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A GENERALIZED SEQUENTIAL OPERATOR-VALUED FUNCTION SPACE INTEGRAL

KUN SOO CHANG, BYOUNG SOO KIM, AND CHEONG HEE PARK

ABSTRACT. In this paper, we define a generalized sequential operator-valued function space integral by using a generalized Wiener measure. It is an extention of the sequential operator-valued function space integral introduced by Cameron and Storvick. We prove the existence of this integral for functionals which involve some product Borel measures.

1. Introduction

In 1968, Cameron and Storvick defined operator-valued function space integrals [3]; the analytic operator-valued function space integral and the sequential operator-valued function space integral. These integrals are based on the Wiener integral associated with the Wiener process. In [2, 5], a measure and an integral associated with a Gaussian Markov process were defined. In [4], we introduced a generalized Wiener measure associated with a Gaussian Markov process, and by using the generalized Wiener measure we defined a generalized analytic operator-valued function space integral as a bounded linear operator from $L_p(\mathbb{R})$ into $L_{p'}(\mathbb{R})$ for $1 < p \leq 2$.

In this paper, by using the generalized Wiener measure, we define a generalized sequential operator-valued function space integral. And we prove the existence of the integral for functionals which involve product Borel measures. Our results extend some results in [3].

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Let $f(t)$ be a real valued continuous function defined on $[a, b]$ and let u and v be functions satisfying the following conditions throughout this paper.

(1) There exist u' and v' which are continuous on $[a, b]$.
(2) $u'(t)v(t) - u(t)v'(t) > 0$ for $a \le t \le b$.
(3) $u \ge 0,\ v > 0$ on $[a, b]$.

In [4], we introduced *the generalized Wiener measure space* $(C_{f(a)}[a, b], \mathcal{S}, m^f_{u,v})$ associated with a Markov process with the mean function f and the covariance function u/v.

$$m^f_{u,v}(\{x \in C_{f(a)}[a,b] \,|\, (x(t_1), \cdots, x(t_n)) \in B\})$$
$$= \int_B \Big(\prod_{i=1}^n \frac{1}{\sqrt{2\pi A(t_{i-1}, t_i)}}$$
$$\cdot \exp\Big(- \sum_{i=1}^n \frac{(\xi_i - f(t_i) - \frac{v(t_i)}{v(t_{i-1})}(\xi_{i-1} - f(t_{i-1})))^2}{2A(t_{i-1}, t_i)}\Big) d\vec{\xi},$$

where $t_0 = a < t_1 < \cdots < t_n \le b$, $\xi_0 = f(a)$, B is a Borel set of $\mathbb{R}^n$ and $A(t_{i-1}, t_i) = \big(u(t_i) - \frac{u(t_{i-1})}{v(t_{i-1})} v(t_i)\big) v(t_i)$ for $i = 1, \cdots, n$.

Let us introduce some notations needed in this paper. $\mathbb{C}, \mathbb{C}^+$ and $\tilde{\mathbb{C}}^+$ are sets of all complex numbers, all complex numbers with positive real part and all nonzero complex numbers with nonnegative real part, respectively. $C[a, b]$ is the class of all continuous functions defined on $[a, b]$ and $C_\tau[a, b]$ is the class of all continuous functions x defined on $[a, b]$ with $x(0) = \tau$. For $1 \le p < \infty$, $L_p(\mathbb{R})$ is the space of complex valued Lebesgue measurable functions ψ on $\mathbb{R}$ such that $|\psi|^p$ is integrable with respect to Lebesgue measure on $\mathbb{R}$ and $\|\psi\|_p = (\int_{\mathbb{R}} |\psi(\xi)|^p d\xi)^{1/p}$. $L_\infty(\mathbb{R})$ is the space of essentially bounded functions ψ on $\mathbb{R}$ with the essential norm $\|\psi\|_\infty$. $\mathcal{L}(L_p(\mathbb{R}), L_{p'}(\mathbb{R}))$ is the space of bounded linear operators from $L_p(\mathbb{R})$ into $L_{p'}(\mathbb{R})$, where $1/p + 1/p' = 1$.

(1) Let $1 < p \le 2$. For $\lambda \in \tilde{\mathbb{C}}^+$, $\psi \in L_p(\mathbb{R})$ and $\xi \in \mathbb{R}$, let
$$(C_\lambda \psi)(\xi) = \int_{\mathbb{R}} \sqrt{\frac{\lambda}{2\pi}} \exp\{-\frac{\lambda(\eta - \xi)^2}{2}\} \psi(\eta) d\eta,$$
where we always choose $\lambda^{\frac{1}{2}}$ with nonnegative real part. Then $C_\lambda \psi \in L_{p'}(\mathbb{R})$ and $\|C_\lambda \psi\|_{p'} \le (\frac{|\lambda|}{2\pi})^{\frac{1}{2} - \frac{1}{p}} \|\psi\|_p$. Hence $C_\lambda \in \mathcal{L}(L_p(\mathbb{R}), L_{p'}(\mathbb{R}))$. And C_λ is analytic in $\mathbb{C}^+$ and strongly continuous on $\tilde{\mathbb{C}}^+$ as a function of λ. Here when $Re\lambda = 0$, the above integral should be interpreted as the limit in the mean just as in the theory of the L_p-Fourier transform [7].

(2) Let $1 \le p < \infty$. For $\psi \in L_p(\mathbb{R})$, a positive real number k and $\xi \in \mathbb{R}$, let $(S_k\psi)(\xi) = \psi(k\xi)$. Then $S_k\psi \in L_p(\mathbb{R})$ and
$$\|S_k\psi\|_p = \left(\int_{\mathbb{R}} |\psi(k\xi)|^p d\xi\right)^{\frac{1}{p}} = \left(\tfrac{1}{k}\right)^{\frac{1}{p}} \|\psi\|_p.$$
Hence $S_k \in \mathcal{L}(L_p(\mathbb{R}), L_p(\mathbb{R}))$ [4].

