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ON SEMI-RIEMANNIAN MANIFOLDS SATISFYING THE SECOND BIANCHI IDENTITY

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ABSTRACT. In this paper we introduce new notions of Ricci-like tensor and many kind of curvature-like tensors such that *concircular, projective, or conformal curvature-like tensors* defined on semi-Riemannian manifolds. Moreover, we give some geometric conditions which are equivalent to the *Codazzi tensor*, the *Weyl tensor*, or the *second Bianchi identity* concerned with such kind of curvature-like tensors respectively and also give a generalization of Weyl's Theorem given in [18] and [19].

1. Introduction

It has been shown in Besse [3] and Gray [12] that there exist a few of classes of Riemannian metrics which generalize the notion of Einstein metrics and are characterized by tensorial conditions. In Riemannian geometry, it is well known that the properties of the *Ricci tensor* S and its covariant derivative ∇S are much more important. Also, the first and the second *Bianchi* identities for the Riemannian curvature tensor R give nice expressions and important roles for the study of geometry.

Now let us denote by (M, g) an n-dimensional semi-Riemannian manifold. In section 2 we prepare some basic formulas on semi-Riemannian manifolds concerned with the first and the second *Bianchi* equation, which can be said respectively the first and the second *Bianchi* identity in terms of components, for the Riemannian curvature tensor R. In sections 3 and 4 we consider such a notion of the *Ricci-like tensor* ∇U

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for the symmetric tensor U of type $(0,2)$ and the *curvature-like tensor* T of type $(0,4)$ on semi-Riemannian manifolds and want to find out some equivalent conditions when the symmetric tensor U or Ric(T) is the *Codazzi tensor* or otherwise the *Weyl tensor.*

In sections 5, 6 and 7 we introduce the new notions of different type of curvature-like tensors such that the *concircular curvature-like tensor* $Y = Y(T, U)$, the *projective curvature-like tensor* $V = V(T, U)$, and the *conformal curvature-like tensor* $B = B(T, U)$ which generalize the *concircular curvature tensor* Z, the *projective curvature tensor* W, and the *conformal curvature tensor* D. With such kind of curvature-like tensors Y, V and B we want to investigate some geometric conditions which are equivalent to the *second Bianchi identity* in terms of the Ricci-like 2-form ϕ_U or the associated curvature-like 2 form Ψ respectively.

Among so many different type of theorems concerned with such curvature-like tensors we want to introduce a main theorem given in section 7. In order to do this let us denote by Ψ_T the associated curvature-like 2 form for the curvature-like tensor T (cf. Section 4), where T is viewed as the 2-form with values in the tensor bundle of type $(0,2)$. When the associated curvature-like 2 form Ψ_B of the *conformal curvature-like tensor* B satisfies $\delta(\Psi_B) = 0$, (M, g) is said to have the *harmonic-like curvature.* Now we introduce the following:

THEOREM 1. *Let M be an n-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let $U = Ric(T)$ be the Ricci-like tensor for T. We assume that T satisfies the second Bianchi identity. If $n \geqq 4$, then the following assertions are equivalent:*

(a) $\nabla U \in C^\infty(\mathcal{A})$.
(b) *The conformal curvature-like tensor $B = B(T, U)$ satisfies the second Bianchi identity.*
(c) *U is the Codazzi tensor.*
(d) *The Ricci-like form ϕ_U satisfies the Codazzi equation.*
(e) *The associated curvature-like form Ψ_B is closed.*
(f) *The associated curvature-like form Ψ_B is coclosed.*
(g) *(M, g) has the harmonic-like curvature.*

When $U - \frac{1}{2(n-1)}(Tr\ U)g$ is the Codazzi tensor, the symmetric tensor U of type $(0,2)$ is said to be the *Weyl tensor.* When the associated curvature-like 2 form Ψ_D of the *conformal curvature tensor* D satisfies $\delta(\Psi_D) = 0$, we say that (M, g) is said to have the *harmonic Weyl tensor.*

If the Riemannian metric g is conformally related to the Riemannian metric g^* which is locally flat, then (M, g) is said to be *conformally flat.* In Riemannian geometry, it is seen by Weyl [18] and [19] that (M, g) is *conformally flat* if and only if $D = 0$ provided $n \geqq 4$, and S is the Weyl tensor provided that $n = 3$. In particular, if $D = 0$ and if $n \geqq 4$, then S is the Weyl tensor (See Yano and Kon [23]). However, conversely if S is the Weyl tensor, we do not know whether we are able to get any information about the curvature tensor. In this paper we give an answer for this problem affirmatively. Now let us apply the *curvature tensor* R, the *Ricc tensor* S and the *conformal curvature tensor* D to Theorem 1. Then as a generalization of Weyl's Theorem given in [18] and [19] we assert the following:

THEOREM 2. *Let M be an n-dimensional semi-Riemannian manifold and let R and S be the curvature tensor and the Ricci tensor on M. If $n \geqq 4$, then the following assertions are equivalent:*

(a) $\nabla S \in C^\infty(\mathcal{A} \oplus \mathcal{I})$.
(b) *The conformal curvature tensor D satisfies the second Bianchi identity.*
(c) *S is the Weyl tensor.*
(d) *The Weyl form ψ_S satisfies the Codazzi equation.*
(e) *The associated curvature-like form Ψ_D is coclosed.*
(f) *(M, g) has the harmonic Weyl tensor.*

2. Semi-Riemannian manifolds

This section is concerned with recalling basic formulas on semi-Riemannian manifolds (See Kobayashi and Nomizu [14] and O'Neill [15]). Let M be an $n(\geqq 2)$-dimensional semi-Riemannian manifold of index s $(0 \leqq s \leqq n)$ equipped with the semi-Riemannian metric tensor g. We can choose a local field $\{E_j\} = \{E_1, \ldots, E_n\}$ of orthonormal frames on a neighborhood of M. Here and in the sequel, the indices $i, j, k, \ldots$ run from 1 to n. With respect to the semi-Riemannian metric we have $g(E_j, E_k) = \varepsilon_j \delta_{jk}$, where

$$\varepsilon_j = -1 \text{ or } 1 \text{ according as } \quad 1 \leqq j \leqq s \quad \text{or} \quad s+1 \leqq j \leqq n.$$

Let $\{\theta_i\}$, $\{\theta_{ij}\}$ and $\{\Theta_{ij}\}$ be the canonical form, the connection form and the curvature form on M, respectively, with respect to the local field $\{E_j\}$ of orthonormal frames. Then we have the following structure equations

$$(2.1)\qquad \begin{aligned} &d\theta_i + \sum_j \varepsilon_j \theta_{ij} \wedge \theta_j = 0, \quad \theta_{ij} + \theta_{ji} = 0, \\ &d\theta_{ij} + \sum_k \varepsilon_k \theta_{ik} \wedge \theta_{kj} = \Theta_{ij}, \quad \Theta_{ij} = -\frac{1}{2}\sum_{kl} \varepsilon_{kl} R_{ijkl} \theta_k \wedge \theta_l, \end{aligned}$$

where $\varepsilon_{ij\cdots k} = \varepsilon_i \varepsilon_j \cdots \varepsilon_k$ and R_{ijkl} denotes the components of the Riemannian curvature tensor R of M. By the equation (2.1), the first Bianchi equation

$$(2.2)\qquad \sum_j \varepsilon_j \Theta_{ij} \wedge \theta_j = 0$$

is given. In terms of components, it implies that we have the relation

$$(2.3)\qquad R_{ijkl} + R_{jkil} + R_{kijl} = 0,$$

which is called the *first Bianchi identity* for the Riemannian curvature tensor R.

Now, relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows;

$$(2.4)\qquad S = \sum_{i,j} \varepsilon_{ij} S_{ij} \theta_i \otimes \theta_j,$$

where $S_{ij} = \sum_k \varepsilon_k R_{kijk} = S_{ji}$. The scalar curvature r of M is also given by

$$(2.5)\qquad r = \sum_j \varepsilon_j S_{jj}.$$

An n-dimensional semi-Riemannian manifold M is said to be *Einstein* if the Ricci tensor S satisfies the condition

$$(2.6)\qquad S_{ij} = \frac{r}{n} \varepsilon_i \delta_{ij}.$$

The components $T_{ij\cdots kl}$ of the covariant derivative of the components $T_{ij\cdots k}$ of the tensor T are defined by

(2.7)

$$\sum_l \varepsilon_l T_{ij\cdots kl} \theta_l = dT_{ij\cdots k} - \sum_l \varepsilon_l (T_{lj\cdots k}\theta_{li} + T_{il\cdots k}\theta_{lj} + \cdots + T_{ij\cdots l}\theta_{lk}),$$

where $\theta_0 = \{\theta_j\}$ and $\theta = \{\theta_{ij}\}$ denote the canonical form and the connection form associated with the orthonormal frame $\{E_j\}$ on M. By the exterior derivative of the third equation of (2.1), we have

$$d\Theta_{ij} = \sum_k \varepsilon_k(\Theta_{ik} \wedge \theta_{kj} - \theta_{ik} \wedge \Theta_{kj}), \tag{2.8}$$

with the help of the property $d^2 = 0$ and the property of the exterior derivative. We can regard $\theta_0 = (\theta_j)$ as a vector in $\mathbb{R}^n$, and $\theta = (\theta_{ij})$ and $\Theta = (\Theta_{ij})$ can be viewed as skew-symmetric matrices of order n. Then the equations (2.2) and (2.8) can be reformed as

$$\Theta \wedge \theta_0 = 0, \quad d\Theta = \Theta \wedge \theta - \theta \wedge \Theta. \tag{2.9}$$

The first equation of (2.9) is called the *first Bianchi equation* and the second one is called the *second Bianchi equation* for the curvature form Θ.

On the other hand, since Θ_{ij} is the 2-form, the left hand side of (2.8) is given by

$$\begin{aligned}
d\Theta_{ij} &= -\frac{1}{2}\sum_{k,l} \varepsilon_{kl}(dR_{ijkl} \wedge \theta_k \wedge \theta_l + R_{ijkl} d\theta_k \wedge \theta_l - R_{ijkl}\theta_k \wedge d\theta_l) \\
&= -\frac{1}{2}\sum_{k,l} \varepsilon_{kl}\{\sum_r \varepsilon_r R_{ijklr}\theta_r \wedge \theta_k \wedge \theta_l \\
&\quad + \sum_r \varepsilon_r(R_{rjkl}\theta_{ri} + R_{irkl}\theta_{rj}) \wedge \theta_k \wedge \theta_l\} \\
&= -\frac{1}{2}\sum_{r,k,l} \varepsilon_{rkl}\{R_{ijklr}\theta_r \wedge \theta_k \wedge \theta_l \\
&\quad + (R_{irkl}\theta_k \wedge \theta_l) \wedge \theta_{rj} - \theta_{ir} \wedge (R_{rjkl}\theta_k \wedge \theta_l)\} \\
&= -\frac{1}{2}\sum_{r,k,l} \varepsilon_{rkl} R_{ijklr}\theta_r \wedge \theta_k \wedge \theta_l + \sum_r \varepsilon_r(\Theta_{ir} \wedge \theta_{rj} - \theta_{ir} \wedge \Theta_{rj}),
\end{aligned}$$

where the first equality follows from the fact that the canonical form is a 1-form, the second one is derived from (2.1) and (2.7), the third one follows from the second equation of (2.1) and fourth one is derived by (2.1). Hence we have

$$d\Theta_{ij} = -\frac{1}{2}\sum_{r,k,l} \varepsilon_{rkl} R_{ijklr}\theta_r \wedge \theta_k \wedge \theta_l + (\Theta \wedge \theta - \theta \wedge \Theta)_{ij}. \tag{2.10}$$

From the second Bianchi equation (2.9) for Θ and (2.10), we have

$$\sum_{r,k,l} \varepsilon_{rkl} R_{ijklr} \theta_r \wedge \theta_k \wedge \theta_l = 0.$$

By the property $R_{ijkl} + R_{ijlk} = 0$ of the Riemannian curvature tensor R, we have

$$R_{ijklh} + R_{ijlhk} + R_{ijhkl} = 0, \tag{2.11}$$

which is called the *second Bianchi identity* for R. Hence we have

$$\sum_h \varepsilon_h R_{hjklh} + S_{jlk} - S_{jkl} = 0, \quad 2\sum_k \varepsilon_k S_{jkk} = r_j. \tag{2.12}$$

On the other hand, the exterior differential dr of the scalar curvature r on M is given by

$$dr = \sum_j \varepsilon_j r_j \theta_j. \tag{2.13}$$

Now, the semi-Riemannian manifold of constant sectional curvature is called a *semi-space form*. Let $M_s^n(c)$ be an n-dimensional semi-space form of constant sectional curvature c and of index s, $0 \leqq s \leqq n$, then the Riemannian curvature tensor R_{ijkl} of $M_s^n(c)$ is given by

$$R_{ijkl} = c\varepsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \tag{2.14}$$

3. Ricci-like tensors

In this section, the concept of Ricci-like tensors for the curvature tensor on the semi-Riemannian manifold is introduced. Let M be an n-dimensional semi-Riemannian manifold with semi-Riemannian metric g. Let R (resp S and r) be the Riemannian curvature tensor (resp. the Ricci tensor and the scalar curvature) on M. We define ∇S by $\nabla S(X, Y, Z) = \nabla_X S(Y, Z)$ for any vector fields X, Y and Z, where ∇ denotes the Riemannian connection of M. Then we have $\nabla S(X, Y, Z) = \nabla S(X, Z, Y)$. Furthermore, it follows from the second Bianchi identity that we have $dr = 2div\ S$.

