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POWER TAIL ASYMPTOTIC RESULTS OF A DISCRETE TIME QUEUE WITH LONG RANGE DEPENDENT INPUT

GANG UK HWANG AND KHOSROW SOHRABY

ABSTRACT. In this paper, we consider a discrete time queueing system fed by a superposition of an ON and OFF source with heavy tail ON periods and geometric OFF periods and a D-BMAP (Discrete Batch Markovian Arrival Process). We study the tail behavior of the queue length distribution and both infinite and finite buffer systems are considered. In the infinite buffer case, we show that the asymptotic tail behavior of the queue length of the system is equivalent to that of the same queueing system with the D-BMAP being replaced by a batch renewal process. In the finite buffer case (of buffer size K), we derive upper and lower bounds of the asymptotic behavior of the loss probability as $K \to \infty$.

1. Introduction

Recent traffic measurements resulting from a number of applications have revealed that there is sufficient statistical evidence of long range dependence (LRD) in the autocorrelations of the number of packets or cell arrivals in a time interval [19]. This implies that the autocorrelation function of the count process decays as a power of the lag time in contrast to that of the classical short range dependent traffic (e.g., Markovian models) [9, 11, 26, 27, 28] where the decay of the autocorrelation function is exponentially fast. There have been a number of works on queueing analysis with long range dependent arrivals (e.g., see [3, 6, 12, 15, 16, 18, 20, 23, 24, 25] and references therein). Recently, Daniel and Blondia [5] analyzed queues with long range dependent traffic based on discrete time

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models. They considered the superposition of an ON-OFF heavy tail source (similar to our model) and a batch renewal source, and obtained the tail behavior of the queue length distribution when the number of servers equals $c, c \geq 1$. Our work generalizes [5] in that we allow a general *correlated* batch Markovian arrival process for the case of a single server.

In this paper, we consider a discrete time queue fed by two traffic sources: an ON and OFF source with heavy tail ON periods and geometric OFF periods and a D-BMAP (Discrete Batch Markovian Arrival Process). For convenience, we refer to the ON and OFF source having heavy tail ON periods as the Heavy Tail source from now on. The OFF periods of the Heavy Tail source are assumed to be geometrically distributed. The ON periods of the Heavy Tail source are generated according to a heavy tail distribution. The Heavy Tail source represents a long range dependent traffic source.

A good example of a Heavy Tail source is a "continuous" source generating variable size packets where the interarrival time of packets is exponentially distributed and the packet sizes have a distribution with a heavy tail. A "discretization" of such a source (as in ATM networks) results in a discrete cell arrival process where the OFF periods are geometrically distributed and the ON periods have a heavy tail discrete distribution [26]. Here, it is assumed that the link rate to source rate ratio is unity, i.e., cells arrive consecutively during the ON periods.

For the ON periods of the Heavy Tail source, we use a distribution which is asymptotically equivalent to a Pareto distribution with a parameter in the range $(2, 3)$ (see the details in section 2). This type of distributions results in an arrival process with a finite mean, an infinite variance and a power law covariance function with a Hurst parameter in the range $(1/2, 1)$ [4].

In our analysis, we first study the tail probabilities of the queue length for an infinite buffer system. Our analysis is based on a spectral decomposition method and Tauberian Theorem for power series [7]. Our result shows that the tail behavior of the queue length distribution is equivalent to that of the same queueing system with the D-BMAP being replaced by a batch renewal process, and the asymptotic decay constant depends only on the *average* ON and OFF periods of the Heavy Tail source and the total utilization of the D-BMAP. Our result in the infinite buffer case is consistent with those in [2, 12], where a similar queueing behavior based on a fluid flow model was shown. Next, we consider the same queueing system except that the buffer size is finite and of size K, and derive upper and lower bounds of the asymptotic behavior of the loss probability as $K \to \infty$. Our result shows that the loss probabilities are asymptotically power-tailed as the buffer size increases, so that buffering does not reduce the loss probability significantly.

The rest of the paper is organized as follows. In Section 2, we present the mathematical model for our system and some preliminaries needed in the analysis. In Section 3, we derive an exact tail probability asymptotic for an infinite buffer system. In Section 4, we study a finite buffer system and derive upper and lower bounds of the asymptotic of the loss probabilities for the system. In Section 5, we provide conclusions.

2. Mathematical description of sources

We consider a discrete time queueing system where the time is divided into slots of equal size and one slot is needed to transmit a cell in the system. We assume our system has an infinite buffer fed by two sources: an ON and OFF source with heavy tail ON periods and geometric OFF periods (called the Heavy Tail source) and a D-BMAP. Refer to Figure 1. To describe the D-BMAP mathematically, we introduce a homogeneous and ergodic discrete-time Markov chain, called the *modulating Markov chain*, in which transitions between states of the chain take place only at the slot boundaries. Denote J_k as the state of the modulating Markov chain at slot k. We assume that the state space of J_k is $\{0, \cdots, M\}$. Given that the modulating Markov chain is in state i, the probability generating function (P.G.F.) of the number D_{k+1} of cells arriving in slot $k+1$ with a transition to state j, is denoted by $d_{ij}(z)$, i.e.,

$$d_{ij}(z) = E[z^{D_{k+1}} 1_{\{J_{k+1}=j\}} | J_k = i],$$

where 1_E is an indicator function taking the value of 1 (0) if the event E occurs (does not occur). Then, the D-BMAP is the arrival process where the probability generating matrix (P.G.M.) of the number of cells generated in a slot is an $(M+1) \times (M+1)$ matrix $\mathbf{D}(z)$ given as follows:

$$\mathbf{D}(z) = \begin{pmatrix} d_{00}(z) & d_{01}(z) & \cdots & d_{0M}(z) \\ d_{10}(z) & d_{11}(z) & \cdots & d_{1M}(z) \\ \vdots & \vdots & \ddots & \vdots \\ d_{M0}(z) & d_{M1}(z) & \cdots & d_{MM}(z) \end{pmatrix}.$$

For the analysis, we assume that, for sufficiently small $\epsilon > 0$ $\mathbf{D}(z)$ exists for $0 \le z \le 1 + \epsilon$. It should be noted that the matrix $\mathbf{D}(1)$ is simply the probability transition matrix of the modulating Markov chain. Since we assume that $\mathbf{D}(1)$ is irreducible, we know that $\mathbf{D}(z)$ is irreducible for

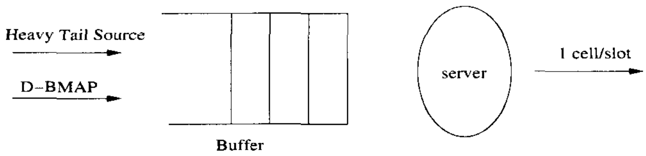

FIGURE 1. The Queueing Model

each z in $0 < z \leq 1$. A good example of the D-BMAP is a superposition of 2-state Markov Modulated Bernoulli Processes (MMBPs) [9, 28].