(3) Let $1 \le p < \infty$. For $\psi \in L_p(\mathbb{R})$, $k \in \mathbb{R}$ and $\xi \in \mathbb{R}$, let $(T_k\psi)(\xi) = \psi(\xi + k)$. Then $T_k\psi \in L_p(\mathbb{R})$ and
$$\|T_k\psi\|_p = \left(\int_{\mathbb{R}} |\psi(\xi + k)|^p d\xi\right)^{\frac{1}{p}} = \|\psi\|_p.$$
Hence $T_k \in \mathcal{L}(L_p(\mathbb{R}), L_p(\mathbb{R}))$ [4].

(4) For $\theta \in L_\infty(\mathbb{R})$ and $\psi \in L_2(\mathbb{R})$, let $(M_\theta\psi)(\xi) = \theta(\xi)\psi(\xi)$. Then $M_\theta\psi \in L_2(\mathbb{R})$ and $\|M_\theta\psi\|_2 \le \|\theta\|_\infty \|\psi\|_2$. Hence $M_\theta \in \mathcal{L}(L_2(\mathbb{R}), L_2(\mathbb{R}))$ [7].

Let $\psi \in L_2(\mathbb{R})$, $\theta \in L_\infty(\mathbb{R})$ and $A(s,t) = (u(t) - \frac{u(s)}{v(s)}v(t))v(t)$ for $a \le s < t \le b$. Then for k, $l \in \mathbb{R}$ and $\lambda \in \mathbb{C}^+$,

$$\begin{aligned}
&\left(\left(T_{-l} \circ S_{\frac{v(t)}{v(s)}} \circ C_{\frac{\lambda}{A(s,t)}} \circ T_k \circ M_\theta\right)\psi\right)(\xi) \\
\text{(1-1)}\quad &= \int_{\mathbb{R}} \sqrt{\frac{\lambda}{2\pi A(s,t)}} \exp\left\{-\frac{\lambda(\zeta - k - \frac{v(t)}{v(s)}(\xi - l))^2}{2A(s,t)}\right\}\psi(\zeta)\theta(\zeta)\,d\zeta
\end{aligned}$$

and

$$\|(T_{-l} \circ S_{\frac{v(t)}{v(s)}} \circ C_{\frac{\lambda}{A(s,t)}} \circ T_k \circ M_\theta)\psi\|_2 \le \left(\tfrac{v(s)}{v(t)}\right)^{1/2} \|\theta\|_\infty \, \|\psi\|_2.$$

Since $C_{\frac{\lambda}{A(s,t)}}$ is analytic in $\mathbb{C}^+$ as a function of λ, $(T_{-l} \circ S_{v(t)/v(s)} \circ C_{\lambda/A(s,t)} \circ T_k \circ M_\theta)\psi$ is analytic in $\mathbb{C}^+$ as a function of λ.

2. A generalized sequential operator-valued function space integral

Let σ be a partition of $[a,b]$ such that $\sigma : a = t_0 < t_1 < \cdots < t_n = b$ and let its norm $\|\sigma\| = \max_{1\le j\le n}(t_j - t_{j-1})$. For $(\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$ and $x \in C_{f(a)}[a,b]$, let

$$z(\sigma, \xi_0, \cdots, \xi_n, t) = \begin{cases} \xi_{j-1} & \text{if } t_{j-1} \le t < t_j \ (j = 1, \cdots, n), \\ \xi_n & \text{if } t = b, \end{cases}$$

$$x_\sigma(t) = \begin{cases} x(t_{j-1}) & \text{if } t_{j-1} \le t < t_j \ (j = 1, \cdots, n), \\ x(b) & \text{if } t = b. \end{cases}$$

Let $B[a,b]$ be the space of real valued functions which are continuous except for a finite number of jump discontinuous points on $[a,b]$.

DEFINITION 2.1. Let $F(x)$ be a complex valued function defined on $B[a,b]$. For any $\lambda \in \mathbb{C}^+$, $\psi \in L_2(\mathbb{R})$ and $\xi \in \mathbb{R}$, let

$$((I^{\sigma\lambda}_{u,v,f}F)\psi)(\xi) = \left(\prod_{i=1}^{n}\sqrt{\frac{\lambda}{2\pi A(t_{i-1},t_i)}}\right)\int_{\mathbb{R}^n} F(z(\sigma,\xi_0,\cdots,\xi_n,\cdot))$$
$$\cdot\,\psi(\xi_n)\exp\left\{-\sum_{i=1}^{n}\frac{\lambda(\xi_i - f(t_i) - \frac{v(t_i)}{v(t_{i-1})}(\xi_{i-1} - f(t_{i-1})))^2}{2A(t_{i-1},t_i)}\right\}d\vec{\xi},$$

where $\xi_0 = \xi + f(a)$. If $I^{\sigma\lambda}_{u,v,f}F$ is a bounded linear operator from $L_2(\mathbb{R})$ into $L_2(\mathbb{R})$ and there exists $w - \lim_{\|\sigma\|\to 0} I^{\sigma\lambda}_{u,v,f}F$, where "$w -$ lim" means the limit with respect to the weak operator topology, then we let $I^{seq\lambda}_{u,v,f}F = w - \lim_{\|\sigma\|\to 0} I^{\sigma\lambda}_{u,v,f}F$. If there exists the weak limit of $I^{seq_{p-iq}}_{u,v,f}F$ as p converges 0^+, then we call the limit the generalized sequential operator valued function space integral of F associated with $\lambda = -iq$ and $m^f_{u,v}$, and we denote it by $J^{seq_{-iq}}_{u,v,f}F$.