Now, let TM (resp. T^*M)be the tangent bundle (resp. cotangent bundle) of M. The space DM consisting of all differentiable forms on the Riemannian manifold M may be regarded as follows; $DM = \sum_{p=0}^{n} D^pM$, where D^pM is the subspace of all p-forms in DM. Let $\mathcal{H} = \mathcal{H}(M,g)$ be the vector subbundle in $D^3M = \bigotimes^3 T^*M$ the fiber of which, at any point x in M, consists of all trilinear mapping ξ of $T_xM \times T_xM \times T_xM$ into $\mathbb{R}$ such that $\xi(X,Y,Z) = \xi(X,Z,Y)$ for any vector fields X, Y and Z at x and $2\sum_j \varepsilon_j \xi(E_j,E_j,X) = \sum_j \varepsilon_j \xi(X,E_j,E_j)$ for any vector X at x and any orthonormal basis $\{E_j\}$ for T_xM. Then $\xi = \nabla S$ is the section of the vector bundle $\mathcal{H}$. We call the section on $C^\infty(\mathcal{H})$ the *Ricci-like tensor* on M. Then naturally a scalar product on the vector bundle $\mathcal{H} = \mathcal{H}(M,g)$ can be defined by

$$< \xi, \eta > = \sum_{i,j,k} \varepsilon_{ijk} \xi(E_i,E_j,E_k)\eta(E_i,E_j,E_k).$$

Let FM be the ring consisting of all smooth functions on M and let T_s^rM be the module over FM consisting of all tensor fields on M of type (r,s). For any integers p and q such that $1 \leqq p < q \leqq s$, the metric contraction reduced by p and q is denoted by $C_{pq} : T_s^rM \to T_{s-2}^rM$ with respect to the orthonormal frame $\{E_j\}$. In terms of the metric contraction, the section ξ in $C^\infty(\mathcal{H})$ satisfies that $\xi(X,Y,Z)$ is symmetric with respect to Y and Z, and $2C_{12}(\xi) = C_{23}(\xi)$.

Given a semi-Riemannian manifold (M,g) one has the following natural bundle homomorphisms associated with D^3M : the *partial alternation* $a : D^3M \to D^3M$, the *partial symmetrization* $b : D^3M \to D^3M$ and the *mapping* $i : T^*M \to \mathcal{H}(M,g)$ such that

$$a(\xi)(X,Y,Z) = \frac{1}{2}\{\xi(X,Y,Z) - \xi(Y,X,Z)\},$$
$$b(\xi)(X,Y,Z) = \frac{1}{3}\{\xi(X,Y,Z) + \xi(Y,Z,X) + \xi(Z,X,Y)\},$$
$$i(\omega)(X,Y,Z) = g(X,Y)\omega(Z) + g(X,Z)\omega(Y) + \frac{2n}{n-2}g(Y,Z)\omega(X)$$

for any ξ in D^3M, any vector fields X, Y and Z at x, any 1-form ω in T^*M. The contraction $c : D^3M \to T^*M$ is defined by $c = C_{12}$, namely it is given by

$$c(\xi)(X) = \sum_j \varepsilon_j \xi(E_j,E_j,X)$$

for any orthonormal frame $\{E_j\}$. By the contraction C_{pq}, we have

$$C_{12}(\xi) = C_{13}(\xi) = \frac{1}{2}C_{23}(\xi), \quad \xi \in C^\infty(\mathcal{H}). \tag{3.1}$$

The subbundles $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{I}$ in $\mathcal{H} = \mathcal{H}(M, g)$ are defined by

$$\mathcal{A} = \mathcal{H} \cap Ker\ a, \quad \mathcal{B} = \mathcal{H} \cap Ker\ b, \quad \mathcal{C} = \mathcal{H} \cap Ker\ c, \quad \mathcal{I} = Im\ i.$$

Then it is easily seen that $\mathcal{A}$ and $\mathcal{B}$ are contained in the vector bundle $\mathcal{C}$. On the other hand, for any 1-form ω in T^*M, we put $i(\omega) = \xi$. Then we have

$$C_{12}(\xi) = \frac{(n-1)(n+2)}{n-2}\omega. \tag{3.2}$$

LEMMA 3.1. *Subbundles $\mathcal{C}$ and $\mathcal{I}$ are orthogonal and subbundles $\mathcal{A}$ and $\mathcal{B}$ are also orthogonal.*

Proof. By definition, for any section ξ in $\mathcal{I}$, there exists a 1-form ω in T^*M such that $i(\omega) = \xi$ so that it satisfies

$$\xi(X, Y, Z) = g(X, Y)\omega(Z) + g(X, Z)\omega(Y) + \frac{2n}{n-2}g(Y, Z)\omega(X)$$

for any vector fields X, Y and Z. Moreover, it satisfies $\xi(X, Y, Z) = \xi(X, Z, Y)$, which implies that

$$\sum_j \varepsilon_j \xi(E_j, E_j, Z) = \sum_j \varepsilon_j \xi(E_j, Z, E_j) = \frac{1}{2}\sum_j \varepsilon_j \xi(Z, E_j, E_j).$$

This implies that it is contained in $\mathcal{H}$. Thus we have $\mathcal{I} \subset \mathcal{H}$. From this fact we have the following property; for any η in $\mathcal{C}$ and any ξ in $\mathcal{I}$ the scalar product is given by

$$\begin{aligned}
< \xi, \eta > &= \sum_{i,j,k} \varepsilon_{ijk} \eta(E_i, E_j, E_k) \xi(E_i, E_j, E_k) \\
&= \sum_{i,j,k} \varepsilon_{ijk} \eta(E_i, E_j, E_k) \{\omega(E_k) g(E_i, E_j) + \omega(E_j) g(E_i, E_k) \\
&\quad + \frac{2n}{n-2}\omega(E_i) g(E_j, E_k)\} \\
&= \sum_i \varepsilon_i \{\omega(E_i) C_{12}(\eta)(E_i) + \omega(E_i) C_{13}(\eta)(E_i) \\
&\quad + \frac{2n}{n-2}\omega(E_i) C_{23}(\eta)(E_i)\}.
\end{aligned}$$

Since η is the section of the subbundle $\mathcal{C}$, we have by definition $C_{12}(\eta) = 0$, from which together with (3.1) it follows that we obtain that $C_{pq}(\eta) = 0$ for any $p, q = 1, 2, 3$. Thus we have $< \xi, \eta >= 0$ and we can show that subbundles $\mathcal{C}$ and $\mathcal{I}$ are orthogonal.

For the orthogonality of subbundles $\mathcal{A}$ and $\mathcal{B}$, the proof is trivial. In fact, any section ξ in $\mathcal{A}$ and η in $\mathcal{B}$, the scalar product $< \xi, \eta >$ is given by

$$< \xi, \eta > = \sum_{i,j,k} \varepsilon_{ijk} \xi(E_i, E_j, E_k) \eta(E_i, E_j, E_k).$$

Since ξ is the section on $\mathcal{A}$, $\xi(X, Y, Z)$ is symmetric with respect to X, Y and Z and hence we have

$$\begin{aligned} < \xi, \eta > &= \frac{1}{3} \sum_{i,j,k} \varepsilon_{ijk} \xi(E_i, E_j, E_k) \{ \eta(E_i, E_j, E_k) \\ &\quad + \eta(E_j, E_k, E_i) + \eta(E_k, E_i, E_j) \} \\ &= 0, \end{aligned}$$

where the last equation is derived from the definition of the subbundle $\mathcal{B}$. It completes the proof. □

PROPOSITION 3.2. *The subbundle $\mathcal{H} = \mathcal{H}(M, g)$ can be orthogonally decomposed as follows:*

$$\mathcal{H} = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{I}.$$

Proof. The proof is similar to that in the Riemannian manifold. See Besse [3] and Gray [12]. So it is simply sketched for later use. By Lemma 3.1, three subbundles $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{I}$ are mutually decomposed. So we may show that subbundle $\mathcal{H}$ can be decomposed into three parts. For any section ξ in $C^\infty(\mathcal{H})$ we put $\xi_\mathcal{I} = \frac{n-2}{(n-1)(n+2)} i(C_{12}(\xi))$. By the definition of the mapping i, $\xi_\mathcal{I}$ is the section in $C^\infty(\mathcal{I})$. Then we have

$$\begin{aligned} \xi_\mathcal{I}(X, Y, Z) =& \frac{n-2}{(n-1)(n+2)} \{ g(X, Y) C_{12}(\xi)(Z) + g(X, Z) C_{12}(\xi)(Y) \\ &+ \frac{2n}{n-2} g(Y, Z) C_{12}(\xi)(X) \}. \end{aligned}$$

Accordingly, by the simple and direct calculation, we see that $C_{12}(\xi_\mathcal{I}) = C_{13}(\xi_\mathcal{I}) = C_{23}(\xi_\mathcal{I})/2 = C_{12}(\xi)$.

Next, we put $\xi_{\mathcal{A}} = b(\xi - \xi_{\mathcal{I}})$. Then it is easily seen that $\xi_{\mathcal{A}}(X,Y,Z)$ is symmetric with respect to Y and Z and $C_{12}(\xi_{\mathcal{A}}) = C_{13}(\xi_{\mathcal{A}}) = C_{23}(\xi_{\mathcal{A}})/2 = C_{12}(\xi)$. Thus $\xi_{\mathcal{A}}$ is the section in $C^\infty(\mathcal{A})$.

We put $\xi_{\mathcal{B}} = \xi - \xi_{\mathcal{I}} - \xi_{\mathcal{A}}$. Then, by the direct calculation, we can show that $\xi_{\mathcal{B}}(X,Y,Z) + \xi_{\mathcal{B}}(Y,Z,X) + \xi_{\mathcal{B}}(Z,X,Y) = 0$ and $C_{12}(\xi_{\mathcal{B}}) = C_{13}(\xi_{\mathcal{B}}) = C_{23}(\xi_{\mathcal{B}}) = 0$, from which it follows that $\xi_{\mathcal{B}}$ is the section in $C^\infty(\mathcal{B})$. It completes the proof. □

Now, let M be an n-dimensional semi-Riemannian manifold with Riemannian connection ∇ and let $\theta_0 = \{\theta_j\}$ and $\theta = \{\theta_{ij}\}$ be the canonical form and the connection form on M associated with the orthonormal frame $\{E_j\}$. Let U be a symmetric tensor of type $(0,2)$ with components $U_{ij}(= U_{ji}) = U(E_i, E_j)$. We define the covariant derivative ∇U of the symmetric tensor U are defined by $\nabla U(X,Y,Z) = \nabla_X U(Y,Z)$. Since U is symmetric, so is ∇U with respect to Y and Z. If it satisfies $C_{12}(\nabla U) = C_{23}(\nabla U)/2$, then ∇U is the section in $C^\infty(\mathcal{H})$.

