \cap \Psi_q^{\alpha_1}$ and so $p \sqsubseteq_f^{\alpha_1} q$.

Now, we define a fuzzy relation $\sqsubseteq_f^*$ by

$$p \sqsubseteq_f^* q \Leftrightarrow m_{\sqsubseteq_f^*}(p, q) > 0$$

where $m_{\sqsubseteq_f^*} : Q \times Q \to [0,1]$ is a function.

Defined by

$$m_{\sqsubseteq_f^*}(p, q) = \inf\{m_\Psi(F) \mid p \in F \text{ for all } F \in \Psi_q\}$$

for all $(p, q) \in Q \times Q$. Then, we have the following theorem.

Theorem 3.4 (1) If Ψ is a fuzzy knowledge space on Q and if any $p, q \in Q$ and $p \sqsubseteq_f^* q$, then there exists $\alpha_2 \in [\alpha_0, 1]$ such that $m_{\sqsubseteq_f^*}(p, q) = \alpha_2$ and $p \in \cap \Psi_q^{\alpha_2}$.

(2) If Ψ is a fuzzy knowledge space on Q and if any $p, q \in Q$ with $p \sqsubseteq_f^* q$ and $m_{\sqsubseteq_f^*}(p, q) = \alpha_2$, then $p \sqsubseteq_f^{\alpha_2} q$.

Proof. (1) By the definition of $p \sqsubseteq_f^* q$, we can choose $\alpha_2 \in [\alpha_0, 1]$ such that $\alpha_2 = p \sqsubseteq_f^* q$. Since Ψ_q is finite,

$$m_\Psi(F) \geq \alpha_2 \text{ for all } F \in \Psi_q \text{ and } p \in F.$$

We then have $p \in \bigcap \Psi_q^{\alpha_2}$.

(2) If $p \sqsubseteq_f^* q$, then by(1) we can take α_2 with $p \in \bigcap \Psi_q^{\alpha_2}$. We then have $p \sqsubseteq_f^{\alpha_2} q$.

Using Theorem 3.3(3) and Theorem 3. 4 (2), we note that if $p, q \in Q$ and $\alpha = \alpha_1 = \alpha_2$, then α-level surmise relations of these two fuzzy surmise relations are equal, that is, $\sqsubseteq_f^{*\alpha_2} = \sqsubseteq_f^{\alpha_1}$. We also will have more properties of fuzzy surmise relation.

Theorem 3.5 If Ψ is a fuzzy knowledge space on Q and any $p \in Q$, then $p \sqsubseteq_f p$ and $p \sqsubseteq_f^* p$.

Proof. Let $p \in Q$. We clearly have $p \in \bigcap \Psi_p$ and so $p \sqsubseteq_f p$. By Theorem 3.2 (3), there exists $\alpha_0 \in (0,1]$ such that $\Psi_p = \Psi_p^{\alpha_0}$. We note that for all $F \in \Psi_p$, we have $m_{\Psi_q}(F) \geq \alpha_0$. Since

$$m_{\sqsubseteq_f^*}(p, q) = \inf\{m_\Psi(F) | p \in F \text{ for all } F \in \Psi_q\},$$

$$m_{\sqsubseteq_f^*}(p, q) \geq \alpha_0 > 0 \text{ and so } p \sqsubseteq_f^* p.$$

From Theorem 3.5, we have the following corollary.

Corollary 3.6 If Ψ is a fuzzy knowledge space on Q and fuzzy surmise relations $\sqsubseteq_f$ and $\sqsubseteq_f^*$ on $Q \times Q$ are reflexive.

Theorem 3.7 If Ψ is a quasi-ordinal fuzzy knowledge space on Q and if $p \sqsubseteq_f^* q$ and $q \sqsubseteq_f^* r$ for all $p, q, r \in Q$, then we have $p \sqsubseteq_f^* r$.

Proof. Let $p \sqsubseteq_f^* q$ and $q \sqsubseteq_f^* r$ for all $p, q, r \in Q$. We then have that

$$m_{\sqsubseteq_f^*}(p, q)$$
$$= \inf\{m_\Psi(F) | p \in F \text{ for all } F \in \Psi_q\} > 0$$

and

$$m_{\sqsubseteq_f^*}(q, r)$$
$$= \inf\{m_\Psi(F) | q \in F \text{ for all } F \in \Psi_r\} > 0$$

Since Ψ_q and Ψ_r are finite, there exist $F_1 \in \Psi_q$ and $F_2 \in \Psi_r$ such that $p \in F_1$, $q \in F_2$ and

$$m_{\sqsubseteq_f^*}(p, q) = m_\Psi(F_1) > 0, \; m_{\sqsubseteq_f^*}(q, r) = m_\Psi(F_2) > 0.$$

Since Ψ is a quasi-ordinal fuzzy knowledge space,

$$p \in F_1 \cap F_2 \in \Psi_q \cap \Psi_r \subset \Psi_r$$

and

$$m_\Psi(F_1 \cap F_2) = \min\{m_\Psi(F_1), m_\Psi(F_2)\} > 0.$$

We note that since F_1 and F_2 are two elements which have smallest degrees $m_\Psi(F_1)$ and $m_\Psi(F_2)$ of

$$m_\Psi(F_1) = \inf\{m_\Psi(F) | p \in F \text{ for all } F \in \Psi_q\}$$

and

$$m_\Psi(F_2) = \inf\{m_\Psi(F) | q \in F \text{ for all } F \in \Psi_r\},$$

respectively. So, we can clearly know that $F_1 \cap F_2$ is the element which is

$$m_\Psi(F_1 \cap F_2) = \inf\{m_\Psi(F) | p \in F \text{ for all } F \in \Psi_r\}$$

We then have $m_{\sqsubseteq_f^*}(p, r) = m_\Psi(F_1 \cap F_2) > 0$ and so $p \sqsubseteq_f^* r$.