In the analysis, we assume that the matrix $\mathbf{D}(z)$ is diagonalizable [17], i.e.,

$$\mathbf{D}(z) = \mathbf{G}(z)\mathbf{\Lambda}(z)\mathbf{H}(z), \mathbf{G}(z)\mathbf{H}(z) = \mathbf{H}(z)\mathbf{G}(z) = I. \tag{1}$$

Here, I denotes the identity matrix of dimension $M+1$ and $\mathbf{\Lambda}(z)$ is a diagonal eigenvalue matrix of $\mathbf{D}(z)$, given by

$$\mathbf{\Lambda}(z) = diag(\lambda_0(z), \cdots, \lambda_M(z)).$$

For each eigenvalue $\lambda_i(z), 0 \leq i \leq M$, let $\mathbf{h}_i(z) = (h_{i0}(z), \cdots, h_{iM}(z))$ and $\mathbf{g}_i(z) = (g_{i0}(z), \cdots, g_{iM}(z))^T$ be the left and right eigenvectors of $\mathbf{D}(z)$ corresponding to $\lambda_i(z)$, respectively. Here the superscript T denotes the transpose of the row vector. We then have

$$\mathbf{H}(z) = \begin{pmatrix} \mathbf{h}_0(z) \\ \vdots \\ \mathbf{h}_M(z) \end{pmatrix}, \mathbf{G}(z) = (\mathbf{g}_0(z), \cdots, \mathbf{g}_M(z)).$$

Without loss of generality, we may assume that $\lambda_0(z)$ is the Perron-Frobenius eigenvalue of the matrix $\mathbf{D}(z)$. We also assume that the eigenvalues and the eigenvectors of $\mathbf{D}(z)$ are twice continuously differentiable in $z \in [0, 1]$. Since $D(1)$ is irreducible, nonnegative and aperiodic by our assumption, we have, by the Perron-Frobenius theory [1]

$$\lambda_0(1) = 1, \qquad |\lambda_i(1)| < 1, \qquad i \neq 0. \tag{2}$$

Let ρ_s be the mean number of arrivals from the D-BMAP during a slot. Then ρ_s satisfies

$$\rho_s = \boldsymbol{\pi} \left. \frac{d}{dz}\mathbf{D}(z) \right|_{z=1} \mathbf{e},$$

where $\boldsymbol{\pi}$ is the steady state probability vector of $\mathbf{D}(1)$ and $\mathbf{e}$ is a column vector of dimension $M+1$, all of whose elements are equal to 1.

Next, we describe the Heavy Tail source mathematically. The Heavy Tail source is defined to be an ON-OFF source having the following property: while the source is in the OFF (ON) period, it generates zero (one) cell per slot. The OFF period is assumed to be geometrically distributed with parameter $1-p(>0)$. The ON period of the Heavy Tail source is assumed to be distributed according to a discrete Pareto distribution whose probability mass function $\{b_n\}_{n\geq 1}$ where for constants $a(>0)$ and s with $2<s<3$, b_n satisfies

$$b_n \sim an^{-s}, \qquad \text{as } n \to \infty.$$

Here, $k_n \sim k'_n$ means that $\lim_{n\to\infty} k_n/k'_n = 1$. Let $B(z)$ be the P.G.F. of $\{b_n\}_{n\geq 1}$, i.e.,

$$B(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad \text{for } 0 \leq z \leq 1.$$

From the asymptotic behavior of $\{b_n\}_{n\geq 1}$ we know the ON period has a finite mean, denoted by $E[B]$, but has infinite variance.

Let ρ be the total traffic intensity of our system, given by

$$\rho = \rho_s + \frac{pE[B]}{1+pE[B]}. \tag{3}$$

For stability, we assume that $\rho < 1$. In addition, we assume that arrivals, if any, occur in the middle of each slot and departures, if any, occur just before the slot boundary.

For simplicity, we use the notation $\sigma = 3 - s(>0)$ in the analysis. Observing that

$$1 - B(z) - (1-z)B'(z) = (1-z)^2 \sum_{m=1}^{\infty} m \left\{ \sum_{n=m+1}^{\infty} b_n \right\} z^{m-1},$$

and using the fact that $b_n \sim an^{-s}$, it is easy to show, by Theorem 5.1 [4], that

$$\lim_{z\to 1^-} (1-z)^{\sigma} \frac{1 - B(z) - (1-z)B'(z)}{(1-z)^2} = \frac{a\Gamma(\sigma)}{s-1}, \tag{4}$$

which will be repeatedly used in the analysis.

3. Analysis of an infinite buffer system

In this section, we will obtain the tail behavior of the queue length of an infinite buffer queueing system fed by a mixture of a D-BMAP and a Heavy Tail source. In the analysis, we assume that high service priority is given to the Heavy Tail source, so that all arriving cells from the D-BMAP are stored in the buffer during ON periods of the Heavy Tail source plus the following slots. Note that this assumption does not impact the tail probability behavior of the queue length in the system.

Now, the analysis for the infinite buffer system is performed in two steps. The first step is to compute the Probability Generating Vector (P.G.V.) of the number of cells at an arbitrary slot in an OFF period of the Heavy Tail source in the steady state by discarding all ON periods of the Heavy Tail source. The second step is to compute the P.G.V. of the number of cells at an arbitrary slot in an ON period of the Heavy Tail source in the steady state, and finally by combining the two results we obtain the P.G.V. of the queue length distribution for our system.

3.1. Derivation of P.G.M. of the OFF periods

In this subsection, we consider the end of slots in OFF periods as embedded points (see Figure 2). Let $\{X_n\}_{n\geq 1}$ be the number of cells in the system at the end of the n-th embedded point, say s_n, in the steady state. Note then that $\{J_{s_n}\}_{n\geq 1}$ is the state of the D-BMAP at the end of the n-th embedded point s_n. From our definitions, it is easy to see that X_n satisfies the evolution equation

$$X_{n+1} = \max(X_n - 1, 0) + A_{n+1},$$

where A_{n+1} denotes the total number of cells from the D-BMAP arriving between the nth and the $(n+1)$th embedded points.