For the Ricci tensor S, ∇S is the section in $C^\infty(\mathcal{H})$. So the symmetric tensor U of type $(0,2)$ such that $\nabla U \in C^\infty(\mathcal{H})$ is the generalization of the Ricci tensor and the property of ∇U is checked. First we investigate the case of $\nabla U \in C^\infty(\mathcal{I})$. The components U_{ijk} of the covariant derivative ∇U of the symmetric tensor U are defined by

$$\sum_k \varepsilon_k U_{ijk}\theta_k = dU_{ij} - \sum_k \varepsilon_k (U_{kj}\theta_{ki} + U_{ik}\theta_{kj}), \tag{3.3}$$

where $U_{ijk}(= U_{jik}) = \nabla U(E_k, E_i, E_j) = \nabla_{E_k} U(E_i, E_j)$.

THEOREM 3.3. *Let M be an n-dimensional semi-Riemannian manifold and let U be a symmetric tensor in D^2M with $u = Tr\ U$. Then the following assertions are equivalent*:

(a) $\nabla U \in C^\infty(\mathcal{I})$.

(b) *The components of ∇U are given by*

$$2(n-1)(n+2)U_{ijk} = \frac{2n}{n-2} u_k \varepsilon_i \delta_{ij} + (n-2)\varepsilon_k (u_j \delta_{ki} + u_i \delta_{kj}). \tag{3.4}$$

Proof. Suppose that $\nabla U \in C^\infty(\mathcal{I})$. Then there exists a 1-form ω such that $i(\omega) = \nabla U$ and

$$\nabla U(X,Y,Z) = g(X,Y)\omega(Z) + g(X,Z)\omega(Y) + \frac{2n}{n-2} g(Y,Z)\omega(X). \tag{3.5}$$

Accordingly we have

$$C_{12}(\nabla U) = n\omega + \omega + \frac{2n}{n-2}\omega = \frac{(n-1)(n+2)}{n-2}\omega,$$

from which together with the above equation, we have

$$\begin{aligned}&(n-1)(n+2)\nabla U(X,Y,Z)\\ &= (n-2)\{C_{12}(\nabla U)(Z)g(X,Y) + C_{12}(\nabla U)(Y)g(X,Z)\}\\ &+ \frac{2n}{n-2}C_{12}(\nabla U)(X)g(Y,Z).\end{aligned}$$

Since ∇U is the section in $C^\infty(\mathcal{I})$, we have

$$2C_{12}(\nabla U) = C_{23}(\nabla U) = \nabla(Tr\ U) = du,$$

and hence we get

$$\begin{aligned}&2(n-1)(n+2)\nabla U(X,Y,Z)\\ &= (n-2)\{Zug(X,Y) + Yug(X,Z)\} + \frac{2n}{n-2}Xug(Y,Z).\end{aligned}$$

In terms of components, the above equation is given as (3.4). Thus the equations (3.4) and (3.5) are equivalent, which implies that (a)⇔(b). It completes the proof. □

The non-trivial example of the symmetric tensors U in D^2M such that $\nabla U \in C^\infty(\mathcal{I})$ are given by Ki and Nakagawa [13]. They showed that there exist infinitesimal many hypersurfaces satisfying $\nabla S \in C^\infty(\mathcal{I})$ of $M^{n+1}(c)$, $n \geqq 3$, where S denotes the Ricci tensor.

We define a 1-form $\phi_U = \{\phi_i\}$ associated with the symmetric tensor U by

$$\phi_i = \sum_j \varepsilon_j U_{ij}\theta_j. \tag{3.6}$$

Then ϕ_U is called a *Ricci-like form* for the symmetric tensor U. The canonical form $\theta_0 = (\theta_j)$ and the Ricci-like form $\phi_U = (\phi_j)$ for U can be regarded as vectors in $\mathbb{R}^n$ and the connection form $\theta = (\theta_{ij})$ can be regarded as skew-symmetric $n \times n$ matrix. We call the equation

$$d\phi_U + \theta \wedge \phi_U = 0 \tag{3.7}$$

the *Codazzi equation* for ϕ_U. The symmetric tensor U of type $(0,2)$ is called the *Codazzi tensor* if it satisfies $\nabla U(X,Y,Z) = \nabla U(Y,X,Z)$ for any vector fields X, Y and Z, namely, in terms of coordinates, if its components of the covariant derivative ∇U of U satisfy

$$U_{ijk} = U_{ikj}. \tag{3.8}$$

For Codazzi tensors, we have many studies, for examples, Berger and Ebin [2], Derdziński [7, 8], Derdziński and Shen [11], Choi, Yang and one of the present authors [17]. Now we assert the following:

THEOREM 3.4. *On the semi-Riemannian manifold M, let U be the symmetric tensor in D^2M and let ϕ_U be the Ricci-like form for U. If it satisfies $2C_{12}(\nabla U) = C_{23}(\nabla U)$, then the following are equivalent:*
(a) $\nabla U \in C^\infty(\mathcal{A})$.
(b) *U is the Codazzi tensor.*
(c) *ϕ_U satisfies the Codazzi equation $d\phi_U + \theta \wedge \phi_U = 0$.*

Proof. We first show (a)⇔(c). Under the assumption $2C_{12}(\nabla U) = C_{23}(\nabla U)$, we see that $\nabla U \in C^\infty(\mathcal{H})$. Since $\phi_U = \{\phi_i\}$ is the Ricci-like form for U, the exterior differentiation of (3.6) is given by

$$\begin{aligned}(d\phi_U)_i &= \sum_j \varepsilon_j (dU_{ij} \wedge \theta_j + U_{ij} d\theta_j) \\ &= \sum_{j,k} \varepsilon_{jk} \{(U_{ijk}\theta_k + U_{kj}\theta_{ki} + U_{ik}\theta_{kj}) \wedge \theta_j + U_{ij}(-\theta_{jk} \wedge \theta_k)\} \\ &= \sum_{j,k} \varepsilon_{jk} (U_{ijk}\theta_k + U_{kj}\theta_{ki}) \wedge \theta_j \\ &= \sum_{j,k} \varepsilon_{jk} U_{ijk} \theta_k \wedge \theta_j - (\theta \wedge \phi_U)_i,\end{aligned}$$

and hence we have

$$(d\phi_U + \theta \wedge \phi_U)_i = \sum_{j,k} \varepsilon_{jk} U_{ijk} \theta_k \wedge \theta_j, \tag{3.9}$$

which implies that $d\phi_U + \theta \wedge \phi_U = 0$ holds on M if and only if $U_{ijk} = U_{ikj}$. It is equivalent to $\nabla U(X,Y,Z) = \nabla U(Y,X,Z)$. Accordingly, the 1-form ϕ_U satisfies the Codazzi equation $d\phi_U + \theta \wedge \phi_U = 0$ if and only if ∇U is symmetric, i.e., $\nabla U \in C^\infty(\mathcal{A})$. The statement (a)⇔(b) is trivial. It completes the proof. □

The simplest examples of Codazzi tensors which are not parallel are the second fundamental forms of hypersurfaces in a space form. The semi-Riemannian manifold is said to *have the harmonic curvature* if the Ricci tensor becomes the Codazzi tensor.

REMARK 3.1. It is trivial that the semi-Riemannian manifold whose Ricci tensor is parallel has the *harmonic curvature.* So the class of semi-Riemannian manifolds which has *harmonic curvature* but not parallel Ricci tensor is much more essential. Berger and Ebin [2] proved that on a compact Riemannian manifold every Codazzi tensor is parallel, if the sectional curvature is non-negative and if there is a point at which the sectional curvature is positive. From this point indefinite Riemannian manifolds are so much valuable in the theory of *harmonic curvature.*

For the subbundle $\mathcal{B}$, we can verify the following.

THEOREM 3.5. *On the semi-Riemannian manifold M, let U be the symmetric tensor in D^2M. Then $\nabla U \in C^\infty(\mathcal{B})$ if and only if ∇U $(X,X,X) = 0$ for any vector field X.*

Proof. If U satisfies $\nabla U \in C^\infty(\mathcal{B})$, then we have by the definition

$$\nabla U(X,Y,Z) + \nabla U(Y,Z,X) + \nabla U(Z,X,Y) = 0$$

for any vector fields X, Y and Z, from which it follows that $\nabla U(X,X,X)$ $= 0$ for any vector field X.

By using the elementary method of Linear Algebra the converse can be derived completely. The method of polarization is repeatedly used. Now we omit the details. □

Next, we define a Weyl equation of the Ricci-like form ϕ_U for the symmetric tensor U of type $(0,2)$. We call such a equation

$$d\phi_U + \theta \wedge \phi_U = \frac{1}{2(n-1)}(du \wedge \theta + u \wedge \Theta) \tag{3.10}$$

the *Weyl equation* for ϕ_U, where $u = C_{12}(U) = Tr\ U$. The symmetric tensor U is called the *Weyl tensor* if its components of the covariant derivative ∇U of U satisfy

$$U_{ijk} - \frac{1}{2(n-1)}u_k\varepsilon_i\delta_{ij} = U_{ikj} - \frac{1}{2(n-1)}u_j\varepsilon_i\delta_{ik}. \tag{3.11}$$

The Ricci-like form ψ_U in T^*M is defined by

$$(3.12)\quad \psi_U = \phi_U - \frac{1}{2(n-1)}u\theta \ ; \ \psi_i = \sum_j \varepsilon_j (U_{ij} - \frac{1}{2(n-1)}u\varepsilon_i\delta_{ij})\theta_j,$$

which is called the *Weyl form* for U. For the Weyl tensor, see Bourguignon [4], Derdziński [9] and Derdziński and Shen [11].

REMARK 3.2. In their paper [13], Ki and Nakagawa gave many examples of Riemannian manifolds whose Ricci tensors are the Weyl tensors, but not the Codazzi tensors. Let M be a Riemannian manifold with the above property. Then the Ricci tensor of the product manifold $M_s^n(c) \times M$ has the same situation.

THEOREM 3.6. *Let M be an n-dimensional semi-Riemannian manifold and let ϕ_U be the Ricci-like form for the symmetric tensor U in D^2M with $u = C_{12}(U)$. If it satisfies $2C_{12}(\nabla U) = C_{23}(\nabla U)$, then the following assertions are equivalent*:

(a) $\nabla U \in C^\infty(\mathcal{A} \oplus \mathcal{I})$.

(b) *U is the Weyl tensor.*

(c) *$U - \frac{1}{2(n-1)}ug$ is the Codazzi tensor.*

(d) *The Ricci-like form ϕ_U satisfies the Weyl equation.*

(e) *The Weyl form $\psi_U = \phi_U - \frac{1}{2(n-1)}u\theta$ satisfies the Codazzi equation.*

(f) *The components of the covariant derivative ∇U of U satisfy* (3.11).

Proof. (f) is the definition of the Weyl tensor. We show that (b)$\Rightarrow$(c). By (3.11) we have

$$\{U_{ij} - \frac{1}{2(n-1)}u\varepsilon_i\delta_{ij}\}_k = \{U_{ik} - \frac{1}{2(n-1)}u\varepsilon_i\delta_{ik}\}_j,$$

which implies that the symmetric tensor $U - \frac{1}{2(n-1)}ug$ is the Codazzi tensor by (3.3), where g denotes the semi-Riemannian metric. So we have (b)$\Rightarrow$(c). It is trivial that the converse (c)$\Rightarrow$(b) holds.

Next we show that (c)$\Rightarrow$(e). Suppose that the symmetric tensor $U - \frac{1}{2(n-1)}ug$ is the Codazzi tensor. For the Ricci-like form ϕ for $U - \frac{1}{2(n-1)}ug$, we see that the 1-form $\phi(U - \frac{1}{2(n-1)}ug)$ satisfies the Codazzi equation.