Let $F(x) = \exp\left\{\int_{[a,b]}(\int_{[a,b]}\theta(s,t,x(s),x(t))d\eta(s))d\beta(t)\right\}$ be defined on $B[a,b]$, where η and β are complex Borel measures on $[a,b]$ and $\theta(s,t,\zeta,\xi)$ is a complex valued Borel measurable function defined on $[a,b]^2\times\mathbb{R}^2$. We suppose that the measures η, β and the function $\theta(s,t,\zeta,\xi)$ satisfy the following conditions.

(2-1) $\|\theta\|_{\infty,1;\eta,\beta} := \int_{[a,b]}(\int_{[a,b]}\|\theta(s,t,\cdot,\cdot)\|_\infty d|\eta|(s))d|\beta|(t) < \infty$, where $|\eta|$ and $|\beta|$ are total variations of η and β respectively.

(2-2) $\theta(s,t,\zeta,\xi)$ is bounded on every compact subset of $[a,b]^2\times\mathbb{R}^2$ and is continuous for $|\eta|\times|\beta|\times l\times l - a.e.\,(s,t,\zeta,\xi) \in [a,b]^2\times\mathbb{R}^2$, where l is the Lebesgue measure on $\mathbb{R}$.

(2-3) $|\eta|\times|\beta|(\{\,(s,s)\mid s\in[a,b]\,\}) = 0$.

Throughout this section, for each $\lambda > 0$ let

$$h_\lambda(s,t,\zeta,\xi) = \lambda^{-1/2}(\zeta - f(t)) + \xi\tfrac{v(t)}{v(s)} + f(t).$$

LEMMA 2.2. *Let $\lambda > 0$ and let $\{\sigma_n\}_{n=1}^\infty$ be a sequence of partitions of $[a,b]$ such that $\|\sigma_n\| \to 0$ as $n\to\infty$. For $m^f_{u,v}\times l - a.e.(x,\xi) \in C_{f(a)}[a,b]\times\mathbb{R}$,*

$$\lim_{n\to\infty} F\big(\lambda^{-1/2}(x_{\sigma_n}(\cdot) - f_{\sigma_n}(\cdot)) + \xi\tfrac{v_{\sigma_n}(\cdot)}{v(a)} + f_{\sigma_n}(\cdot)\big)$$
$$= F\big(\lambda^{-1/2}(x(\cdot) - f(\cdot)) + \xi\tfrac{v(\cdot)}{v(a)} + f(\cdot)\big)$$

in the uniform topology on $B[a,b]$.

Proof. For each $\lambda > 0$, define

$$H_\lambda : [a,b]^2 \times C_{f(a)}[a,b] \times \mathbb{R} \to [a,b]^2 \times \mathbb{R}^2$$

by $H_\lambda(s,t,x,\xi) = \big(s,t,h_\lambda(a,s,x(s),\xi),h_\lambda(a,t,x(t),\xi)\big)$. Since H_λ is continuous, $(\theta \circ H_\lambda)(s,t,x,\xi)$ is a Borel measurable function. And $N = \{(s,t,\zeta,\xi) \in [a,b]^2 \times \mathbb{R}^2 \,|\, \theta$ is discontinuous at $(s,t,\zeta,\xi)\}$ is a $|\eta| \times |\beta| \times l \times l$- null set by (2-2).

Now let us show that $H_\lambda^{-1}(N)$ is a $|\eta| \times |\beta| \times m_{u,v}^f \times l$-null set. We consider the (s,t,ξ)-section of $H_\lambda^{-1}(N)$.

$$\begin{aligned}
&[H_\lambda^{-1}(N)]^{(s,t,\xi)}\\
&= \{x \in C_{f(a)}[a,b] \,|\, (s,t,x,\xi) \in H_\lambda^{-1}(N)\}\\
&= \big\{x \in C_{f(a)}[a,b] \,\big|\, \big(s,t,\ h_\lambda(a,s,x(s),\xi),\ h_\lambda(a,t,x(t),\xi)\,\big) \in N\big\}\\
&= \big\{x \in C_{f(a)}[a,b] \,\big|\, \big(\,h_\lambda(a,s,x(s),\xi),\ h_\lambda(a,t,x(t),\xi)\,\big) \in N^{(s,t)}\big\}\\
&= \big\{x \in C_{f(a)}[a,b] \,\big|\, \big(x(s),x(t)\big) \in M\,\big\},
\end{aligned}$$

where $M = \lambda^{\frac{1}{2}}\big[N^{(s,t)} - (\xi\frac{v(s)}{v(a)} + f(s), \xi\frac{v(t)}{v(a)} + f(t))\big] + (f(s),f(t))$. Since N is a $|\eta| \times |\beta| \times l \times l$-null set, M is $l \times l$-null set for $|\eta| \times |\beta| - a.e.\ (s,t) \in [a,b]^2$ and all $\xi \in \mathbb{R}$. By the condition (2-3) and the definition of the generalized Wiener measure, $m_{u,v}^f([H_\lambda^{-1}(N)]^{(s,t,\xi)}) = 0$ for $|\eta| \times |\beta| \times l - a.e.(s,t,\xi)$. By the Fubini theorem, $|\eta| \times |\beta| \times m_{u,v}^f \times l(H_\lambda^{-1}(N)) = 0$. Therefore $|\eta| \times |\beta|(\,(H_\lambda^{-1}(N))^{(x,\xi)}\,) = 0$ for $m_{u,v}^f \times l$-a.e.$(x,\xi) \in C_{f(a)}[a,b] \times \mathbb{R}$. Hence for $m_{u,v}^f \times l$-a.e. (x,ξ)

$$\begin{aligned}
&\lim_{n\to\infty} \int_{[a,b]} \int_{[a,b]} \theta\Big(s,t,\lambda^{-\frac{1}{2}}(x_{\sigma_n}(s) - f_{\sigma_n}(s)) + \xi\tfrac{v_{\sigma_n}(s)}{v(a)} + f_{\sigma_n}(s),\\
&\qquad\qquad \lambda^{-\frac{1}{2}}(x_{\sigma_n}(t) - f_{\sigma_n}(t)) + \xi\tfrac{v_{\sigma_n}(t)}{v(a)} + f_{\sigma_n}(t)\Big) d\eta(s)d\beta(t)\\
&= \int_{[a,b]} \int_{[a,b]} \theta\Big(s,t,h_\lambda(a,s,x(s),\xi),h_\lambda(a,t,x(t),\xi)\Big) d\eta(s)d\beta(t)
\end{aligned}$$

by the conditions (2-1), (2-2) and Dominated Convergence Theorem. This completes the proof. □