Let C_{n+1} be the number of cells from the D-BMAP during (s_n, s_{n+1}), given that there is an ON period of the Heavy Tail source between s_n and s_{n+1}. Define $c_{ij}(z)$ by

$$c_{ij}(z) = E[z^{C_{n+1}} I\{J_{s_{n+1}} = j\} | J_{s_n} = i, \text{ an ON period is in } (s_n, s_{n+1})].$$

Let $\mathbf{C}(z)$ be an $(M+1) \times (M+1)$ matrix whose (i,j)-th component is $c_{ij}(z)$. Then, $\mathbf{C}(z)$ is the P.G.M. of the number of cells generated from the D-BMAP during an ON period of the Heavy Tail source plus the

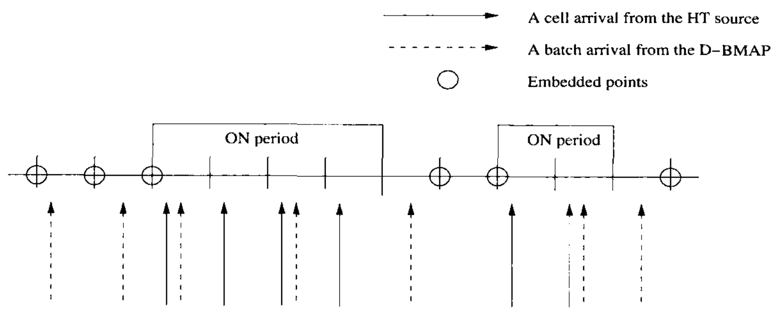

FIGURE 2. Embedded Points of Our System

following slot. Then

$$\begin{aligned}\mathbf{C}(z) &= \sum_{n=1}^{\infty} b_n[\mathbf{D}(z)]^{n+1} = \sum_{n=1}^{\infty} b_n \mathbf{G}(z)\mathbf{\Lambda}(z)^{n+1}\mathbf{H}(z)\\ &= \mathbf{G}(z)diag(\sum_{n=1}^{\infty} b_n \lambda_i(z)^{n+1})\mathbf{H}(z)\\ &= \mathbf{G}(z)\tilde{\mathbf{\Lambda}}(z)\mathbf{\Lambda}(z)\mathbf{H}(z), \end{aligned} \tag{5}$$

where $\tilde{\mathbf{\Lambda}}(z) = diag(\tilde{\lambda}_i(z))$ and $\tilde{\lambda}_i(z) = \sum_{n=1}^{\infty} b_n[\lambda_i(z)]^n$.

Let $\mathbf{A}(z)$ be the P.G.M. of the sequence $\{(A_n, J_{s_n})\}$ whose (i,j)-th component is

$$E[z^{A_{n+1}} I\{J_{s_{n+1}} = j\} | J_{s_n} = i].$$

Then, by the definition of $\mathbf{A}(z)$ it satisfies

$$\begin{aligned}\mathbf{A}(z) &= p\mathbf{C}(z) + (1-p)\mathbf{D}(z)\\ &= \mathbf{G}(z)[p\tilde{\mathbf{\Lambda}}(z)\mathbf{\Lambda}(z) + (1-p)\mathbf{\Lambda}(z)]\mathbf{H}(z),\end{aligned} \tag{6}$$

where in the second equation (1) and (5) are used. Let $\tilde{\rho} = E[A_n]$ be the mean number of arrivals between two consecutive embedded points. Then, the fact that $\boldsymbol{\pi}$ is also the steady state probability vector of $\mathbf{A}(1)$ yields

$$\tilde{\rho} = \boldsymbol{\pi} \frac{d}{dz}\mathbf{A}(z)\Big|_{z=1} \mathbf{e} = (1 + pE[B])\rho_s. \tag{7}$$

From (3), (7) and our stability condition $\rho < 1$ it follows that

$$1 - \rho = 1 - \rho_s - \frac{pE[B]}{1+pE[B]} = \frac{1-\rho_s(1+pE[B])}{1+pE[B]} = \frac{1-\tilde{\rho}}{1+pE[B]} > 0$$

and consequently, we obtain $\tilde{\rho} < 1$. In addition, note that $\{(X_n, J_{s_n})\}_{n\geq 1}$ is a Markov chain whose transition probability matrix is of the M/G/1 type as follows:

$$\begin{pmatrix} \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \cdots \\ \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \cdots \\ 0 & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $(M+1)\times(M+1)$ matrices $\mathbf{A}_i, i \geq 0$ are the Taylor series expansion of the P.G.M. $\mathbf{A}(z)$. Since $\mathbf{A}(1)$ is irreducible and $\tilde{\rho} < 1$, the Markov chain $\{(X_n, J_{s_n})\}_{n\geq 1}$ has the stationary distribution [21].

Define $X_i(z), i = 0, \cdots, M$, by

$$X_i(z) = \sum_{l=0}^{\infty} x_{li} z^l,$$

where

$$x_{li} = P\{X = l, J_s = i\},$$

and X and J_s are the steady state versions of X_n and J_{s_n}, respectively. Let row vectors $\mathbf{x}_l$ be defined by $\mathbf{x}_l = (x_{l0}, \cdots, x_{lM})$ for $l \geq 0$. Then, the P.G.V. $\mathbf{X}(z)$ of the steady state distribution for the Markov chain $\{(X_n, J_{s_n})\}_{n\geq 1}$, defined by

$$\mathbf{X}(z) = (X_0(z), \cdots, X_M(z)),$$

satisfies

$$\begin{aligned} \mathbf{X}(z) &= (z-1)\mathbf{x}_0\mathbf{A}(z)[zI - \mathbf{A}(z)]^{-1} \\ &= (z-1)\mathbf{x}_0\mathbf{G}(z)[p\tilde{\mathbf{\Lambda}}(z)\mathbf{\Lambda}(z) + (1-p)\mathbf{\Lambda}(z)] \\ &\qquad \times[zI - p\tilde{\mathbf{\Lambda}}(z)\mathbf{\Lambda}(z) - (1-p)\mathbf{\Lambda}(z)]^{-1}\mathbf{H}(z) \text{ by (6)} \\ &= (z-1)\mathbf{x}_0\mathbf{G}(z) diag\left(\frac{p\tilde{\lambda}_i(z)\lambda_i(z) + (1-p)\lambda_i(z)}{z - p\tilde{\lambda}_i(z)\lambda_i(z) - (1-p)\lambda_i(z)}\right)\mathbf{H}(z), \end{aligned}$$

so that

$$\begin{aligned} X_j(z) &= (z-1)\sum_i\sum_k x_{0k}g_{ik}(z)\frac{p\tilde{\lambda}_i(z)\lambda_i(z) + (1-p)\lambda_i(z)}{z - p\tilde{\lambda}_i(z)\lambda_i(z) - (1-p)\lambda_i(z)}h_{ij}(z) \\ &= \sum_i\sum_k x_{0k}g_{ik}(z)\frac{z-1}{z-t_i(z)}t_i(z)h_{ij}(z), \qquad (8) \end{aligned}$$

where

$$t_i(z) = p\tilde{\lambda}_i(z)\lambda_i(z) + (1-p)\lambda_i(z). \qquad (9)$$

For the analysis we need the following lemma:

LEMMA 3.1.

$$\lim_{z\to 1^-}(1-z)^\sigma \frac{1-\lambda_0(z)-(1-z)\lambda_0'(z)}{(1-z)^2}=0, \tag{10}$$

$$\lim_{z\to 1^-}(1-z)^\sigma \frac{1-\tilde{\lambda_0}(z)-(1-z)\tilde{\lambda_0}'(z)}{(1-z)^2}=(\rho_s)^{2-\sigma}\frac{a\Gamma(\sigma)}{s-1}, \tag{11}$$

$$\lim_{z\to 1^-}(1-z)^\sigma \frac{1-t_0(z)-(1-z)t_0'(z)}{(1-z)^2}=p(\rho_s)^{2-\sigma}\frac{a\Gamma(\sigma)}{s-1}, \tag{12}$$

$$\lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}\left\{\frac{z-1}{z-t_0(z)}\right\}=\frac{p}{(1-t_0'(1))^2}(\rho_s)^{2-\sigma}\frac{a\Gamma(\sigma)}{s-1}, \tag{13}$$

$$\lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}\left\{\frac{z-1}{z-t_i(z)}\right\}=0, \qquad \text{for } i\neq 0. \tag{14}$$

Proof. See Appendix. □

Now we are ready to examine the behavior of $\mathbf{X}'(z)$ near $z=1$. By letting $\phi_{ij}(z)=\sum_k x_{0k}g_{ik}(z)t_i(z)h_{ij}(z)$ we obtain

$$\begin{aligned}
&\lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}X_j(z)\\
&=\lim_{z\to 1^-}(1-z)^\sigma \sum_i\left\{\frac{d\phi_{ij}(z)}{dz}\frac{z-1}{z-t_i(z)}+\phi_{ij}(z)\frac{d}{dz}\left\{\frac{z-1}{z-t_i(z)}\right\}\right\}\\
&=\lim_{z\to 1^-}\phi_{0j}(z)(1-z)^\sigma\frac{d}{dz}\left\{\frac{z-1}{z-t_0(z)}\right\}\\
&=\sum_k x_{0k}g_{0k}(1)t_0(1)h_{0j}(1)\frac{p\rho_s^{2-\sigma}}{(1-t_0'(1))^2}\frac{a\Gamma(\sigma)}{s-1} \text{ by (13)}\\
&=\sum_k x_{0k}h_{0j}(1)\frac{p\rho_s^{2-\sigma}}{(1-t_0'(1))^2}\frac{a\Gamma(\sigma)}{s-1} \text{ since } g_{0k}(1)=t_0(1)=1\\
&=(1-\tilde{\rho})\frac{h_{0j}(1)p\rho_s^{2-\sigma}}{(1-t_0'(1))^2}\frac{a\Gamma(\sigma)}{s-1} \text{ since } \textstyle\sum_k x_{0k}=1-\tilde{\rho}\\
&=\frac{h_{0j}(1)p\rho_s^{2-\sigma}}{(1-\tilde{\rho})}\frac{a\Gamma(\sigma)}{s-1}.
\end{aligned} \tag{15}$$

Here, in the second equation we use

$$1-t_0'(1)=1-\rho_s(1+pE[B])=1-\tilde{\rho} \tag{16}$$

and $\frac{d\phi_{ij}(z)}{dz} < \infty$ for $z \in [0,1]$ to have

$$\lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}\phi_{ij}(z)\frac{z-1}{z-t_i(z)} = 0, i = 0,\cdots,M.$$

We further use (14) to have

$$\lim_{z\to 1^-}(1-z)^\sigma \phi_{ij}(z)\frac{d}{dz}\left\{\frac{1-z}{z-t_i(z)}\right\} = 0, \quad i \neq 0.$$

In the last equation, we use (16) again.

For later use, we need the following lemma.

LEMMA 3.2. *The asymptotic behavior of the tail probabilities of X is*

$$\sum_{k\geq n}\mathbf{x}_k\mathbf{e} \sim \frac{ap\rho_s^{s-1}}{(1-\rho_s(1+pE[B]))(s-1)(s-2)}n^{2-s}.$$

Proof. Since $\sum_j h_{0j}(1) = 1$, from (15)

$$\text{(17)} \qquad \lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}\mathbf{X}(z)\mathbf{e} = \lim_{z\to 1^-}(1-z)^\sigma \sum_j \frac{d}{dz}X_j(z) = \frac{p\rho_s^{2-\sigma}}{(1-\tilde{\rho})}\frac{a\Gamma(\sigma)}{s-1}.$$

Hence, by Theorem 5.1 we get

$$\sum_{k=1}^{n} k\mathbf{x}_k\mathbf{e} \sim \frac{p\rho_s^{2-\sigma}}{(1-\tilde{\rho})\Gamma(\sigma+1)}\frac{a\Gamma(\sigma)}{s-1}n^\sigma.$$

From Lemma 5.2 we get

$$\sum_{k\geq n}\mathbf{x}_k\mathbf{e} \sim \frac{p\rho_s^{2-\sigma}}{(1-\tilde{\rho})\Gamma(\sigma+1)}\frac{a\Gamma(\sigma)}{s-1}\frac{\sigma}{1-\sigma}n^{\sigma-1}.$$

Then our lemma immediately follows from the fact that $\sigma = 3-s$. □

3.2. Deriving P.G.M. for ON periods

In this subsection, we derive the P.G.V. of the number of cells during ON periods of the Heavy Tail source. Let Y be the random variable representing the number of cells at the end of an arbitrary slot in ON periods in the steady state, and $\mathbf{Y}(z)$ be the P.G.V. of (Y, J_t) where J_t is the steady state version of J_{t_n} and $\{J_{t_n}\}$ is the sequence obtained from $\{J_n\}$ by considering it only at the end epochs $\{t_n\}$ of slots in ON periods (similarly as we obtained the sequence $\{J_{s_n}\}$).