On the other hand, we have

$$\phi\{U - \frac{1}{2(n-1)}ug\}_i = \sum_j \varepsilon_j\{U_{ij} - \frac{1}{2(n-1)}u\varepsilon_i\delta_{ij}\}\theta_j,$$

and hence we get

$$\phi\{U - \frac{1}{2(n-1)}ug\} = \phi_U - \frac{1}{2(n-1)}u\theta = \psi_U,$$

from which it follows that ψ_U satisfies the Codazzi equation. This means that (e) holds. The converse (e)$\Rightarrow$(a) is proved from the above equation. We show that (e)$\Leftrightarrow$(b). For the Ricci-like form ϕ_U for the symmetric tensor U, the Weyl form ψ_U is given by $\psi_U = \phi_U - \frac{1}{2(n-1)}u\theta$. Accordingly we obtain

$$\begin{aligned}
&d\psi_U + \theta\wedge\psi_U \\
&= d\phi_U - \frac{1}{2(n-1)}(du\wedge\theta + ud\theta) + \theta\wedge\{\phi_U - \frac{1}{2(n-1)}u\theta\} \\
&= d\phi_U - \frac{1}{2(n-1)}\{du\wedge\theta + u(-\theta\wedge\theta + \Theta)\} + \theta\wedge\{\phi_U - \frac{1}{2(n-1)}u\theta\} \\
&= d\phi_U + \theta\wedge\phi_U - \frac{1}{2(n-1)}(du\wedge\theta + u\Theta).
\end{aligned}$$

Accordingly, ψ_U satisfies the Codazzi equation if and only if ϕ_U satisfies the Weyl equation.

Last we assert the fact that (c)$\Leftrightarrow$(a). Suppose that ∇U is the section in $C^\infty(\mathcal{H})$. Then $\nabla U(X,Y,Z)$ is symmetric with respect to Y and Z. On the other hand, suppose that ∇U is the section in $C^\infty(\mathcal{A})$, $\nabla U(X,Y,Z)$ is symmetric with respect to X and Y and hence ∇U is symmetric. Accordingly, we have $U_{ijk} = U_{ikj}$. On the other hand, we have $C_{12}(U) = C_{23}(U)/2$, and hence we have $C_{12}(U) = C_{23}(U) = 0$. Thus U is the Codazzi tensor and therefore it is the Weyl tensor. So, in this case we prove (a)$\Rightarrow$(c). In the case $\nabla U \in C^\infty(\mathcal{I})$, by Theorem 3.3 we have

$$2(n-1)(n+2)U_{ijk} = 2nu_k\varepsilon_i\delta_{ij} + (n-2)\varepsilon_k(u_j\delta_{ki} + u_i\delta_{kj}).$$

Accordingly we have

$$\begin{aligned}
&2(n-1)(n+2)(U_{ijk} - U_{ikj}) \\
&= \{2nu_k\varepsilon_i\delta_{ij} + (n-2)\varepsilon_k(u_j\delta_{ki} + u_i\delta_{kj})\} \\
&\quad - \{2nu_j\varepsilon_i\delta_{ik} + (n-2)\varepsilon_j(u_k\delta_{ji} + u_i\delta_{kj})\} \\
&= (n+2)(u_k\varepsilon_i\delta_{ij} - u_j\varepsilon_i\delta_{ik}).
\end{aligned}$$

By (3.11), the above equation means that $U - \frac{1}{2(n-1)}ug$ is the Codazzi tensor and we can prove (a)⇒(c). Next we suppose (c) holds. The section $\xi = \nabla U$ in $C^\infty(\mathcal{H})$ is decomposed into three terms $\xi = \xi_\mathcal{I} + \xi_\mathcal{A} + \xi_\mathcal{B}$, where $\xi_\mathcal{I} \in C^\infty(\mathcal{I})$, $\xi_\mathcal{A} \in C^\infty(\mathcal{A})$ and $\xi_\mathcal{B} \in C^\infty(\mathcal{B})$. By the proof of Proposition 3.2, the first term $\xi_\mathcal{I}$ is given by $\xi_\mathcal{I} = \frac{n-2}{(n-1)(n+2)} i(c(\xi))$, where $c(\xi) = C_{12}(\nabla U) = C_{13}(\nabla U) = C_{23}(\nabla U)/2 = du/2$, $u = Tr\ U = C_{12}(U)$. Accordingly we have

$$\begin{aligned}
&\xi_\mathcal{I}(X,Y,Z)\\
&= \frac{n-2}{2(n-1)(n+2)} i(du)(X,Y,Z)\\
&= \frac{1}{2(n-1)(n+2)}[(n-2)\{g(X,Y)du(Z)\\
&\quad + g(Y,Z)du(Y)\} + 2ng(Y,Z)du(X)].
\end{aligned}$$

Furthermore, $\xi_\mathcal{A}$ is given by $\xi_\mathcal{A} = b(\xi - \xi_\mathcal{I})$. The section $\xi_\mathcal{A}$ is contained in $C^\infty(\mathcal{A})$. In fact, we get

$$\begin{aligned}
&\xi_\mathcal{A}(X,Y,Z)\\
&= b(\xi - \xi_\mathcal{I})(X,Y,Z)\\
&= \frac{1}{3}\{(\xi - \xi_\mathcal{I})(X,Y,Z) + (\xi - \xi_\mathcal{I})(Y,Z,X) + (\xi - \xi_\mathcal{I})(Z,X,Y)\}\\
&= \frac{1}{3}\{\xi(X,Y,Z) + \xi(Y,Z,X) + \xi(Z,X,Y)\}\\
&\quad - \frac{1}{3}\{\xi_\mathcal{I}(X,Y,Z) + \xi_\mathcal{I}(Y,Z,X) + \xi_\mathcal{I}(Z,X,Y)\},
\end{aligned}$$

which means that it is symmetric with respect to X and Y, because $\xi(X,Y,Z)$ and $\xi_\mathcal{I}(X,Y,Z)$ are both symmetric with respect to Y and Z. Thus $\xi_\mathcal{A}$ is the section in $C^\infty(\mathcal{A})$. On the other hand, by the assumption, we have

$$\xi(X,Y,Z) - \xi(Y,X,Z) = \frac{1}{2(n-1)}\{g(X,Z)du(Y) - g(Y,Z)du(X)\}.$$

Substituting this equation into the above $\xi_\mathcal{I}$ and $\xi_\mathcal{A}$ and calculating directly, we can obtain $\xi_\mathcal{I}(X,Y,Z) + \xi_\mathcal{A}(X,Y,Z) = \xi(X,Y,Z)$, which implies that $\xi_\mathcal{B}$ is the 0-form. Thus $\xi = \nabla U \in C^\infty(\mathcal{I} \oplus \mathcal{A})$. It completes the proof. □

THEOREM 3.7. *Let M be an n-dimensional semi-Riemannian manifold and let U be the symmetric tensor in D^2M. If it satisfies $2C_{12}(\nabla U) = C_{23}(\nabla U)$, then $\nabla U \in C^\infty(\mathcal{A} \oplus \mathcal{B})$ if and only if $u = C_{12}(U)$ is constant.*

Proof. Suppose that $\nabla U \in C^\infty(\mathcal{A} \oplus \mathcal{B})$. Then we have $2C_{12}(\nabla U) = 2C_{13}(\nabla U) = C_{23}(\nabla U)$. If $\nabla U \in C^\infty(\mathcal{A})$, then $C_{12}(\nabla U) = C_{13}(\nabla U) = C_{23}(\nabla U)$, so we have $C_{12}(\nabla U) = 0$. on the other hand, if $\nabla U \in C^\infty(\mathcal{B})$, then $C_{12}(\nabla U) + C_{13}(\nabla U) + C_{23}(\nabla U) = 0$, so we have $C_{12}(\nabla U) = 0$. Because of $C_{23}(\nabla U) = du$, and $u = C_{12}(U)$ is constant.

Conversely, we have $C_{12}(\nabla U) = C_{13}(\nabla U) = C_{23}(\nabla U) = 0$. From the assumption ∇U is the section in $C^\infty(\mathcal{H})$ and hence by Proposition 3.2 it can be decomposed as $\nabla U = \xi_\mathcal{I} + \xi_\mathcal{A} + \xi_\mathcal{B}$, where $\xi_\mathcal{I} \in C^\infty(\mathcal{I})$, $\xi_\mathcal{A} \in C^\infty(\mathcal{A})$ and $\xi_\mathcal{B} \in C^\infty(\mathcal{B})$. By the above discussion, we have $C_{pq}(\xi_\mathcal{A}) = 0$ and $C_{pq}(\xi_\mathcal{B}) = 0$. Since the contraction is linear, we get $C_{pq}(\xi_\mathcal{I}) + C_{pq}(\xi_\mathcal{A}) + C_{pq}(\xi_\mathcal{B}) = 0$, and hence we get $C_{pq}(\xi_\mathcal{I}) = 0$, By Theorem 3.3, we obtain $\xi_\mathcal{I} = 0$. It completes the proof. □

4. Curvature-like tensors

In this section the concept of Bianchi identities for the curvature-like tensor on the semi-Riemannian manifold is introduced. Let M be an n-dimensional semi-Riemannian manifold of index s, $0 \leqq s \leqq n$, with Riemannian connection ∇. We denote by TM the tangent bundle of M. Let T be a quadrilinear mapping of $TM \times TM \times TM \times TM$ into $\mathbb{R}$ satisfying the curvature-like properties:

(a) $$T(X,Y,Z,U) = -T(Y,X,Z,U) = -T(X,Y,U,Z),$$

(b) $$T(X,Y,Z,U) = T(Z,U,X,Y),$$

(c) $$T(X,Y,Z,U) + T(Y,Z,X,U) + T(Z,X,Y,U) = 0.$$

Then T is called the *curvature-like tensor* on M. Let T_{ijkl} be the components of T associated with the orthonormal frame $\{E_j\}$, then the components T_{ijkl} are given by $T_{ijkl} = T(E_i, E_j, E_k, E_l)$. By the conditions (a), (b) and (c), following properties of the components of T hold corresponding to the conditions (a), (b) and (c):

$$T_{ijkl} = -T_{jikl} = -T_{ijlk}, \tag{4.1}$$

$$T_{ijkl} = T_{klij} = T_{lkji}, \tag{4.2}$$

$$T_{ijkl} + T_{jkil} + T_{kijl} = 0. \tag{4.3}$$

If the components T_{ijkl} of a tensor T in $D^4M = \bigotimes^4 T^*M$ satisfy (4.1), (4.2) and (4.3), then it becomes a curvature-like tensor.

For an orthonormal frame $\{E_j\}$, let $\theta_0 = \{\theta_j\}$, $\theta = \{S_{ij}\}$ and $\Theta = \{\Theta_{ij}\}$ be the canonical form, the connection form and the curvature form on M. In the same way as we associate a curvature form Θ to the Riemannian curvature tensor R, we associate a 2-form $\Phi_T = \{\Phi_{ij}\}$ to the curvature-like tensor T in the following

$$\Phi_{ij} = \sum_{k,l} \varepsilon_{kl} T_{ijkl} \theta_k \wedge \theta_l, \tag{4.4}$$

which is the analogue to the curvature form Θ except for the coefficient. So it is called the *curvature-like form* for the curvature-like tensor T. The canonical form $\theta_0 = (\theta_j)$ can be regarded as a vector in $\mathbb{R}^n$ and the connection form $\theta = (\theta_{ij})$ and the *curvature-like form* $\Phi_T = (\Phi_{ij})$ can be regarded as $n \times n$ skew-symmetric matrices. Then, corresponding to the first equation of (2.9), the equation

$$\Phi_T \wedge \theta_0 = 0 \tag{4.5}$$

is called the *first Bianchi equation* for the curvature-like form Φ_T. By (4.4) and (4.5) the tensor T satisfies the first Bianchi equation if and only if its components satisfies (4.3). So Φ_T always satisfies the first Bianchi equation. So (4.3) is called the *first Bianchi identity* for T.

Corresponding to the second equation of (2.9), we call the equation

$$d\Phi_T = \Phi_T \wedge \theta - \theta \wedge \Phi_T \tag{4.6}$$

the *second Bianchi equation* for the curvature-like form Φ_T.