LEMMA 2.3 Let $\lambda > 0$ and let $\sigma; a = t_0 < t_1 < \cdots < t_n = b$. Then $F(\lambda^{-1/2}(x_\sigma(\cdot) - f_\sigma(\cdot)) + \xi\frac{v_\sigma(\cdot)}{v(a)} + f_\sigma(\cdot))$ is measurable as a function of (x,ξ), and for $m_{u,v}^f \times l$-a.e. (x,ξ)

$$\big|F\big(\lambda^{-1/2}(x_\sigma(\cdot) - f_\sigma(\cdot)) + \xi\tfrac{v_\sigma(\cdot)}{v(a)} + f_\sigma(\cdot)\big)\big| \le \exp\big\{\|\theta\|_{\infty,1;\eta,\beta}\big\}.$$

Proof. To show the measurability, consider the following equalities (2-4)

$$
\begin{aligned}
&F\big(\lambda^{-1/2}(x_\sigma(\cdot)-f_\sigma(\cdot))+\xi\tfrac{v_\sigma(\cdot)}{v(a)}+f_\sigma(\cdot)\big)\\
&=\exp\bigg\{\int_{[a,b]}\int_{[a,b]}\theta\Big(s,t,\lambda^{-\frac{1}{2}}(x_\sigma(s)-f_\sigma(s))+\xi\tfrac{v_\sigma(s)}{v(a)}+f_\sigma(s),\\
&\qquad\lambda^{-\frac{1}{2}}(x_\sigma(t)-f_\sigma(t))+\xi\tfrac{v_\sigma(t)}{v(a)}+f_\sigma(t)\Big)d\eta(s)d\beta(t)\bigg\}\\
&=\exp\bigg\{\sum_{i,j=1}^{n}\int_{[t_{j-1},t_j)}\int_{[t_{i-1},t_i)}\theta\Big(s,t,h_\lambda(a,t_{i-1},x(t_{i-1}),\xi),\\
&\qquad h_\lambda(a,t_{j-1},x(t_{j-1}),\xi)\Big)d\eta(s)d\beta(t)\\
&\quad+\sum_{i=1}^{n}\int_{\{b\}}\int_{[t_{i-1},t_i)}\theta\Big(s,b,h_\lambda(a,t_{i-1},x(t_{i-1}),\xi),\\
&\qquad h_\lambda(a,b,x(b),\xi)\Big)d\eta(s)d\beta(t)\\
&\quad+\sum_{j=1}^{n}\int_{[t_{j-1},t_j)}\int_{\{b\}}\theta\Big(b,t,h_\lambda(a,b,x(b),\xi),\\
&\qquad h_\lambda(a,t_{j-1},x(t_{j-1}),\xi)\Big)d\eta(s)d\beta(t)\\
&\quad+\int_{\{b\}}\int_{\{b\}}\theta\Big(b,b,h_\lambda(a,b,x(b),\xi),h_\lambda(a,b,x(b),\xi)\Big)d\eta(s)d\beta(t)\bigg\}.
\end{aligned}
$$

For each $1\le i\le n$ and $1\le j\le n$, define $H_{i,j}(s,t,x,\xi)$ by $H_{i,j}(s,t,x,\xi)$ $=\big(s,t,h_\lambda(a,t_{i-1},x(t_{i-1}),\xi),h_\lambda(a,t_{j-1},x(t_{j-1}),\xi)\big)$ on $[t_{i-1},t_i)\times[t_{j-1},t_j)\times C_{f(a)}[a,b]\times\mathbb{R}$. Since $H_{i,j}$ is a continuous function, $(\theta\circ H_{i,j})(s,t,x,\xi)$ is Borel measurable. And by the Fubini theorem,

$$
\begin{aligned}
&\int_{[t_{j-1},t_j)}\int_{[t_{i-1},t_i)}(\theta\circ H_{i,j})(s,t,x,\xi)d\eta(s)\,d\beta(t)\\
&=\int_{[t_{j-1},t_j)}\int_{[t_{i-1},t_i)}\theta\Big(s,t,h_\lambda(a,t_{i-1},x(t_{i-1}),\xi),\\
&\qquad h_\lambda(a,t_{j-1},x(t_{j-1}),\xi)\Big)d\eta(s)d\beta(t)
\end{aligned}
$$

is measurable as a function of (x,ξ) on $C_{f(a)}[a,b]\times\mathbb{R}$. For the other integrals in the equation (2-4), the measurability is proved similarly. Hence the proof for the measurability is completed.