By the GASTA property [8], we have

$$\begin{aligned}\mathbf{Y}(z) &= [\mathbf{x}_0(z-1)+\mathbf{X}(z)]\sum_{n=1}^{\infty}\frac{1}{E[B]}\left[\sum_{k=n}^{\infty}b_k\right]\mathbf{D}(z)^n \\ &= \frac{1}{E[B]}[\mathbf{x}_0(z-1)+\mathbf{X}(z)]\sum_{n=1}^{\infty}\left[\sum_{k=n}^{\infty}b_k\right]\mathbf{G}(z)\mathbf{\Lambda}(z)^n\mathbf{H}(z) \\ &= \frac{1}{E[B]}[\mathbf{x}_0(z-1)+\mathbf{X}(z)]\mathbf{G}(z)diag\left(\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}b_k\lambda_i(z)^n\right)\mathbf{H}(z) \\ &= \frac{1}{E[B]}[\mathbf{x}_0(z-1)+\mathbf{X}(z)]\mathbf{G}(z)\tilde{\tilde{\mathbf{\Lambda}}}(z)\mathbf{H}(z), \qquad (18)\end{aligned}$$

where $\tilde{\tilde{\mathbf{\Lambda}}}(z) = diag(\tilde{\tilde{\lambda}}_i(z))$ and $\tilde{\tilde{\lambda}}_i(z) = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} b_k\lambda_i(z)^n$.

As in subsection 3.1, we need the following lemma to examine the behavior of $\mathbf{Y}'(z)$ near $z=1$.

LEMMA 3.3.

$$\lim_{z\to 1^-}(1-z)^{\sigma}\frac{d}{dz}\tilde{\tilde{\lambda}}_0(z) = (\rho_s)^{1-\sigma}\frac{a\Gamma(\sigma)}{s-1}, \qquad (19)$$

$$\lim_{z\to 1^-}(1-z)^{\sigma}\frac{d}{dz}\tilde{\tilde{\lambda}}_i(z) = 0, \qquad \textit{for } i\neq 0. \qquad (20)$$

Proof. See Appendix. □

Now multiplying a column vector $\mathbf{e}$ on both sides of (18), we obtain

$$\begin{aligned}\mathbf{Y}(z)\mathbf{e} &= \frac{1}{E[B]}(z-1)\sum_j x_{0j}\sum_i\sum_k g_{ij}(z)\tilde{\tilde{\lambda}}_i(z)h_{ik}(z) \\ &\quad+\frac{1}{E[B]}\sum_j X_j(z)\sum_i\sum_k g_{ij}(z)\tilde{\tilde{\lambda}}_i(z)h_{ik}(z) \\ &= \frac{1}{E[B]}(z-1)\sum_j x_{0j}\sum_i\sum_k g_{ij}(z)\tilde{\tilde{\lambda}}_i(z)h_{ik}(z) \\ &\quad+\frac{1}{E[B]}\sum_j X_j(z)\left\{\sum_k g_{0j}(z)\tilde{\tilde{\lambda}}_0(z)h_{0k}(z)\right. \qquad (21) \\ &\quad\left.+\sum_{i\neq 0}\sum_k g_{ij}(z)\tilde{\tilde{\lambda}}_i(z)h_{ik}(z)\right\}.\end{aligned}$$

Combining (19), (20) and (21) yields

$$
\begin{aligned}
&\lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}\mathbf{Y}(z)\mathbf{e}\\
&= \frac{1}{E[B]}\sum_j \left\{ \sum_k g_{0j}(1)\tilde{\tilde{\lambda}}_0(1)h_{ik}(1) \lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}X_j(z) \right.\\
&\qquad\qquad \left. + X_j(1)g_{0j}(1)\sum_k h_{0k}(1) \lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}\tilde{\tilde{\lambda}}_0(z) \right\}\\
&\quad + \frac{1}{E[B]}\sum_j \sum_{i\neq 0}\sum_k g_{ij}(1)\tilde{\tilde{\lambda}}_i(1)h_{ik}(1) \lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}X_j(z)\\
&= \frac{1}{E[B]}\sum_j \left\{ \sum_i \sum_k g_{ij}(1)\tilde{\tilde{\lambda}}_i(1)h_{ik}(1) \lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}X_j(z) \right.\\
&\qquad\qquad \left. + X_j(1)g_{0j}(1) \lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}\tilde{\tilde{\lambda}}_0(z) \right\} \qquad (22)\\
&= \frac{1}{E[B]}\sum_j E[B] \lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}X_j(z)\\
&\quad + \frac{1}{E[B]}\sum_j X_j(1) \lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}\tilde{\tilde{\lambda}}_0(z)\\
&= \frac{1}{E[B]}\sum_j E[B] \lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}X_j(z) + \frac{1}{E[B]}\rho_s^{1-\sigma}\frac{a\Gamma(\sigma)}{s-1}.
\end{aligned}
$$

Here, in the first equation we use the facts that

$$\lim_{z\to 1^-}(1-z)^\sigma \tfrac{d}{dz}\left\{\tfrac{1}{E[B]}(z-1)\textstyle\sum_j x_{0j}\sum_i\sum_k g_{ij}(z)\tilde{\tilde{\lambda}}_i(z)h_{ik}(z)\right\} = 0,$$

$$\lim_{z\to 1^-}(1-z)^\sigma \tfrac{1}{E[B]}(z-1)\textstyle\sum_j X_j(z)\sum_k \tfrac{d}{dz}\left\{g_{0j}(z)h_{0k}(z)\right\}\tilde{\tilde{\lambda}}_0(z) = 0,$$

$$\lim_{z\to 1^-}(1-z)^\sigma \tfrac{1}{E[B]}\textstyle\sum_j X_j(z)\sum_{i\neq 0}\sum_k \tfrac{d}{dz}\left\{g_{ij}(z)\tilde{\tilde{\lambda}}_i(z)h_{ik}(z)\right\} = 0.$$

In the second equation we use the fact that $\sum_k h_{0k}(1) = 1$, and in the third equation we use the fact that

$$
\begin{aligned}
\sum_i\sum_k g_{ij}(1)\tilde{\tilde{\lambda}}_i(1)h_{ik}(1) &= (\mathbf{G}(1)\tilde{\tilde{\mathbf{\Lambda}}}(1)\mathbf{H}(1)\mathbf{e})_j\\
&= (\sum_{n=1}^{\infty}\sum_{k=n}^{\infty} b_n\mathbf{D}(1)^n\mathbf{e})_j = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} b_n = E[B],
\end{aligned}
$$

and $g_{0j}(1) = 1$. In the last equation, we use $\sum_j X_j(1) = 1$ and (19).