PROPOSITION 4.1. *On the semi-Riemannian manifold M, let T be the curvature-like tensor in D^4M on M. Then the following assertions are equivalent:*

(a) *The curvature-like form Φ_T satisfies the second Bianchi equation (4.6).*

(b) *Its components satisfy*

$$T_{ijklh} + T_{ijlhk} + T_{ijhkl} = 0. \tag{4.7}$$

Proof. Since Φ_{ij} is the 2-form, the left hand side of (4.4) is given by

$$\begin{aligned} d\Phi_{ij} &= \sum_{k,l} \varepsilon_{kl}(dT_{ijkl} \wedge \theta_k \wedge \theta_l + T_{ijkl} d\theta_k \wedge \theta_l - T_{ijkl}\theta_k \wedge d\theta_l) \\ &= \sum_{k,l} \varepsilon_{kl}\{\sum_r \varepsilon_r T_{ijklr}\theta_r \wedge \theta_k \wedge \theta_l \\ &\quad + \sum_r \varepsilon_r (T_{rjkl}\theta_{ri} + T_{irkl}\theta_{rj}) \wedge \theta_k \wedge \theta_l\} \\ &= \sum_{r,k,l} \varepsilon_{rkl}\{T_{ijklr}\theta_r \wedge \theta_k \wedge \theta_l \\ &\quad + (T_{irkl}\theta_k \wedge \theta_l) \wedge \theta_{rj} - \theta_{ir} \wedge (T_{rjkl}\theta_k \wedge \theta_l)\} \\ &= \sum_{r,k,l} \varepsilon_{rkl} T_{ijklr}\theta_r \wedge \theta_k \wedge \theta_l + \sum_r \varepsilon_r(\Phi_{ir} \wedge \theta_{rj} - \theta_{ir} \wedge \Phi_{rj}), \end{aligned}$$

where the first equality follows from the fact that the canonical form is a 1-form, the second one is derived from (2.1) and (2.7), the third one follows from the second equation of (2.1) and fourth one is derived by (4.4). Hence we have

$$d\Phi_{ij} = \sum_{r,k,l} \varepsilon_{rkl} T_{ijklr}\theta_r \wedge \theta_k \wedge \theta_l + (\Phi \wedge \theta - \theta \wedge \Phi)_{ij}. \tag{4.8}$$

It implies that (a) is equivalent to (b). This completes the proof. □

The equation (4.7) is called the *second Bianchi identity* for the curvature-like tensor T.

REMARK 4.1. Of course, as is well known, the Riemannian curvature tensor R satisfies the first and the second Bianchi identities.

REMARK 4.2. By Proposition 4.1, it is seen that the parallel curvature -like tensor on M satisfies the second Bianchi equation. The converse does not necessarily hold. See Example 6.1.

Next, we define a associated curvature-like form Ψ_T for the curvature-like tensor T. The 2-form Ψ_T with values in the bundle D^2M is defined by

$$\Psi_T = \sum_{i,j} \varepsilon_{ij} \Phi_{ij} \theta_i \wedge \theta_j, \tag{4.9}$$

which is called the *associated curvature-like form* for the curvature-like tensor T. If $d\Psi_T = 0$, then the associated curvature-like form Ψ_T is said to be *closed*.

THEOREM 4.2. *Let M be an n-dimensional semi-Riemannian manifold. For any curvature-like tensor T with components T_{ijkl}, the following assertions are equivalent:*

(a) *The associated curvature-like form Ψ_T is closed, i.e., $d\Psi_T = 0$.*
(b) *The curvature-like form Φ_T satisfies the second Bianchi equation* (4.6).
(c) *The components of the covariant derivative ∇T of T satisfy the second Bianchi identity* (4.7).

Proof. By Proposition 4.1, the assertions (b) and (c) are equivalent. Next we show that the conditions (a) and (b) are equivalent. Since the associated curvature-like form Ψ_T is given by (4.9), we have

$$\begin{aligned} d\Psi_T &= \sum_{i,j} \varepsilon_{ij}(d\Phi_{ij} \wedge \theta_i \wedge \theta_j + \Phi_{ij} \wedge d\theta_i \wedge \theta_j - \Phi_{ij} \wedge \theta_i \wedge d\theta_j) \\ &= \sum_{i,j} \varepsilon_{ij}(d\Phi_{ij} \wedge \theta_i \wedge \theta_j - \sum_k \varepsilon_k \Phi_{ij} \wedge \theta_{ik} \wedge \theta_k \wedge \theta_j \\ &\quad + \sum_k \varepsilon_k \Phi_{ij} \wedge \theta_i \wedge \theta_{jk} \wedge \theta_k) \\ &= \sum_{i,j} \varepsilon_{ij}(d\Phi_{ij} + \sum_k \varepsilon_k \theta_{ik} \wedge \Phi_{kj} - \sum_k \varepsilon_k \Phi_{ik} \wedge \theta_{kj}) \wedge \theta_i \wedge \theta_j \\ &= \sum_{i,j} \varepsilon_{ij}(d\Phi_T + \theta \wedge \Phi_T - \Phi_T \wedge \theta)_{ij} \wedge \theta_i \wedge \theta_j, \end{aligned}$$

where the first equality follows from the fact that Φ_T is the 2-form and the second one is derived by (2.1). From the above equation, the conditions (a) and (b) are equivalent. It completes the proof. □

Now, let T be the curvature-like tensor in D^4M and let Ψ_T be the associated curvature-like form for T. The mapping $\delta : D^4M \to D^3M$ defined by the divergence : $\delta(\Psi_T) = -C_{15}(\nabla T)$, where C_{pq} is the metric contraction functioned by $C_{pq} : T^r_s M \to T^r_{s-2}M$. This is a generalization of the well known differential operators on $\mathbb{R}^3$. For the orthonormal frame $\{E_j\}$, in terms of coordinates, the components of $\delta(\Psi_T)$ is given by $\delta(\Psi_T)_{ijk} = -\sum_l \varepsilon_l T_{lijkl}$. If $\delta(\Psi_T) = 0$, then the associated curvature-like form Ψ_T is said to be *coclosed.*

REMARK 4.3. On the semi-Riemannian manifold (M, g) with the Riemannian connection ∇, it has a formal adjoint $\nabla^* : T^*M \times T^r_s M \to T^r_s M$

defined by as follows: for any vector fields $X_1, \ldots, X_r$ and any α in $T^*M \times T^r_s M$, $\nabla^*\alpha$ is defined by

$$(\nabla^*\alpha)(X_1, \ldots, X_r) = -\sum_k \varepsilon_k \nabla_{E_k}\alpha(E_k, X_1, \ldots, X_r),$$

where $\{E_k\}$ is the orthonormal frame. Namely, $(\nabla^*\alpha)(X_1, \ldots, X_r)$ is the opposite of the trace with respect to g of D^sM valued 2-form

$$(X, Y) \to (\nabla_X\alpha)(Y, X_1, \ldots, X_r).$$

For the exterior differential $d : D^rM \to D^{r+1}M$, we define by $\delta : D^rM \to D^{r-1}M$ the formal adjoint. For the orthonormal basis $\{E_j\}$ for T_xM at any point x, the components of $\delta(\Psi_T)$ are given by

$$\delta(\Psi_T(T))(X, Y, Z) = -\sum_k \varepsilon_k \nabla_{E_k}\Psi_T(T)(E_k, X, Y, Z).$$

Accordingly, the above δ operator on the semi-Riemannian manifolds is the formal analogue of the adjoint operator to the exterior differential d on the Riemannian manifold.

The semi-Riemannian manifold (M, g) is said to *have the harmonic-like curvature* for T if $\delta(\Psi_T) = 0$. In particular, if $T = R$, then we see that (M, g) *has the harmonic curvature.*

Now, we want to introduce the concept of the Ricci-like tensors for the curvature-like tensor on the semi-Riemannian manifold is introduced. Let M be an n-dimensional semi-Riemannian manifold with semi-Riemannian metric g and with curvature-like tensor T with components T_{ijkl}. The tensor $Ric(T)$ associated with T is defined by $Ric(T)$ (X, Y), where $Ric(T)(X, Y)$ denotes the trace of the map $\{Z \to T(Z, X)$ $Y\}$ and $T(Z, X)Y$ is a vector field defined by $T(X, Y, Z, W) = g(T(X, Y)$ $Z, W)$ for any vector fields X, Y, Z and W. Then $Ric(T)$ is called the *Ricci-like tensor* for T. By the definition of the Ricci-like tensor, $Ric(T)$ is a symmetric tensor of type $(0, 2)$ and its components T_{ij} are given by

$$T_{ij} = \sum_k \varepsilon_k T_{kijk}.$$

It easily seen by (4.1) and (4.2) that we have $T_{ij} = T_{ji}$. The *scalar-like curvature* t associated with T is defined by $t = C_{12}(Ric(T)) = \sum_{j,k} \varepsilon_{jk} T_{kjjk}$.

LEMMA 4.3. *Let M be an n-dimensional semi-Riemannian manifold. For any curvature-like tensor T with components T_{ijkl}, let Φ_T, Ψ_T or $\phi_{Ric(T)}$ be the curvature-like form, the associated curvature-like form or the Ricci-like form for T. If Φ_T satisfies the second Bianchi equation $d\Phi_T = \Phi_T \wedge \theta - \theta \wedge \Phi_T$, then we have*

$$(4.10) \qquad \delta(\Psi_T)_{ijk} = T_{ijk} - T_{ikj},$$

$$(4.11) \qquad d\phi_{Ric(T)} + \theta \wedge \phi_{Ric(T)} = 0 \quad \textit{if and only if} \quad \delta(\Psi_T) = 0,$$

where T_{ijk} denote the components of the covariant derivative $\nabla(Ric(T))$ of $Ric(T)$.

Proof. It is trivial by (3.9) and (4.7) and the definition of δ. □

As the direct consequence of Lemma 4.3, we can prove the following.

PROPOSITION 4.4. *Let M be an n-dimensional semi-Riemannian manifold. For any curvature-like tensor T, let Φ_T, Ψ_T or $\phi_{Ric(T)}$ be the curvature-like form, the associated curvature-like form or the Ricci-like form for T. If Φ_T satisfies the second Bianchi equation $d\Phi_T = \Phi_T \wedge \theta - \theta \wedge \Phi_T$, then Ψ_T is coclosed if and only if $Ric(T)$ is the Codazzi tensor.*

We suppose that the curvature-like tensor T satisfies the second Bianchi identity (4.7). Then, by Lemma 4.3, we see that $\sum_l \varepsilon_l T_{lijkl} = T_{ikj} - T_{ijk}$. Accordingly, we have $2\sum_l \varepsilon_l T_{lil} = \sum_l \varepsilon_l T_{lli} = t_i$, where $t_i = C_{23}(\nabla Ric(T))$. So the Ricci-like tensor $Ric(T)$ satisfies $2C_{12}(\nabla U) = C_{23}(\nabla U)$, where $U = Ric(T)$. Accordingly, by Theorem 3.4, we can prove the following.

THEOREM 4.5. *Let M be an n-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let $\phi_{Ric(T)}$ be the Ricci-like form for $Ric(T)$. Then the following assertions are equivalent:*

(a) $\nabla(Ric(T)) \in C^\infty(\mathcal{A})$.

(b) *$\phi_{Ric(T)}$ satisfies the Codazzi equation $d\phi_{Ric(T)} + \theta \wedge \phi_{Ric(T)} = 0$.*

(c) *$Ric(T)$ is the Codazzi tensor.*

(d) *(M, g) has the harmonic-like curvature.*

By Theorem 3.6, we have the following.

THEOREM 4.6. *Let M be an n-dimensional semi-Riemannian manifold and let T (resp. $Ric(T)$ and t) be the curvature-like tensor (resp.*

the Ricci-like tensor and scalar-like curvature for T). Let $\phi_{Ric(T)}$ be the Ricci-like form for $Ric(T)$. Then the following assertions are equivalent:

(a) $\nabla(Ric(T)) \in C^\infty(\mathcal{A} \oplus \mathcal{I})$.