By the condition (2-1), $\|\theta(s,t,\cdot,\cdot)\|_\infty < \infty$ for $|\eta| \times |\beta| - a.e.(s,t)$. Therefore by the Fubini theorem, $|\theta(s,t,\zeta,\xi)| \le \|\theta(s,t,\cdot,\cdot)\|_\infty$ for almost all (s,t,ζ,ξ). If we let

$$N_{i,j} = \{\ (s,t,\zeta,\xi) \in [t_{i-1},t_i) \times [t_{j-1},t_j) \times \mathbb{R}^2 \mid |\theta(s,t,\zeta,\xi)| > \|\theta(s,t,\cdot,\cdot)\|_\infty\},$$

then by the similar method as in the proof of Lemma 2.2, we can show that $|\eta| \times |\beta| \times m^f_{u,v} \times l(H^{-1}_\lambda(N_{i,j})) = 0$. Hence for $m^f_{u,v} \times l$-a.e. (x,ξ),

$$|\theta(s,t,h_\lambda(x(t_{i-1}),\xi,a,t_{i-1}),h_\lambda(x(t_{j-1}),\xi,a,t_{j-1}))| \le \|\theta(s,t,\cdot,\cdot)\|_\infty$$

for $|\eta| \times |\beta|$-a.e. $(s,t) \in [t_{i-1},t_i) \times [t_{j-1},t_j)$.

$$\int_{[t_{j-1},t_j)} \int_{[t_{i-1},t_i)} \left| \theta\Big(s,t,h_\lambda(a,t_{i-1},x(t_{i-1}),\xi), h_\lambda(a,t_{j-1},x(t_{j-1}),\xi)\Big) \right| d|\eta|(s)d|\beta|(t)$$

$$\le \int_{[t_{j-1},t_j)} \int_{[t_{i-1},t_i)} \|\theta(s,t,\cdot,\cdot)\|\, d|\eta|(s)d|\beta|(t).$$

If we apply this process for the other integrals in the equation (2-4), then we obtain

$$\left|F\big(\lambda^{-1/2}(x_\sigma(\cdot) - f_\sigma(\cdot)) + \xi\tfrac{v_\sigma(\cdot)}{v(a)} + f_\sigma(\cdot)\big)\right| \le \exp\left\{\|\theta\|_{\infty,1;\eta,\beta}\right\}.$$

□

THEOREM 2.4. *Let g and h be complex valued functions defined on $[a,b] \times [a,b] \times \mathbb{R}$ such that $\int_{[a,b]}(\int_{[a,b]} \|g(s,t,\cdot)\|_\infty d|\eta|(s))d|\beta|(t) < \infty$ and $\int_{[a,b]}(\int_{[a,b]} \|h(s,t,\cdot)\|_\infty d|\eta|(s))d|\beta|(t) < \infty$, where η and β are complex Borel measures on $[a,b]$. Let $\theta(s,t,\zeta,\xi) = f(s,t,\zeta) + g(s,t,\xi)$ and $F(x) = \exp\{\int_{[a,b]}(\int_{[a,b]} \theta(s,t,x(s),x(t))d\eta(s))d\beta(t)\}$. If the measures η, β and the function $\theta(s,t,\zeta,\xi)$ satisfy the conditions (2-2) and (2-3), then for any partition $\sigma; \{t_i\}_{i=0}^n$ of $[a,b]$ and $\psi \in L_2(\mathbb{R})$, $I^{\sigma\lambda}_{u,v,f}F$ is analytic in $\mathbb{C}^+$ as a function of λ and*

$$\|(I^{\sigma\lambda}_{u,v,f}F)\psi\|_2 \le \sqrt{\tfrac{v(a)}{v(b)}}\|\psi\|_2 \exp\left\{ \int_{[a,b]} \int_{[a,b]} (\ \|g(s,t,\cdot)\|_\infty + \|h(s,t,\cdot)\|_\infty\ )d|\eta|(s)\, d|\beta|(t) \right\}.$$

Proof. For $\lambda \in \mathbb{C}^+$, $\vec{t} = (t_0, \cdots, t_n)$, $\xi_0 = \xi + f(a)$ and $\vec{\xi} = (\xi_1, \xi_n) \in \mathbb{R}^n$, let

$$w_n(\lambda, \vec{t}, \vec{\xi}) = \left(\prod_{i=1}^{n} \sqrt{\frac{\lambda}{2\pi A(t_{i-1}, t_i)}} \right) \cdot \exp\left\{ -\sum_{i=1}^{n} \frac{\lambda(\xi_i - f(t_i) - \frac{v(t_i)}{v(t_{i-1})}(\xi_{i-1} - f(t_{i-1})))^2}{2A(t_{i-1}, t_i)} \right\}.$$

Then

$$((I_{u,v,f}^{\sigma_\lambda} F)\psi)(\xi) = \int_{\mathbb{R}^n} F(z(\sigma, \xi_0, \xi_1, \cdots, \xi_n, \cdot))\psi(\xi_n) w_n(\lambda, \vec{t}, \vec{\xi}) d\vec{\xi},$$

where

$$\begin{aligned}
&F\big(z(\sigma, \xi_0, \xi_1, \cdots, \xi_n, \cdot)\big) \\
&= \exp\Big\{ \int_{[a,b]} \Big(\int_{[a,b]} g\big(s, t, z(\sigma, \xi_0, \xi_1, \cdots, \xi_n, s)\big)\, d\eta(s) \Big) d\beta(t) \\
&\qquad + \int_{[a,b]} \Big(\int_{[a,b]} h\big(s, t, z(\sigma, \xi_0, \xi_1, \cdots, \xi_n, t)\big)\, d\eta(s) \Big) d\beta(t) \Big\} \\
&= \exp\Big\{ \sum_{i=1}^{n} \int_{[a,b]} \Big(\int_{[t_{i-1}, t_i)} \dot{g}(s, t, \xi_{i-1}) d\eta(s) \Big) d\beta(t) \\
&\qquad + \int_{[a,b]} \Big(\int_{\{b\}} g(s, t, \xi_n) d\eta(s) \Big) d\beta(t) \\
&\qquad + \sum_{j=1}^{n} \int_{[t_{j-1}, t_j)} \Big(\int_{[a,b]} h(s, t, \xi_{j-1}) d\eta(s) \Big) d\beta(t) \\
&\qquad + \int_{\{b\}} \Big(\int_{[a,b]} h(s, t, \xi_n) d\eta(s) \Big) d\beta(t).
\end{aligned}$$