3.3. Tail asymptotic

In this subsection, by combining the results in 3.1 and 3.2 we derive the tail behavior for our system. For doing this, let $Q(z)$ be the P.G.F. of the number of cells at the end of an arbitrary slot in the steady state, and q_k be the probability that there are k cells at an arbitrary slot in the steady state. Then, by investigating $\{X, J_s\}$ and $\{Y, J_t\}$ it is easy to show that

$$Q(z) = \frac{1}{pE[B]+1}\mathbf{X}(z)\mathbf{e} + \frac{pE[B]}{pE[B]+1}\mathbf{Y}(z)\mathbf{e}. \tag{23}$$

To examine the tail behavior of the queue length, we first investigate the behavior of $Q'(z)$ near $z = 1$. From (17), (22) and (23),

$$\begin{aligned}
&\lim_{z\to 1^-}(1-z)^{\sigma}\frac{d}{dz}Q(z)\\
&= \frac{1}{pE[B]+1}\lim_{z\to 1^-}(1-z)^{\sigma}\frac{d}{dz}\mathbf{X}(z)\mathbf{e}\\
&\quad + \frac{pE[B]}{pE[B]+1}\lim_{z\to 1^-}(1-z)^{\sigma}\frac{d}{dz}\mathbf{Y}(z)\mathbf{e}\\
&= \frac{1}{pE[B]+1}\lim_{z\to 1^-}(1-z)^{\sigma}\frac{d}{dz}\mathbf{X}(z)\mathbf{e}\\
&\quad + \frac{p}{pE[B]+1}\left\{E[B]\lim_{z\to 1^-}(1-z)^{\sigma}\frac{d}{dz}\mathbf{X}(z)\mathbf{e} + \rho_s^{1-\sigma}\frac{a\Gamma(\sigma)}{s-1}\right\}\\
&= \lim_{z\to 1^-}(1-z)^{\sigma}\frac{d}{dz}\mathbf{X}(z)\mathbf{e} + \frac{p}{pE[B]+1}\rho_s^{1-\sigma}\frac{a\Gamma(\sigma)}{s-1}\\
&= \frac{p(\rho_s)^{2-\sigma}}{1-\tilde{\rho}}\frac{a\Gamma(\sigma)}{s-1} + \frac{p}{pE[B]+1}(\rho_s)^{1-\sigma}\frac{a\Gamma(\sigma)}{s-1}\\
&= \frac{p}{pE[B]+1}\left\{\frac{(\rho_s)^{2-\sigma}}{1-\rho} + (\rho_s)^{1-\sigma}\right\}\frac{a\Gamma(\sigma)}{s-1}.
\end{aligned} \tag{24}$$

Let a constant ζ be defined by

$$\zeta = \frac{p}{pE[B]+1}\left\{\frac{(\rho_s)^{2-\sigma}}{1-\rho} + (\rho_s)^{1-\sigma}\right\}\frac{a\Gamma(\sigma)}{s-1}.$$

Then, by Theorem 5.1 we have

$$\sum_{k=1}^{n} kq_k \sim \frac{\zeta}{\Gamma(\sigma+1)}n^{\sigma},$$

from which, by using Lemma 5.2, we finally get

$$\sum_{k \geq n} q_k \sim \frac{\zeta}{\Gamma(\sigma+1)} \frac{\sigma}{1-\sigma} n^{\sigma-1}$$
$$= \frac{(3-s)\zeta}{(s-2)\Gamma(\sigma+1)} n^{2-s}.$$

Recalling $\sigma = 3 - s$, we have our main result.

THEOREM 3.4. *The asymptotic behavior of the tail probabilities of our system is*

$$\sum_{k \geq n} q_k \sim \frac{a}{(s-1)(s-2)} \frac{p}{pE[B]+1} \left\{ \frac{(\rho_s)^{2-\sigma}}{1-\rho} + (\rho_s)^{1-\sigma} \right\} n^{2-s}.$$

REMARK 1. Our result above shows that the correlation in the D-BMAP does not have any impact on the tail behavior of the buffer distribution. That is, our result is identical to that reported in [5] assuming i.i.d. batches (i.e., a batch renewal process).

4. Finite buffer system

In this section, we consider a system with a finite buffer of size K, and derive lower and upper bounds on the loss probability asymptotic as $K \to \infty$. Similar studies include Jelenkovic [13] who considered the loss probability asymptotic behavior of the GI/GI/1 queueing system under subexponentiality and later Jelenkovic and Momcilovic [14] studied the asymptotic loss probability in a finite buffer fluid queue with heterogeneous heavy tail ON and OFF processes.

The arrival processes are the same as given in Section 2. For the analysis, as in Section 3 we assume that high service priority is given to the Heavy Tail source, so that all arriving cells from D-BMAP are stored in the buffer, if possible, during ON periods of the Heavy Tail source plus the following slots. Note that this assumption does not have any impact on the loss probability of the system.

For a finite buffer system, let $\tilde{X}_n$ be the number of cells in the (finite) system at the n-th embedded point s_n introduced in subsection 3.1. Then the evolution of the queueing system $\{(\tilde{X}_n, J_{s_n})\}$ is defined by the following Lindley equation:

$$\tilde{X}_{n+1} = \min\{\max(\tilde{X}_n - 1, 0) + A_{n+1}, K\}.$$

To go further, we need the following theorem which was shown in [10] under the following two assumptions. Let $d_{ij}(z)$ be defined as in Section 2. We assume that

$$d_{ij}(0) = 0, \qquad 0 \le i \le M, j \ne M,$$

and that the total offered load ρ is less than 1. Note that, in [10] they assumed that $d_{ij}(z)$ depends only on j, but this is not needed in deriving the following theorem. The physical interpretation of the above assumption is that there is only one state, say M, for which the source transmits no cell.