(b) *$Ric(T)$ is the Weyl tensor.*

(c) *$Ric(T) - \frac{1}{2(n-1)}tg$ is the Codazzi tensor.*

(d) *The Ricci-like form $\phi_{Ric(T)}$ satisfies the Weyl equation.*

(e) *The Weyl form $\psi_{Ric(T)} = \phi_{Ric(T)} - \frac{1}{2(n-1)}t\theta$ satisfies the Codazzi equation.*

(f) *The components T_{ijk} of the covariant derivative $\nabla Ric(T)$ of $Ric(T)$ satisfy*

$$T_{ijk} - \frac{1}{2(n-1)}t_k\varepsilon_i\delta_{ij} = T_{ikj} - \frac{1}{2(n-1)}t_j\varepsilon_i\delta_{ik}.$$

Applying Theorem 4.6 to the case Riemannian curvature tensor R, we can prove the following.

THEOREM 4.7. *Let M be an n-dimensional semi-Riemannian manifold and let R (resp. S and r) be the curvature tensor (resp. the Ricci tensor and scalar curvature). Let ϕ_S be the Ricci form. Then the following assertions are equivalent:*

(a) $\nabla S \in C^\infty(\mathcal{A} \oplus \mathcal{I})$.

(b) *S is the Weyl tensor.*

(c) *$S - \frac{1}{2(n-1)}rg$ is the Codazzi tensor.*

(d) *The Ricci form ϕ_S satisfies the Weyl equation.*

(e) *The Weyl form $\psi_S = \phi_S - \frac{1}{2(n-1)}r\theta$ satisfies the Codazzi equation.*

(f) *The components S_{ijk} of the covariant derivative ∇S of S satisfy*

$$S_{ijk} - \frac{1}{2(n-1)}r_k\varepsilon_i\delta_{ij} = S_{ikj} - \frac{1}{2(n-1)}r_j\varepsilon_i\delta_{ik}.$$

THEOREM 4.8. *Let M be an n-dimensional semi-Riemannian manifold and let R (resp. S) be the curvature tensor (resp. the Ricci tensor). Let ϕ_s be the Ricci form. If the scalar curvature is constant. Then the following assertions are equivalent:*

(a) $\nabla S \in C^\infty(\mathcal{A})$.

(b) *S is the Codazzi tensor.*

(c) *The components S_{ijk} of the covariant derivative ∇S satisfy $S_{ijk} = S_{ikj}$.*

(d) (M, g) *has the harmonic curvature.*

5. The concircular curvature tensor Z

This section is devoted to the investigation of the concircular curvature tensor defined on the semi-Riemannian manifold. Let M be an $n(\geqq 3)$-dimensional semi-Riemannian manifold of index s, $0 \leqq s \leqq n$, with the Riemannian connection ∇ and let R (resp. S or r) be the Riemannian curvature tensor (resp. the Ricci tensor or the scalar curvature) on M.

Now, let Z be the *concircular curvature tensor* with components Z_{ijkl} on M, which is defined by

$$Z_{ijkl} = R_{ijkl} - \frac{r}{n(n-1)}\varepsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \tag{5.1}$$

About the concircular curvature tensor on Riemannian manifold, see Yano and Bochner [22] etc. As is easily seen, Z is the curvature-like tensor on M.

REMARK 5.1. The semi-Riemannian manifold is of constant curvature if and only if the concircular curvature tensor vanishes identically.

Now, let T be the curvature-like tensor and let U be the symmetric tensor of type $(0, 2)$. We put $u = C_{12}(U)$. For such a pair (T, U), we define the tensor $Y = Y(T, U)$ with components Y_{ijkl} by

$$Y_{ijkl} = T_{ijkl} - \frac{u}{n(n-1)}\varepsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}), \tag{5.2}$$

which is called the *concircular curvature-like tensor* for T and U. It is trivial that $Y = Y(T, U)$ is the curvature-like tensor. Calculating directly we have

$$Y_{ijkl} + Y_{jkil} + Y_{kijl} = T_{ijkl} + T_{jkil} + T_{kijl}. \tag{5.3}$$

From this equation and (4.3), we have

$$Y_{ijkl} + Y_{jkil} + Y_{kijl} = 0. \tag{5.4}$$

Thus we can prove the following.

PROPOSITION 5.1. *Let M be an $n(\geqq 2)$-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let U be the symmetric tensor in D^2M. Then the concircular curvature-like tensor $Y = T(T,U)$ satisfies the first Bianchi identity.*

Next, we give a necessary and sufficient condition for the concircular curvature-like tensor $Y = T(T,U)$ to satisfy the second Bianchi identity in terms of the scalar-like curvature u, where u is given by $u = C_{12}(U)$ for the metric contraction C_{12}.

LEMMA 5.2. *Let M be an $n(\geqq 3)$-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let U be the symmetric tensor in D^2M. If T satisfies the second Bianchi identity, then the concircular curvature-like tensor $Y = T(T,U)$ satisfies the second Bianchi identity if and only if the scalar-like curvature $u = C_{12}(U)$ is constant.*

Proof. By (5.2), the components Y_{ijklh} of the covariant derivative ∇Y of Y are given by

$$Y_{ijklh} = T_{ijklh} - \frac{1}{n(n-1)} u_h \varepsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \tag{5.5}$$

Thus we have

$$\begin{aligned} &Y_{ijklh} + Y_{ijlhk} + Y_{ijhkl} \\ &= T_{ijklh} + T_{ijlhk} + T_{ijhkl} - \frac{1}{n(n-1)}\{\varepsilon_j(u_l\delta_{jk} - u_k\delta_{jl})\varepsilon_i\delta_{ih} \\ &\quad + \varepsilon_j(u_h\delta_{jl} - u_l\delta_{jh})\varepsilon_i\delta_{ik} + \varepsilon_j(u_k\delta_{jh} - u_h\delta_{jk})\varepsilon_i\delta_{il}\}. \end{aligned} \tag{5.6}$$

We suppose that T satisfies the second Bianchi identity $T_{ijklh} + T_{ijlhk} + T_{ijhkl} = 0$. If Y satisfies $Y_{ijklh} + Y_{ijlhk} + Y_{ijhkl} = 0$. Then we have by (5.6)

$$(n-2)\varepsilon_j(u_i\delta_{jk} - u_k\delta_{ji}) = 0, \quad (n-1)(n-2)u_i = 0,$$

from which it implies that the scalar-like curvature u is constant. The converse is trivial by (5.6). It completes the proof. □

By Lemma 5.2 and Theorem 3.7, we can prove

THEOREM 5.3. *Let M be an $n(\geqq 3)$-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let U be the symmetric tensor in D^2M. We assume that T satisfies the second Bianchi* *identity. If it satisfies $2C_{12}(\nabla U) = C_{23}(\nabla U)$, then the following assertions are equivalent:*

(a) $\nabla U \in C^\infty(\mathcal{A} \oplus \mathcal{B})$.
(b) *The concircular curvature-like tensor $Y = Y(T, U)$ satisfies the second Bianchi identity.*
(c) *The scalar-like curvature u is constant.*

Applying Theorem 5.3 to the case where the concircular curvature tensor Z, we can verify the following.

THEOREM 5.4. *Let M be an $n(\geqq 3)$-dimensional semi-Riemannian manifold and let R (resp. S or r) be the curvature tensor (resp. the Ricci tensor or the scalar curvature) on M. Then the following assertions are equivalent:*

(a) $\nabla S \in C^\infty(\mathcal{A} \oplus \mathcal{B})$.
(b) *The concircular curvature tensor Z satisfies the second Bianchi identity.*
(c) *The scalar curvature r is constant.*

6. The projective curvature tensor W

This section is devoted to investigate the semi-Riemannian manifold whose projective curvature tensor satisfies the second Bianchi identity. Let M be an n-dimensional semi-Riemannian manifold of index s, $0 \leqq s \leqq n$, with the Riemannian connection ∇ and let R (resp. S or r) be the Riemannian curvature tensor (resp. the Ricci tensor or the scalar curvature) on M.

Now, let W be the *projective curvature tensor* with components W_{ijkl} on M, which is defined by

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-1}\varepsilon_i(S_{jk}\delta_{il} - S_{jl}\delta_{ik}). \tag{6.1}$$

In the Riemannian manifold whose projective curvature tensor is flat, there exists a one to one correspondence between this neighborhood and a domain in a Euclidean space such that any geodesic in the Riemannian manifold corresponds the straight line in the Euclidean space(See Yano and Bochner [22] for examples). As it is easily seen, W is the curvature-like tensor on M.

REMARK 6.1. If M is Einstein, then the Ricci tensor satisfies (2.6), which yields that the projective curvature tensor W satisfies

$$W_{ijkl} = R_{ijkl} - \frac{1}{n(n-1)} r\varepsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}).$$

This yields that the projective curvature tensors of Einstein Riemannian manifolds are the concircular curvature ones [22]. In particular, if M is of constant curvature, then the projective curvature tensor vanishes identically.

Now, let T be the curvature-like tensor and let U be the symmetric tensor of type $(0,2)$. We put $u = C_{12}(U)$. For such a (T,U), we define the tensor $V = V(T,U)$ with components V_{ijkl} by

$$V_{ijkl} = T_{ijkl} - \frac{1}{(n-1)} \varepsilon_i(U_{jk}\delta_{il} - U_{jl}\delta_{ik}), \tag{6.2}$$

which is called the *projective curvature-like tensor* for T and U. It is trivial that $V = V(T,U)$ is the curvature-like tensor. Accordingly, by (4.1) and (4.2), we have

$$V_{ijkl} + V_{jkil} + V_{kijl} = T_{ijkl} + T_{jkil} + T_{kijl}.$$

From this equation and (4.3), we have

$$V_{ijkl} + V_{jkil} + V_{kijl} = 0. \tag{6.3}$$

Thus we can prove the following.

PROPOSITION 6.1. *Let M be an $n(\geqq 2)$-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let U be the symmetric tensor in D^2M. Then the projective curvature-like tensor $V = V(T,U)$ satisfies the first Bianchi identity.*

Next, we give a necessary condition for the projective curvature-like tensor $V = V(T,U)$ which satisfies the second Bianchi identity. Let Φ_V and Ψ_V be the curvature-like form and the associated curvature-like form for V, let ϕ_U be the Ricci-like form for the symmetric tensor U of type $(0,2)$ and let ψ_U be the Weyl form for U defined by $\psi_U = \phi_U - \frac{1}{2(n-1)} u\theta$, where $u = C_{12}(U)$ and θ is the connection form on M.

LEMMA 6.2. *Let M be an $n(\geqq 3)$-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let U be the symmetric tensor in D^2M. If any two assertions in the following ones hold, then another one holds true:*

(a) *T satisfies the second Bianchi identity.*
(b) *The projective curvature-like tensor $V = V(T,U)$ satisfies the second Bianchi identity.*
(c) *U is the Codazzi tensor.*

Proof. By (6.2), the components V_{ijklh} of the covariant derivative ∇V of V are given by

$$V_{ijklh} = T_{ijklh} - \frac{1}{n-1}\varepsilon_i(U_{jkh}\delta_{il} - U_{jlh}\delta_{ik}). \tag{6.4}$$

Accordingly, we have

(6.5)

$$\begin{aligned} &V_{ijklh} + V_{ijlhk} + V_{ijhkl} \\ &= T_{ijklh} + T_{ijlhk} + T_{ijhkl} \\ &+ \frac{1}{n-1}\varepsilon_i\{(U_{jkl} - U_{jlk})\delta_{ih} + (U_{jlh} - U_{jhl})\delta_{ik} + (U_{jhk} - U_{jkh})\delta_{il}\}. \end{aligned}$$

Suppose that the assertions (a) and (b) hold. Then we have

$$\varepsilon_i\{(U_{jkl} - U_{jlk})\delta_{ih} + (U_{jlh} - U_{jhl})\delta_{ik} + (U_{jhk} - U_{jkh})\delta_{il}\} = 0.$$

Putting $i = h$, multiplying ε_i and summing up with respect to the index i, we have $(n-2)(U_{jlh} - U_{jhl}) = 0$, which means that U is the Codazzi tensor. The others are trivial. It completes the proof. □

We say that (M, g) has *the harmonic projective-like tensor* if $\delta(\Psi_V) = 0$. In particular, (M, g) is said to have *the harmonic projective tensor* if $\delta(\Psi_W) = 0$.