Finally, we will define a fuzzy surmise function and discuss some properties of them.

Definition 3.8 Let $F(\wp(Q))$ be the family of all fuzzy sets of $\wp(Q)$. A fuzzy mapping $\delta_f : Q \to F(\wp(Q))$ is called a fuzzy surmise function if it has the following properties ;

(i) $m_{\delta_f(q)}(E) \leq \chi_E(q)$,

(ii) for each $E \in \wp(Q)$, $\exists E' \in \wp(Q)$ with $E' \subset E$ and $\min\{m_{\delta_f(q)}(E), \chi_E(p)\} \leq m_{\delta_f(p)}(E')$,

(iii) for each $E, E' \in \wp(Q)$ with $E \subset E'$,

$$1 - \min\{m_{\delta_f(q)}(E), m_{\delta_f(q)}(E')\} \geq \chi_{E \cap E'}$$

where $p, q \in Q$ and χ_E is the characteristic function of E on Q.

We note that a mapping $\delta : Q \to \wp(\wp(Q))$ is called a surmise function if it has the following properties ;

(i) $W \in \delta(q) \to q \in W$,

(ii) $W \in \delta(q) \wedge p \in W \to \exists W' \in \delta(p) (W' \subset W)$,

(iii) $W, W' \in \delta(q) \wedge W \subset W' \to W = W'$.

Let $\alpha \in (0,1]$ and we consider the α-level surmise function δ_f^α of a fuzzy surmise function δ_f defined by

$$\delta_f^\alpha(q) = \{E \in \wp(Q) \mid m_{\delta_f(q)}(E) \geq \alpha\}.$$

We then easily know that Then we will prove that δ_f^α is a function from Q to $\wp(\wp(Q))$, and will prove that these α-level surmise functions are surmise functions.

Theorem 3.9 If $\delta_f : Q \to F(\wp(Q))$ is a fuzzy surmise function and $0 < \alpha \leq 1$, then the α-level surmise function $\delta_f^\alpha : Q \to \wp(\wp(Q))$ is a surmise function.

Proof. We will prove the conditions of the definition of surmise functions.

(i) If $F \in \delta_f^\alpha(q)$, then we have

$$\chi_F(q) \geq m_{\delta_f(q)}(F) \geq \alpha > 0.$$

Since χ_F is the characteristic function of F, $\chi_F(q)=1$ and hence $q\in F$. Thus, the condition (i) of the definition of surmise functions is proved.

(ii) Let $F\in\delta_f^\alpha(q)$ and $p\in F$. By the condition (ii) of a fuzzy surmise function, there exists $F'\in\mathfrak{F}(Q)(F'\subset F)$ and

$$\min\{m_{\delta_f(q)}(F), \chi_F(p)\}\le\delta_f(p)(F').$$

Since $m_{\delta_f(q)}(F)\ge\alpha$ and $\chi_F(p)=1$, we have

$$\alpha\le\min\{m_{\delta_f(q)}(F), \chi_F(p)\}\le\delta_f(p)(F')$$

and so $F'\in\delta_f^\alpha(p)$. Thus, the condition (ii) of the definition of surmise functions is proved.

(iii) Let $F\subset F'$ and $F, F'\in\delta_f^\alpha(q)$. We then have $m_{\delta_f(q)}(F)\ge\alpha$ and $m_{\delta_f(q)}(F')\ge\alpha$ and so

$$\alpha\le\min\{m_{\delta_f(q)}(F), m_{\delta_f(q)}(F')\}.$$

Thus, we have

$$\begin{aligned}\chi_{F'\cap F^c} &\le 1-\min\{m_{\delta_f(q)}(F), m_{\delta_f(q)}(F')\}\\ &=1-\alpha<1,\end{aligned}$$

where F^c is the complement of a set F. This means that $\chi_{F'\cap F^c}=0$ and hence $F'\cap F^c=\emptyset$. Since $F\subset F'$, $F=F'$. Thus, the condition (iii) of the definition of surmise functions is proved.

4. Remarks

Using new concepts of fuzzy knowledge space theory, we can apply to the connection of quasi-order fuzzy knowledge spaces with fuzzy surmise relations and to a similar connection between fuzzy knowledge spaces and fuzzy surmise functions. In future, we will study some properties of fuzzy knowledge spaces which are well-graded, compatability, the maximal mesh and entailment, etc.

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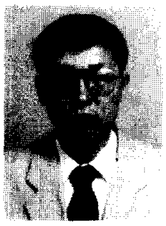

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