Recall that $\mathbf{x}_l = (x_{l0}, \cdots, x_{lM}), x_{li} = P\{X = l, J_s = i\}$ for the infinite buffer system. We then have [10]:

THEOREM 4.1. *The loss probability $P_{loss}^{(K)}$ is given in terms of $\mathbf{x}_k$:*

$$P_{loss}^{(K)} = \frac{(1-\tilde{\rho})\sum_{k=K+1} \mathbf{x}_k \mathbf{A}_0 \mathbf{e}}{\tilde{\rho}\sum_{k=0}^{K} \mathbf{x}_k \mathbf{A}_0 \mathbf{e}}.$$

From now on we assume our system satisfies the above two assumptions. Then, observing that

$$\min_i (\mathbf{A}_0)_{iM}\mathbf{e} \le \mathbf{A}_0\mathbf{e} \le \max_i (\mathbf{A}_0)_{iM}\mathbf{e},$$

we have

$$\frac{1-\tilde{\rho}}{\tilde{\rho}} \frac{\min_i(\mathbf{A}_0)_{iM}}{\max_i(\mathbf{A}_0)_{iM}} \frac{\sum_{k=K+1}\mathbf{x}_k\mathbf{e}}{\sum_{k=0}^{K}\mathbf{x}_k\mathbf{e}} \le P_{loss}^{(K)} \le \frac{1-\tilde{\rho}}{\tilde{\rho}} \frac{\max_i(\mathbf{A}_0)_{iM}}{\min_i(\mathbf{A}_0)_{iM}} \frac{\sum_{k=K+1}\mathbf{x}_k\mathbf{e}}{\sum_{k=0}^{K}\mathbf{x}_k\mathbf{e}}. \tag{25}$$

In addition, by Lemma 3.2 we have

$$\begin{aligned} \frac{\sum_{k=K+1}\mathbf{x}_k\mathbf{e}}{\sum_{k=0}^{K}\mathbf{x}_k\mathbf{e}} &= \sum_{l=1}^{\infty}\left\{\sum_{k=K+1}\mathbf{x}_k\mathbf{e}\right\}^l \\ &= \frac{ap\rho_s^{s-1}}{(1-\rho_s(1+pE[B]))(s-1)(s-2)}K^{2-s} + o(K^{2-s}). \end{aligned} \tag{26}$$

From (25) and (26) we obtain

THEOREM 4.2. *The cell loss rate $P_{loss}^{(K)}$ satisfies*

$$\frac{\min_i(\mathbf{A}_0)_{iM}}{\max_i(\mathbf{A}_0)_{iM}} L \le \lim_{K\to\infty} \frac{P_{loss}^{(K)}}{K^{2-s}} \le \frac{\max_\mathbf{i}(\mathbf{A}_0)_{iM}}{\min_i(\mathbf{A}_0)_{iM}} L,$$

where a constant L is given by

$$L = \frac{1-\rho_s(1+pE[B])}{\rho_s(1+pE[B])} \frac{ap\rho_s^{s-1}}{(1-\rho_s(1+pEG))(s-1)(s-2)}.$$

REMARK 2. As a special case, if we have a renewal arrival process instead of the D-BMAP [5], Theorem 4.2 gives an exact asymptotic for the loss probabilities as the buffer size goes to ∞.

5. Conclusion

In this paper, we provided an exact asymptotic expression on the tail probabilities of a queue with an infinite buffer fed by a superposition of a Heavy Tail source and a D-BMAP. In this case, we showed that the tail behavior of the queue length distribution is equivalent to that of the same queueing system with the D-BMAP being replaced by a batch renewal process. The impact of the D-BMAP is depicted only through its mean.

We gave upper and lower bounds of the asymptotic of the loss probabilities for a queue with a finite buffer under the same traffic environment as the buffer size goes to ∞.

Appendix

A.1. Proof of Lemma 3.1

The equation (10) immediately follows from our assumption that $\lambda_0'(z)$ and $\lambda_0''(z)$ are finite and continuous in $[0,1]$.

Noting that

$$\begin{aligned}
&1-\tilde{\lambda_0}(z)-(1-z)\tilde{\lambda_0}'(z)\\
=\;&1-\tilde{\lambda_0}(z)-\{1-\lambda_0(z)\}B'(\lambda_0(z))\\
&+\{1-\lambda_0(z)-(1-z)\lambda_0'(z)\}B'(\lambda_0(z)),
\end{aligned}$$

we have

$$\begin{aligned}
&\lim_{z\to 1^-}(1-z)^\sigma\frac{1-\tilde{\lambda_0}(z)-(1-z)\tilde{\lambda_0}'(z)}{(1-z)^2}\\
&=\lim_{z\to 1^-}(1-z)^\sigma\left\{\frac{(1-\lambda_0(z))^2}{(1-z)^2}\frac{1-B(\lambda_0(z))-\{1-\lambda_0(z)\}B'(\lambda_0(z))}{(1-\lambda_0(z))^2}\right.\\
&\qquad\left.+\frac{1-\lambda_0(z)-(1-z)\lambda_0'(z)}{(1-z)^2}B'(\lambda_0(z))\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{z\to 1^-} \left\{ \left\{ \frac{1-\lambda_0(z)}{1-z} \right\}^{2-\sigma} (1-\lambda_0(z))^\sigma \right. \\
&\qquad \times \frac{1-B(\lambda_0(z)) - \{1-\lambda_0(z)\}B'(\lambda_0(z))}{(1-\lambda_0(z))^2} \\
&\qquad \left. + (1-z)^\sigma \frac{1-\lambda_0(z)-(1-z)\lambda_0'(z)}{(1-z)^2} B'(\lambda_0(z)) \right\} \\
&= (\lambda_0'(1))^{2-\sigma} \frac{a\Gamma(\sigma)}{s-1} \\
&= (\rho_s)^{2-\sigma} \frac{a\Gamma(\sigma)}{s-1}.
\end{aligned}
$$

Here, in the third equation we use (4), (10) and the Lebesgue Dominated Convergence Theorem, and in the fourth equation we use $\lambda_0'(1) = \rho_s$. The last equation completes the proof of (11).

Note that

$$
\begin{aligned}
&\lim_{z\to 1^-} (1-z)^\sigma \frac{1-t_0(z)-(1-z)t_0'(z)}{(1-z)^2} \\
&= \lim_{z\to 1^-} (1-z)^\sigma \left\{ p \frac{1-\tilde{\lambda}_0(z)\lambda_0(z) - (1-z)\frac{d}{dz}\{\tilde{\lambda}_0(z)\lambda_0(z)\}}{(1-z)^2} \right. \\
&\qquad \left. + (1-p)\frac{1-\lambda_0(z)-(1-z)\lambda_0'(z)}{(1-z)^2} \right\}
\end{aligned}
$$

and that

$$
\begin{aligned}
&\lim_{z\to 1^-} (1-z)^\sigma p \frac{1-\tilde{\lambda}_0(z)\lambda_0(z) - (1-z)\frac{d}{dz}\{\tilde{\lambda}_0(z)\lambda_0(z)\}}{(1-z)^2} \\
&= p \lim_{z\to 1^-} (1-z)^\sigma \left\{ \frac{(1-\tilde{\lambda}_0(z))(1-\lambda_0(z))}{(1-z)^2} \right. \\
&\qquad + \frac{1-\tilde{\lambda}_0(z)-(1-z)\tilde{\lambda}_0{}'(z)}{(1-z)^2}\lambda_0(z) \\
&\qquad \left. + \frac{1-\lambda_0(z)-(1-z)\lambda_0'(z)}{(1-z)^2}\tilde{\lambda}_0(z) \right\}, \\
&= p(\rho_s)^{2-\sigma}\frac{a\Gamma(\sigma)}{s-1} \qquad \text{by (10) and (11).}
\end{aligned}
$$

Then (12) immediately follows by combining the two results above.