Let

$$\phi_n(\xi_n) = \exp\Big\{ \int_{[a,b]} \Big(\int_{\{b\}} g(s, t, \xi_n) d\eta(s) \Big) d\beta(t) + \int_{\{b\}} \Big(\int_{[a,b]} h(s, t, \xi_n) d\eta(s) \Big) d\beta(t) \Big\},$$

$$g_\lambda^1(\xi_{n-1}) = \int_{\mathbb{R}} \sqrt{\tfrac{\lambda}{2\pi A(t_{n-1},t_n)}}\psi(\xi_n)\phi_n(\xi_n)$$
$$\cdot \exp\left\{-\frac{\lambda(\xi_n - f(t_n) - \frac{v(t_n)}{v(t_{n-1})}(\xi_{n-1} - f(t_{n-1})))^2}{2A(t_{n-1},t_n)}\right\}d\xi_n.$$

Then by (1-1), $g_\lambda^1 = (T_{-f(t_{n-1})} \circ S_{v(t_n)/v(t_{n-1})} \circ C_{\lambda/A(t_{n-1},t_n)} \circ T_{f(t_n)} \circ M_{\phi_n})\psi$ and g_λ^1 is analytic in $\mathbb{C}^+$ as a function of λ. Since

$$|\phi_n(\xi_n)| \le \exp\left\{\int_{[a,b]}\left(\int_{\{b\}}\|g(s,t,\cdot)\|_\infty d|\eta|(s)\right)d|\beta|(t) + \int_{\{b\}}\left(\int_{[a,b)}\|h(s,t,\cdot)\|_\infty d|\eta|(s)\right)d|\beta|(t)\right\},$$

$$\|g_\lambda^1\|_2 \le \sqrt{\tfrac{v(t_{n-1})}{v(t_n)}}\|\psi\|_2 \cdot \exp\left\{\int_{[a,b]}\left(\int_{\{b\}}\|g(s,t,\cdot)\|_\infty d|\eta|(s)\right)d|\beta|(t) + \int_{\{b\}}\left(\int_{[a,b]}\|h(s,t,\cdot)\|_\infty d|\eta|(s)\right)d|\beta|(t)\right\}.$$

For each $1 \le k \le n$, let

$$\phi_{k-1}(\xi_{k-1}) = \exp\left\{\int_{[a,b]}\left(\int_{[t_{k-1},t_k)} g(s,t,\xi_{k-1})d\eta(s)\right)d\beta(t) + \int_{[t_{k-1},t_k)}\left(\int_{[a,b]} h(s,t,\xi_{k-1})d\eta(s)\right)d\beta(t)\right\}.$$

By the induction on k from 2 to n, let

$$g_\lambda^k(\xi_{n-k}) = \int_{\mathbb{R}} \sqrt{\frac{\lambda}{2\pi A(t_{n-k},t_{n-k+1})}}$$
$$\cdot \exp\left\{-\frac{\lambda(\xi_{n-k+1} - f(t_{n-k+1}) - \frac{v(t_{n-k+1})}{v(t_{n-k})}(\xi_{n-k} - f(t_{n-k})))^2}{2A(t_{n-k},t_{n-k+1})}\right\}$$
$$\cdot g_\lambda^{k-1}(\xi_{n-k+1})\,\phi_{n-k+1}(\xi_{n-k+1})\,d\xi_{n-k+1}.$$

Then $g_\lambda^k = (T_{-f(t_{n-k})} \circ S_{v(t_{n-k+1})/v(t_{n-k})} \circ C_{\lambda/A(t_{n-k},t_{n-k+1})} \circ T_{f(t_{n-k+1})} \circ M_{\phi_{n-k+1}}) g_\lambda^{k-1}$ by (1-1) and g_λ^k is analytic in $\mathbb{C}^+$ as a function of λ.

$$\begin{aligned}\|g_\lambda^k\|_2 \leq & \sqrt{\frac{v(t_{n-k})}{v(t_{n-k+1})}} \|g_\lambda^{k-1}\|_2 \, \|\phi_{n-k+1}\|_\infty \\ & \cdot \exp\left\{ \int_{[a,b]} \left(\int_{[t_{n-k+1}, t_{n-k+2})} \|g(s,t,\cdot)\|_\infty d|\eta|(s) \right) d|\beta|(t) \right. \\ & \left. + \int_{[t_{n-k+1}, t_{n-k+2})} \left(\int_{[a,b]} \|h(s,t,\cdot)\|_\infty d|\eta|(s) \right) d|\beta|(t) \right\}.\end{aligned}$$

Hence by the induction on k from 2 to n

$$\begin{aligned}\|g_\lambda^n\|_2 \leq & \sqrt{\frac{v(a)}{v(t_1)}} \|g_\lambda^{n-1}\|_2 \exp\left\{ \int_{[a,b]} \left(\int_{[t_1,t_2)} \|g(s,t,\cdot)\|_\infty \right.\right. \\ & \left.\left. d|\eta|(s) \right) d|\beta|(t) + \int_{[t_1,t_2)} \left(\int_{[a,b]} \|h(s,t,\cdot)\|_\infty d|\eta|(s) \right) d|\beta|(t) \right\} \\ \leq & \sqrt{\frac{v(a)}{v(b)}} \|\psi\|_2 \exp\left\{ \int_{[a,b]} \left(\int_{[t_1,b]} \|g(s,t,\cdot)\|_\infty d|\eta|(s) \right) d|\beta|(t) \right. \\ & \left. + \int_{[t_1,b]} \left(\int_{[a,b]} \|h(s,t,\cdot)\|_\infty d|\eta|(s) \right) d|\beta|(t) \right\},\end{aligned}$$

and g_λ^n is analytic in $\mathbb{C}^+$ as a function of λ.