THEOREM 6.3. *Let M be an $n(\geqq 3)$-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let U be the symmetric tensor in D^2M. We assume that T satisfies the second Bianchi identity. If it satisfies $2C_{12}(\nabla U) = C_{23}(\nabla U)$, then the following assertions are equivalent:*

(a) *$\nabla U \in C^\infty(\mathcal{A})$.*
(b) *The projective curvature-like tensor $V = V(T,U)$ satisfies the second Bianchi identity.*
(c) *U is the Codazzi tensor.*
(d) *The Ricci-like form ϕ_U satisfies the Codazzi equation.*

Proof. By Theorem 3.4, the assertions (a), (c) and (d) are equivalent. Also by Lemma 6.2, we see (c)$\Leftrightarrow$(b). □

By applying Theorem 6.3 to the case where $U = Ric(T)$, we can give the condition for (M, g) to have the harmonic projective-like tensor is given.

COROLLARY 6.4. *Let M be an $n(\geqq 3)$-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let $U = Ric(T)$ be the Ricci-like tensor for T. We assume that T satisfies the second Bianchi identity. If it satisfies $2C_{12}(\nabla U) = C_{23}(\nabla U)$, then the following assertions are equivalent:*

(a) $\nabla U \in C^{\infty}(\mathcal{A})$.
(b) *The projective curvature-like tensor $V = V(T, U)$ satisfies the second Bianchi identity.*
(c) *U is the Codazzi tensor.*
(d) *The Ricci-like form ϕ_U satisfies the Codazzi equation.*
(e) *The associated curvature-like form Ψ_V is closed.*
(f) *The associated curvature-like form Ψ_V is coclosed.*
(g) *(M, g) has the harmonic projective-like tensor.*

Proof. By Theorem 4.2, V satisfies the second Bianchi identity if and only if the form Ψ_V is closed. Namely the assertions (b) and (e) are equivalent. Accordingly, in order to prove Corollary 6.4, it is sufficient to show that (c)$\Leftrightarrow$(f). Since T satisfies the second Bianchi identity, we have

$$\sum_h \varepsilon_h T_{hjklh} + T_{jlk} - T_{jkl} = 0, \tag{6.6}$$

where $T_{ij} = \sum_k \varepsilon_k T_{kijk}$ are the components of the Ricci-like tensor $U = Ric(T)$ and T_{ijk} are the components of the covariant derivative $\nabla U = \nabla(Ric(T))$ of $U = Ric(T)$. Putting $i = h$, multiplying ε_i and summing up with respect to the index i in (6.4) and taking account of (6.6) we have

$$\begin{aligned} \sum_h \varepsilon_h V_{hjklh} &= \sum_h \varepsilon_h T_{hjklh} - \frac{1}{n-1}(T_{jkl} - T_{jlk}) \\ &= \frac{n-2}{n-1}(T_{jkl} - T_{jlk}), \end{aligned} \tag{6.7}$$

from which it follows that (c)$\Leftrightarrow$(f). It completes the proof. □

For the Riemannian curvature tensor R, the Ricci tensor $S = Ric(R)$, the scalar curvature r and the projective curvature tensor W are defined by (6.1). Applying Corollary 6.4 to these situations, we can prove the following.

THEOREM 6.5. *Let M be an $n(\geqq 3)$-dimensional semi-Riemannian manifold and let R, S and r be the curvature tensor, the Ricci tensor and the scalar curvature on M. Then the following assertions are equivalent:*
(a) $\nabla S \in C^\infty(\mathcal{A})$.
(b) *The projective curvature tensor W satisfies the second Bianchi identity.*
(c) *S is the Codazzi tensor.*
(d) *The Ricci form ϕ_S satisfies the Codazzi equation.*
(e) *The associated curvature-like form Ψ_W is closed.*
(f) *The associated curvature-like form Ψ_W is coclosed.*
(g) *(M, g) has the harmonic projective tensor.*

Now let us give the following example.

EXAMPLE 6.1 ([1], [6]). For any integer p $(\geqq 2)$ an indefinite complex hypersurface $M(p, \lambda)$ of a $(2n+1)$-dimensional indefinite complex Euclidean space C_n^{2n+1} of index $2n$ can be defined as follows; Let $(z^A) = (z^j, z^{j^*}, z^{2n+1}) = (z^1, \ldots, z^{2n+1})$ be a complex coordinate of C_n^{2n+1} and let λ be a complex number such that $|\lambda| = 1$. Then $M(p, \lambda)$ is an indefinite complete complex hypersurface of index $2n$ defined by

$$z^{2n+1} = \sum_j f_j(z^j + \lambda z^{j^*}), \quad j^* = n + j, \quad f_j(z) = z^p.$$

Then for the components h_{AB} of the second fundamental form we have

$$\begin{aligned} &h_{ij} = p(p-1)\delta_{ij}{\mu_i}^{p-2}, \quad h_{i^*j} = p(p-1)\lambda\delta_{ij}{\mu_i}^{p-2}, \\ &h_{i^*j^*} = p(p-1)\lambda^2\delta_{ij}{\mu_i}^{p-2}, \quad \mu_i = z^i + \lambda z^{i^*}, \end{aligned} \tag{6.8}$$

and then for the components h_{ABC} of the covariant derivatives of the second fundamental form we have

$$\begin{aligned} h_{ijk} &= p(p-1)(p-2)\delta_{ij}\delta_{ik}{\mu_i}^{p-3}, \\ h_{i^*jk} &= p(p-1)(p-2)\lambda\delta_{ij}\delta_{ik}{\mu_i}^{p-3}, \\ h_{i^*j^*k} &= p(p-1)(p-2)\lambda^2\delta_{ij}\delta_{ik}{\mu_i}^{p-3}, \\ h_{i^*j^*k^*} &= p(p-1)(p-2)\lambda^3\delta_{ij}\delta_{ik}{\mu_i}^{p-3}. \end{aligned} \tag{6.9}$$

Then the example above is so much related to the following remark. By the following remark we know that the projective curvature tensor W for Example 6.1 satisfies the second Bianchi identity.

REMARK 6.2. If the curvature-like tensor T is parallel, then it satisfies the second Bianchi identity. However, the converse is not necessarily true. Example 6.1 is the counter example provided $p \geqq 3$. In such an example, by (6.8) we know the following facts: the complex Ricci tensor of M is flat provided that $|\lambda| = 1$, but the curvature tensor is not flat. Accordingly, since the real Ricci tensor is also flat, the scalar curvature r is zero on M. Also it is seen that if $p \geqq 3$, then M is not locally symmetric.

Similarly, if the projective curvature tensor W is parallel, then it satisfies the second Bianchi identity, However, the converse is not necessary true. In fact, if $p \geqq 3$ and if $|\lambda| = 1$, then this indefinite complex hypersurface M in C_n^{2n+1} is Ricci flat but not locally symmetric. So the projective curvature tensor W coincides with the Riemannian curvature tensor R on M, but it is not parallel, that is $\nabla W = \nabla R \neq 0$. However, it satisfies the second Bianchi identity.

7. The conformal curvature tensor D

In this section we want to investigate the semi-Riemannian manifolds whose conformal curvature tensor satisfies the second Bianchi identity. Let M be an n-dimensional semi-Riemannian manifold of index s, $0 \leqq s \leqq n$, with the Riemannian connection ∇ and let R (resp. S or r) be the Riemannian curvature tensor (resp. the Ricci tensor or the scalar curvature) on M.

Now, let D be the conformal curvature tensor with components D_{ijkl} on M, which is given by

(7.1)

$$\begin{aligned} D_{ijkl} = R_{ijkl} &- \frac{1}{n-2}(\varepsilon_i S_{jk}\delta_{il} - \varepsilon_i S_{jl}\delta_{ik} + \varepsilon_j S_{il}\delta_{jk} - \varepsilon_j S_{ik}\delta_{jl}) \\ &+ \frac{r}{(n-1)(n-2)}\varepsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \end{aligned}$$

The Riemannian manifold whose conformal curvature tensor is flat is investigated by Ryan [16], Weyl [18, 19]. See Yano [20, 21] and Yano and Bochner [22] for examples. As it is easily seen, D is the curvature-like tensor on M.

REMARK 7.1. If M is Einstein, then the Ricci tensor satisfies (2.6), which yields by (2.6) that the conformal curvature tensor D satisfies

$$D_{ijkl} = R_{ijkl} - \frac{r}{n(n-1)}\varepsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}).$$

This yields that the conformal curvature tensors of Einstein Riemannian manifolds are the concircular curvature ones [22]. In particular, if M is of constant curvature, then the conformal curvature tensor vanishes identically.

Now, let T be the curvature-like tensor and let U be the symmetric tensor of type $(0,2)$. We put $u = C_{12}(U)$. For such (T,U), we define the tensor $B = B(T,U)$ with components B_{ijkl} by

(7.2)
$$\begin{aligned} B_{ijkl} = T_{ijkl} &- \frac{1}{n-2}(\varepsilon_i U_{jk}\delta_{il} - \varepsilon_i U_{jl}\delta_{ik} + \varepsilon_j U_{il}\delta_{jk} - \varepsilon_j U_{ik}\delta_{jl}) \\ &+ \frac{u}{(n-1)(n-2)}\varepsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}), \end{aligned}$$

which is called the *conformal curvature-like tensor* for T and U. It is trivial that B is the curvature-like tensor. Calculating straightforward, by (7.2), we have

$$B_{ijkl} + B_{jkil} + B_{kijl} = T_{ijkl} + T_{jkil} + T_{kijl}.$$

From this equation and (4.3), we have

$$B_{ijkl} + B_{jkil} + B_{kijl} = 0. \tag{7.3}$$

Thus we have the following.

PROPOSITION 7.1. *Let M be an n-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let U be the symmetric tensor in D^2M. Then the conformal curvature-like tensor $B = B(T,U)$ satisfies the second Bianchi identity.*

First, we give a necessary condition for the conformal curvature-like tensor B to satisfy the second Bianchi identity. Let Φ_B and Ψ_B the curvature-like form and the associated curvature-like form for B, let ϕ_U be the Ricci-like form with the symmetric tensor U of type $(0,2)$ and let ψ_U be the Weyl form for U defined by $\psi_U = \phi_U - \frac{1}{2(n-1)}u\theta$, where $u = C_{12}(U)$ and θ is the connection form on M.