From (11) and (12) we have

$$\begin{aligned}
& \lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}\left\{\frac{z-1}{z-t_0(z)}\right\} \\
= \; & \lim_{z\to 1^-}(1-z)^\sigma \frac{1-t_0(z)-(1-z)t_0'(z)}{(z-t_0(z))^2} \\
= \; & \lim_{z\to 1^-}(1-z)^\sigma \frac{(1-z)^2}{(z-t_0(z))^2}\frac{1-t_0(z)-(1-z)t_0'(z)}{(1-z)^2} \\
= \; & \frac{1}{(1-t_0'(1))^2}\lim_{z\to 1^-}(1-z)^\sigma \frac{1-t_0(z)-(1-z)t_0'(z)}{(1-z)^2} \\
= \; & \frac{p}{(1-t_0'(1))^2}(\rho_s)^{2-\sigma}\frac{a\Gamma(\sigma)}{s-1} \qquad \text{by (12)}
\end{aligned}$$

which completes the proof of (13).

When $i \neq 0$, from (2) we know

$$|\tilde{\lambda}_i(1)| \le \sum_{n=1}^{\infty} b_n|\lambda_i(1)|^n < 1, \text{ and } \quad |t_i(1)| < 1,$$

which give that

$$\lim_{z\to 1^-}\frac{d}{dz}\left\{\frac{z-1}{z-t_i(z)}\right\} = \frac{1}{1-t_i(1)},$$

and consequently we have (14):

$$\lim_{z\to 1^-}(1-z)^\sigma\frac{d}{dz}\left\{\frac{z-1}{z-t_i(z)}\right\} = 0.$$

A.2. Proof of Lemma 3.3

Observing that

$$\begin{aligned}
\tilde{\tilde{\lambda}}_i(z) &= \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} b_k\lambda_i(z)^n = \sum_{k=1}^{\infty}\sum_{n=1}^{k} b_k\lambda_i(z)^n \\
&= \sum_{k=1}^{\infty} b_k\lambda_i(z)\frac{1-\lambda_i(z)^k}{1-\lambda_i(z)} = \lambda_i(z)\frac{1-B(\lambda_i(z))}{1-\lambda_i(z)},
\end{aligned}$$

when $i = 0$, we have

$$\begin{aligned}
&\lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}\left\{\frac{1-B(\lambda_0(z))}{1-\lambda_0(z)}\right\}\\
&= \lim_{z\to 1^-}(1-z)^\sigma \lambda_0'(z)\frac{d}{d\lambda_0(z)}\left\{\frac{1-B(\lambda_0(z))}{1-\lambda_0(z)}\right\}\\
&= \lim_{z\to 1^-}\left(\frac{1-z}{1-\lambda_0(z)}\right)^\sigma \lambda_0'(z)(1-\lambda_0(z))^\sigma \frac{d}{d\lambda_0(z)}\left\{\frac{1-B(\lambda_0(z))}{1-\lambda_0(z)}\right\}\\
&= (\rho_s)^{1-\sigma}\frac{a\Gamma(\sigma)}{s-1},
\end{aligned}$$

so that

$$\begin{aligned}
&\lim_{z\to 1^-}(1-z)^\sigma \frac{d}{dz}\tilde{\tilde{\lambda}}_0(z)\\
&= \lim_{z\to 1^-}(1-z)^\sigma \left\{\lambda_0'(z)\frac{1-B(\lambda_0(z))}{1-\lambda_0(z)} + \lambda_0(z)\frac{d}{dz}\left\{\frac{1-B(\lambda_0(z))}{1-\lambda_0(z)}\right\}\right\}\\
&= \lim_{z\to 1^-}(1-z)^\sigma \lambda_0(z)\frac{d}{dz}\left\{\frac{1-B(\lambda_0(z))}{1-\lambda_0(z)}\right\}\\
&= (\rho_s)^{1-\sigma}\frac{a\Gamma(\sigma)}{s-1}.
\end{aligned}$$

When $i \neq 0$, from the fact that $\lim_{z\to 1^-} \tilde{\tilde{\lambda}}_i'(z)$ is finite, (20) immediately follows.

A.3. Tauberian theorem

Here, we describe the Tauberian Theorem for power series and the following lemma which play a central role in our analysis.

THEOREM 5.1. *Let $q_k \geq 0$ and suppose that $Q(z) = \sum_{k=1}^{\infty} q_k z^k$ converges for $0 \leq z < 1$. If $L(z)$ varies slowly at infinity and $0 \leq \sigma < \infty$, then the two relations are equivalent:*

$$Q(z) \sim \frac{1}{(1-z)^\sigma} L(\frac{1}{1-z}), \qquad z \to 1^-$$

and

$$q_0 + q_1 + \cdots + q_n \sim \frac{1}{\gamma(\sigma+1)} n^\sigma L(n), \qquad n \to \infty.$$

Proof. See [7], p.423. □

LEMMA 5.2. *For $0 < \gamma < \alpha$, if $\sum_{k=1}^{n} k^\alpha x_k \sim n^\gamma$, then $\sum x_k$ converges, and $\sum_{k\geq n} x_k \sim \gamma/(\alpha-\gamma) n^{\gamma-\alpha}$.*

Proof. See [22] 3.3 (c), p.59. □

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Gang Uk Hwang
Division of Applied Mathematics
Korea Advanced Institute of Science and Technology
Taejon, 305-701, Korea
E-mail: guhwang@amath.kaist.ac.kr

Khosrow Sohraby
Telecommunications Networking
University of Missouri - Kansas City
5100 Rockhill Road
Kansas City, MO 64110, U.S.A.
E-mail: sohrabyk@umkc.edu