$$\begin{aligned}((I_{u,v,f}^{\sigma_\lambda} F)\psi)(\xi) = g_\lambda^n(\xi_0) \exp & \left\{ \int_{[a,b]} \int_{[t_0,t_1)} g(s,t,\xi_0) d\eta(s) \, d\beta(t) \right. \\ & \left. + \int_{[t_0,t_1)} \int_{[a,b]} h(s,t,\xi_0) d\eta(s) \, d\beta(t) \right\}.\end{aligned}$$

Hence

$$\begin{aligned}\|(I_{u,v,f}^{\sigma_\lambda})\psi\|_2 \leq & \sqrt{\frac{v(a)}{v(b)}} \|\psi\|_2 \\ & \cdot \exp\left\{ \int_{[a,b]} \left(\int_{[a,b]} \|g(s,t,\cdot)\|_\infty + \|h(s,t,\cdot)\|_\infty d|\eta|(s) \right) d|\beta|(t) \right\}\end{aligned}$$

and $I_{u,v,f}^{\sigma_\lambda} F$ is analytic in $\mathbb{C}^+$ as a function of λ. □

THEOREM 2.5. *Let $\lambda > 0$ and let $\{\sigma_n\}_{n=1}^{\infty}$ be a sequence of partitions of $[a,b]$ such that $\|\sigma_n\| \to 0$ as $n \to \infty$. Then for the function F in Theorem 2.4, $w - \lim_{\|\sigma_n\|\to 0} I^{(\sigma_n)_\lambda}_{u,v,f}F$ exists and equals $I^{\lambda}_{u,v,f}F$. Here $w - \lim$ means the limit in the weak topology and*

$$((I^{\lambda}_{u,v,f}F)\psi)(\xi) = \int_{C_{f(a)}[a,b]} F\big(\lambda^{-\frac{1}{2}}(x(\cdot) - f(\cdot)) + \xi\tfrac{v(\cdot)}{v(a)} + f(\cdot)\big) \cdot \psi\big(\lambda^{-\frac{1}{2}}(x(b) - f(b)) + \xi\tfrac{v(b)}{v(a)} + f(b)\big)dm^{f}_{u,v}(x).$$

Proof. By the Wiener integration formula and change of variables,

$$\int_{C_{f(a)}[a,b]} F(\lambda^{-1/2}(x_{\sigma_n}(\cdot) - f_{\sigma_n}(\cdot)) + \xi\tfrac{v_{\sigma_n}(\cdot)}{v(a)} + f_{\sigma_n}(\cdot)) \cdot \psi(\lambda^{-1/2}(x(b) - f(b)) + \xi\tfrac{v(b)}{v(a)} + f(b))\, dm^{f}_{u,v}(x) = ((I^{(\sigma_n)_\lambda}_{u,v,f}F)\psi)(\xi).$$

Also by Lemma 2.2, Lemma 2.3 and Wiener integration formula,

$$\lim_{n\to\infty}\int_{C_{f(a)}[a,b]} F(\lambda^{-1/2}(x_{\sigma_n}(\cdot) - f_{\sigma_n}(\cdot)) + \xi\tfrac{v_{\sigma_n}(\cdot)}{v(a)} + f_{\sigma_n}(\cdot)) \cdot \psi(\lambda^{-1/2}(x(b) - f(b)) + \xi\tfrac{v(b)}{v(a)} + f(b))\, dm^{f}_{u,v}(x) = ((I^{\lambda}_{u,v,f}F)\psi)(\xi)$$

and $\|\,(I^{(\sigma_n)_\lambda}_{u,v,f}F)\psi)\,\| \le \exp\{\|\theta\|_{\infty,1;\eta,\beta}\}\, \|\psi\|_2$. Hence

$$\lim_{n\to\infty}((I^{(\sigma_n)_\lambda}_{u,v,f}F)\psi)(\xi) = ((I^{\lambda}_{u,v,f}F)\psi)(\xi).$$

We have the desired weak convergence by Theorem 13.44 in [6]. □

THEOREM 2.6. *Under the hypotheses of Theorem 2.4, there exists $I^{seq_\lambda}_{u,v,f}F$ as a bounded linear operator from $L_2(\mathbb{R})$ into $L_2(\mathbb{R})$ and*

$$\|(I^{seq_\lambda}_{u,v,f}F)\psi\|_2 \le \sqrt{\tfrac{v(a)}{v(b)}}\|\psi\|_2 \exp\left\{\int_{[a,b]}\left(\int_{[a,b]}\|g(s,t,\cdot)\|_\infty + \|h(s,t,\cdot)\|_\infty d|\eta|(s)\right)d|\beta|(t)\right\}.$$

Moreover, $I^{seq_\lambda}_{u,v,f}F$ is the analytic extention in $\mathbb{C}^+$ of $I^{\lambda}_{u,v,f}F$ and $I^{seq_\lambda}_{u,v,f}F = I^{an_\lambda}_{u,v,f}F$, where $I^{an_\lambda}_{u,v,f}F$ is the generalized analytic operator-valued function space integral introduced in [4].