LEMMA 7.2. *Let M be an $n(\geqq 3)$-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let U be the symmetric tensor in D^2M. If any two assertions in the following hold, then another one holds true:*

(a) *T satisfies the second Bianchi identity.*
(b) *The conformal curvature-like tensor $B = B(T, U)$ satisfies the second Bianchi identity.*
(c) *U is the Weyl tensor.*

Proof. By (7.2), the components B_{ijklh} of the covariant derivative ∇B of B are given by

(7.4)

$$B_{ijklh} = T_{ijklh} - \frac{1}{n-2}(\varepsilon_i U_{jkh}\delta_{il} - \varepsilon_i U_{jlh}\delta_{ik} + \varepsilon_j U_{ilh}\delta_{jk} - \varepsilon_j U_{ikh}\delta_{jl}) + \frac{1}{(n-1)(n-2)} u_h \varepsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}).$$

Accordingly, we have

(7.5)

$$\begin{aligned} & B_{ijklh} + B_{ijlhk} + B_{ijhkl} \\ = {} & T_{ijklh} + T_{ijlhk} + T_{ijhkl} \\ & + \frac{1}{n-2}[\{(U_{jkl} - U_{jlk}) - \frac{1}{2(n-1)}\varepsilon_j(U_l\delta_{jk} - u_k\delta_{jl})\}\varepsilon_i\delta_{ih} \\ & + \{(U_{jlh} - U_{jhl}) - \frac{1}{2(n-1)}\varepsilon_j(U_h\delta_{jl} - u_l\delta_{jh})\}\varepsilon_i\delta_{ik} \\ & + \{(U_{jhk} - U_{jkh}) - \frac{1}{2(n-1)}\varepsilon_j(U_k\delta_{jh} - u_h\delta_{jk})\}\varepsilon_i\delta_{il} \\ & - \varepsilon_j\delta_{jk}\{(U_{ilh} - U_{ihl}) - \frac{1}{2(n-1)}\varepsilon_i(U_h\delta_{il} - u_l\delta_{ih})\} \\ & - \varepsilon_j\delta_{jl}\{(U_{ihk} - U_{ikh}) - \frac{1}{2(n-1)}\varepsilon_i(U_k\delta_{ih} - u_h\delta_{ik})\} \\ & - \varepsilon_j\delta_{jh}\{(U_{ikl} - U_{ilk}) - \frac{1}{2(n-1)}\varepsilon_i(U_l\delta_{ik} - u_k\delta_{il})\}]. \end{aligned}$$

Suppose that (a) and (b) hold. By (7.5), we have

$$\begin{aligned} (7.6) \qquad & \{(U_{jkl} - U_{jlk}) - \frac{1}{2(n-1)}\varepsilon_j(U_l\delta_{jk} - u_k\delta_{jl})\}\varepsilon_i\delta_{ih} \\ & + \{(U_{jlh} - U_{jhl}) - \frac{1}{2(n-1)}\varepsilon_j(U_h\delta_{jl} - u_l\delta_{jh})\}\varepsilon_i\delta_{ik} \end{aligned}$$

$$+\{(U_{jhk}-U_{jkh})-\frac{1}{2(n-1)}\varepsilon_j(U_k\delta_{jh}-u_h\delta_{jk})\}\varepsilon_i\delta_{il}$$
$$-\varepsilon_j\delta_{jk}\{(U_{ilh}-U_{ihl})-\frac{1}{2(n-1)}\varepsilon_i(U_h\delta_{il}-u_l\delta_{ih})\}$$
$$-\varepsilon_j\delta_{jl}\{(U_{ihk}-U_{ikh})-\frac{1}{2(n-1)}\varepsilon_i(U_k\delta_{ih}-u_h\delta_{ik})\}$$
$$-\varepsilon_j\delta_{jh}\{(U_{ikl}-U_{ilk})-\frac{1}{2(n-1)}\varepsilon_i(U_l\delta_{ik}-u_k\delta_{il})\}=0.$$

Putting $i=h$, multiplying ε_i and summing up with respect to the index i in (7.6), we obtain

(7.7)
$$(n-3)\{(U_{jkl}-U_{jlk})-\frac{1}{2(n-1)}\varepsilon_j(u_l\delta_{jk}-u_k\delta_{jl})\}$$
$$-\varepsilon_j\delta_{jk}(\sum_h\varepsilon_hU_{hlh}-\frac{1}{2}U_l)+\varepsilon_j\delta_{jl}(\sum_h\varepsilon_hU_{hkh}-\frac{1}{2}U_k)=0.$$

Again, putting $j=l$, multiplying ε_j and summing up with respect to the index j in (7.7), we obtain

$$2\sum_l\varepsilon_lU_{lkl}=u_k, \tag{7.8}$$

from which together with (7.7) it follows that we get

$$(U_{jkl}-U_{jlk})-\frac{1}{2(n-1)}\varepsilon_j(u_l\delta_{jk}-u_k\delta_{jl})=0.$$

Thus, this means that U is the weyl tensor and the assertion (c) is derived. The others are trivial. It completes the proof. □

We say that (M,g) has *the harmonic Weyl-like tensor* if $\delta(\Psi_B)=0$. In particular, (M,g) is said to have *the harmonic Weyl tensor* if $\delta(\Psi_D)=0$.

LEMMA 7.3. *Let M be an $n(\geqq 4)$-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let U be the symmetric tensor in D^2M. We assume that T satisfies the second Bianchi identity. If $U=Ric(T)$, then the following assertions are equivalent:*

(a) *The associated curvature-like form Ψ_B is closed.*

(b) *The associated curvature-like form Ψ_B is coclosed.*

(c) *(M,g) has the harmonic Weyl-like tensor.*

Proof. By Theorem 4.2, under the assertion (a) the symmetric tensor U is the Weyl tensor. So we have $2C_{12}(\nabla U) = C_{23}(\nabla U)$, i.e., $2\sum_k \varepsilon_k U_{kjk} = \sum_k \varepsilon_k U_{kkj}$. Moreover, by Lemma 7.2, the conformal curvature-like tensor B satisfies the second Bianchi identity.

As the first step of the proof, we show (a)⇒(b). Since the tensors B and T satisfy the second Bianchi identity. Putting $i = l$, multiplying ε_i and summing up with respect to the index i in (7.2), we have $B_{jk} = T_{jk} - U_{jk}$, where $B_{jk} = \sum_i \varepsilon_i B_{ijki}$ and $T_{jk} = \sum_i \varepsilon_i T_{ijki}$ are components of the tensors $Ric(B)$ and $Ric(T)$. By the assumption $Ric(T) = U$, we have $T_{jk} = U_{jk}$, which means that $B_{jk} = 0$, from which together with the fact that B satisfies the second Bianchi identity, it follows that we obtain $\sum_h \varepsilon_h B_{hjklh} = 0$. It yields that (b) is derived. Conversely, we assume the property (b). Putting $i = h$, multiplying ε_i and summing up with respect to the index i in (7.4) and taking account of the fact $B_{jk} = 0$, we have

$$\begin{aligned}
\sum_h \varepsilon_h B_{hjklh} &= T_{jkl} - T_{jlk} + \frac{1}{(n-1)(n-2)}\varepsilon_j(U_l\delta_{jk} - u_k\delta_{jl}) \\
&- \frac{1}{n-2}\{U_{jkl} - U_{jlk} + \varepsilon_j(\sum_h \varepsilon_h U_{hlh}\delta_{jk} - \sum_h \varepsilon_h U_{hkh}\delta_{jl})\} \\
&= \frac{1}{n-2}\{(n-3)(U_{jkl} - U_{jlk}) - \varepsilon_j(\sum_h \varepsilon_h U_{hlh}\delta_{jk} - \sum_h \varepsilon_h U_{hkh}\delta_{jl}) \\
&+ \frac{1}{n-1}\varepsilon_j(u_l\delta_{jk} - u_k\delta_{jl})\} = 0,
\end{aligned}$$

where the last equality is derived from the assumption (b). Putting $j = l$, multiplying ε_j and summing up with respect to the index j in the above equation, we have $2\sum_k \varepsilon_k U_{kjk} = \sum_k \varepsilon_k U_{kkj}$. From these two equations, we have

$$U_{jkl} - U_{jlk} - \frac{1}{2(n-1)}\varepsilon_j(u_l\delta_{jk} - u_k\delta_{jl}) = 0,$$

from which it follows that (b)⇒(a). It completes the proof. □

LEMMA 7.4. *Under the situation of Lemma 7.2, if $Ric(T)$ is the Codazzi tensor, then U is the Codazzi tensor if and only if Ψ_B is closed.*

Proof. Putting $i = h$ in (7.4), multiplying ε_i and summing up with respect to the index i, we have

$$\sum_h \varepsilon_h B_{hjklh} = -\frac{1}{n-2}(U_{jkl} - U_{jlk}) + \frac{n-3}{2(n-1)(n-2)}\varepsilon_j(u_l\delta_{jk} - u_k\delta_{jl}).$$

If $d\Psi_B = 0$, then we get $(U_{jkl} - U_{jlk}) = \frac{n-3}{2(n-1)}\varepsilon_j(u_l\delta_{jk} - u_k\delta_{jl})$, and hence we have $u_k = 0$ and $U_{jkl} - U_{jlk} = 0$. If $U_{jkl} - U_{jlk} = 0$, then $u_k = 0$ and hence we obtain $d\Psi_B = 0$. It completes the proof. □

REMARK 7.2. By Lemma 7.3, if the *conformal curvature-like tensor* B vanishes identically on M, then U is the Weyl tensor. This yields, in Riemannian geometry, the well known result that *the conformal curvature tensor* $D = 0$ implies that S is the Weyl tensor (see Yano and Kon [23]). However, the converse on the Riemannian manifold does not necessarily hold. Example 6.1 shows its counter meaning. In such an example, since components of the second fundamental form is given by (6.8), the following facts can be guaranteed ; the complex Ricci tensor S of M is flat provided $|\lambda| = 1$, but the curvature tensor R is not flat. Accordingly, since the real Ricci tensor is also flat, the scalar curvature is zero on M, which means that S is the Weyl tensor. However we have $D = R \neq 0$.

The direct consequence of Lemmas 7.3 and 7.4 we can prove the following.

THEOREM 7.5. *Let M be an n-dimensional semi-Riemannian manifold and let T be the curvature-like tensor and let $U = Ric(T)$ be the Ricci-like tensor for T. We assume that T satisfies the second Bianchi identity. If $n \geqq 4$, then the following assertions are equivalent:*

(a) $\nabla U \in C^\infty(\mathcal{A})$.

(b) *The conformal curvature-like tensor $B = B(T, U)$ satisfies the second Bianchi identity.*

(c) *U is the Codazzi tensor.*

(d) *The Ricci-like form ϕ_U satisfies the Codazzi equation.*

(e) *The associated curvature-like form Ψ_B is closed.*

(f) *The associated curvature-like form Ψ_B is coclosed.*

(g) *(M, g) has the harmonic-like curvature.*

For the Riemannian curvature tensor R, the Ricc tensor $S = \mathrm{Ric}(R)$ and the scalar curvature r, we can define the conformal curvature tensor D defined by (7.1). Applying Lemma 7.3 and Theorem 7.5 to these situations, we can prove the following:

THEOREM 7.6. *Let M be an n-dimensional semi-Riemannian manifold and let R and S be the curvature tensor and the Ricci tensor on M. If $n \geqq 4$, then the following assertions are equivalent:*

(a) $\nabla S \in C^\infty(\mathcal{A} \oplus \mathcal{I})$.

(b) *The conformal curvature tensor D satisfies the second Bianchi identity.*

(c) *S is the Weyl tensor.*

(d) *The Weyl form ψ_S satisfies the Codazzi equation.*

(e) *The associated curvature-like form Ψ_D is coclosed.*

(f) *(M, g) has the harmonic Weyl tensor.*

COROLLARY 7.7. *Let M be an n-dimensional semi-Riemannian manifold and let R, S and r be the curvature tensor, the Ricci tensor and the scalar curvature on M. If $n \geqq 4$ and if r is a constant, then the following assertions are equivalent:*

(a) $\nabla S \in C^\infty(\mathcal{A})$.

(b) *The conformal curvature tensor D satisfies the second Bianchi identity.*

(c) *S is the Codazzi tensor.*

(d) *The Ricci form ϕ_S satisfies the Codazzi equation.*

(e) *The associated curvature-like form Ψ_D is closed.*

(f) *The associated curvature-like form Ψ_D is coclosed.*

(g) *(M, g) has the harmonic curvature.*

REMARK 7.3. As it is well known theorem due to Weyl [18, 19], if the *conformal curvature tensor* $D = 0$ and if $n \geqq 4$, then the Ricci tensor S is the Weyl tensor. The terminology "Weyl tensors" is named after this property (see Yano and Kon [23]). Conversely, according to Theorem 7.6, if S is the Weyl tensor and if $n \geqq 4$, then D satisfies the second Bianchi identity. If $D = 0$, then it is trivial that it satisfies the second Bianchi identity. Thus Theorem 7.6 is a generalization of Weyl's theorem.

REMARK 7.4. As it was shown in Example 6.1 and Remark 6.2, if $p \geqq 3$, then the Ricci tensor of the indefinite complex hypersurface

M in C_n^{2n+1} is flat, but it is not locally symmetric. So the *conformal curvature tensor* D coincides with the Riemannian curvature tensor R on M, because the scalar curvature r is also vanishing. But we know that it is not parallel, that is, $\nabla D = \nabla R \neq 0$. However, it satisfies the second Bianchi identity.

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