Proof. Let $\{\sigma_n\}_{n=0}^{\infty}$ be any sequence of partitions of $[a,b]$ such that $\|\sigma_n\| \to 0$ as $n \to \infty$, and let $\psi \in L_2(\mathbb{R})$. By Theorem 2.5, $w-\lim_{n\to\infty} I_{u,v,f}^{(\sigma_n)\lambda} F = I_{u,v,f}^{\lambda} F$ for each $\lambda > 0$. And by Theorem 2.4, $I_{u,v,f}^{(\sigma_n)\lambda} F$ is analytic in $\mathbb{C}^+$ as a function of λ and

$$\|(I_{u,v,f}^{(\sigma_n)\lambda} F)\psi\|_2 \le \sqrt{\tfrac{v(a)}{v(b)}}\|\psi\|_2 \exp\left\{\int_{[a,b]}\left(\int_{[a,b]}\|g(s,t,\cdot)\|_\infty + \|h(s,t,\cdot)\|_\infty d|\eta|(s)\right)d|\beta|(t)\right\}.$$

Hence by Theorem 3 in [3], there exists a function $K_\lambda(F)$ such that

$$\lim_{n\to\infty}\langle (I_{u,v,f}^{(\sigma_n)\lambda} F)\psi, \phi\rangle = \langle K_\lambda(F)\psi,\phi\rangle$$

for each $\lambda \in \mathbb{C}^+$ and ψ, $\phi \in L_2(\mathbb{R})$. And $K_\lambda(F)\psi$ is analytic in $\mathbb{C}^+$. By the Riesz's Theorem [1], $K_\lambda(F) \in \mathcal{L}(L_2(\mathbb{R}), L_2(\mathbb{R}))$ and

$$\|K_\lambda(F)\psi\|_2 \le \sqrt{\tfrac{v(a)}{v(b)}}\|\psi\|_2 \exp\left\{\int_{[a,b]}\left(\int_{[a,b]}\|g(s,t,\cdot)\|_\infty + \|h(s,t,\cdot)\|_\infty d|\eta|(s)\right)d|\beta|(t)\right\}.$$

Hence $I_{u,v,f}^{seq\lambda} F = K_\lambda(F)$ for each λ in $\mathbb{C}^+$. Since $I_{u,v,f}^{seq\lambda} F$ is analytic in $\mathbb{C}^+$ and $I_{u,v,f}^{seq\lambda} F = I_{u,v,f}^{\lambda} F$ for each $\lambda > 0$, $I_{u,v,f}^{seq\lambda} F$ is analytic extention in $\mathbb{C}^+$ of $I_{u,v,f}^{\lambda} F$. □

THEOREM 2.7. *Under the hypothesis of Theorem 2.4, $J_{u,v,f}^{seq_{-iq}} F$ and $J_{u,v,f}^{an_{-iq}} F$ exist and $J_{u,v,f}^{seq_{-iq}} F = J_{u,v,f}^{an_{-iq}} F$ for almost all $q \neq 0$.*

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal sequence in $L_2(\mathbb{R})$. For each n and m, $\langle (I_{u,v,f}^{seq\cdot} F)e_n, e_m\rangle$ is analytic and bounded in $\mathbb{C}^+$ by Theorem 2.6. And $\lim_{p\to 0^+}\langle (I_{u,v,f}^{seq_{p-iq}} F)e_n, e_m\rangle$ exists for all q except a Lebesgue-null set $N_{n,m} \subset \mathbb{R}$ by an application of the Fatou's theorem for $\langle (_{u,v,f}^{seq\cdot} F)e_n, e_m\rangle$. Hence for all n, $m = 1, 2, \cdots$, $\lim_{p\to 0^+}\langle (I_{u,v,f}^{seq_{p-iq}} F)e_n, e_m\rangle$ exists for all q except a Lebesgue-null set $N = \cup_{n,m=1}^{\infty} N_{n,m}$. Hence for each ψ and $\phi \in L_2(\mathbb{R})$, $\lim_{p\to 0^+}\langle (I_{u,v,f}^{seq_{p-iq}} F)\psi, \phi\rangle$ exists for almost all q in $\mathbb{R}$. And

$$|\langle (I_{u,v,f}^{seq_{p-iq}} F)\psi,\phi\rangle| \le \|(I_{u,v,f}^{seq_{p-iq}} F)\psi\|_2\,\|\phi\|_2 \le \sqrt{\tfrac{v(a)}{v(b)}}\|\psi\|_2\,\|\phi\|_2 \cdot \exp\left\{\int_{[a,b]}\left(\int_{[a,b]}\|g(s,t,\cdot)\|_\infty + \|h(s,t,\cdot)\|_\infty d|\eta|(s)\right)d|\beta|(t)\right\}.$$

Therefore

$$|\lim_{p\to 0^+}\langle (I_{u,v,f}^{seq_{p-iq}}F)\psi,\phi\rangle| \le \sqrt{\tfrac{v(a)}{v(b)}}\exp\left\{\int_{[a,b]}\left(\int_{[a,b]}\|g(s,t,\cdot)\|_\infty + \|h(s,t,\cdot)\|_\infty d|\eta|(s)\right)d|\beta|(t)\right\}\|\psi\|_2\,\|\phi\|_2.$$

By the Riesz's theorem [1], there exists a bounded linear operator f_q from $L_2(\mathbb{R})$ into $L_2(\mathbb{R})$ such that

$$\lim_{p\to 0^+}\langle (I_{u,v,f}^{seq_{p-iq}}F)\psi,\phi\rangle = \langle f_q(\psi),\phi\rangle$$

and

$$\|f_q(\psi)\|_2 \le \sqrt{\tfrac{v(a)}{v(b)}}\exp\left\{\int_{[a,b]}\int_{[a,b]}\|g(s,t,\cdot)\|_\infty + \|h(s,t,\cdot)\|_\infty d|\eta|(s)d|\beta|(t)\right\}\|\psi\|_2.$$

Hence the generalized sequential operator-valued function space integral $J_{u,v,f}^{seq_{-iq}}F$ exists for almost all q in $\mathbb{R}$, and $J_{u,v,f}^{seq_{-iq}}F = f_q$. □

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Kun Soo Chang and Byoung Soo Kim
Department of Mathematics
Yonsei University
Seoul 120-749, Korea
E-mail: kunchang@yonsei.ac.kr
mathkbs@dreamwiz.com

Cheong Hee Park
Department of Computer Science
University of Minnesota
Minneapolis, MN 55455, U.S.A.
E-mail: chpark@cs.umn.edu