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CONJUGATE POINTS ON THE QUATERNIONIC HEISENBERG GROUP

CHANGRIM JANG, JUNKON KIM, YEONWOOK KIM, AND KEUN PARK

ABSTRACT. Let N be the quaternionic Heisenberg group equipped with a left-invariant metric. We characterize all the conjugate points along the geodesics on N.

1. Introduction

Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $\langle , \rangle$ and N be its unique simply connected 2-step nilpotent Lie group with the left invariant metric induced by $\langle , \rangle$ on $\mathcal{N}$. The center of $\mathcal{N}$ is denoted by $\mathcal{Z}$. Then $\mathcal{N}$ can be expressed as the direct sum of $\mathcal{Z}$ and its orthogonal complement $\mathcal{Z}^{\perp}$.

For $Z \in \mathcal{Z}$, a skew symmetric linear transformation $j(Z) : \mathcal{Z}^{\perp} \to \mathcal{Z}^{\perp}$ is defined by $j(Z)X = (\mathrm{ad}X)^* Z$ for $X \in \mathcal{Z}^{\perp}$. Or, equivalently,

$$\langle j(Z)X, Y \rangle = \langle [X, Y], Z \rangle$$

for $X, Y \in \mathcal{Z}^{\perp}$. A 2-step nilpotent Lie group N is said to be *of Heisenberg type* if

$$j(Z)^2 = -|Z|^2 \,\mathrm{id}$$

for all $Z \in \mathcal{Z}$.

The classical Heisenberg groups are examples of Heisenberg type. That is, let $n \geq 1$ be any integer and let $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ be any basis of $R^{2n} = \mathcal{V}$. Let $\mathcal{Z}$ be an 1-dimensional vector space spanned by $\{Z\}$. Define

$$[X_i, Y_i] = -[Y_i, X_i] = Z$$

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for any $i = 1, 2, \cdots, n$ with all other brackets are zero. Give on $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ the inner product such that the vectors $\{X_i, Y_i, Z | i = 1, 2, \cdots, n\}$ form an orthonormal basis. The simply connected 2-step nilpotent group of Heisenberg type, N which is determined by $\mathcal{N}$ and equipped with a left-invariant metric induced by the inner product in $\mathcal{N}$ is called the $(2n+1)$-dimensional Heisenberg group.

Another example is the quaternionic Heisenberg group of dimension $4n+3$. Let $\mathcal{N} = \mathcal{Z}^{\perp} \oplus \mathcal{Z}$ be a $(4n+3)$-dimensional real vector space with basis $\{X_i, Y_i, V_i, W_i | 1 \leq i \leq n\}$ of $\mathcal{Z}^{\perp}$ and $\{\xi_1, \xi_2, \xi_3\}$ of $\mathcal{Z}$. Define a Lie bracket on $\mathcal{N}$ as follows:

$$[X_i, Y_i] = \xi_1 = [V_i, W_i], \ [Y_i, X_i] = -\xi_1 = [W_i, V_i],$$
$$[X_i, V_i] = \xi_2 = [W_i, Y_i], \ [V_i, X_i] = -\xi_2 = [Y_i, W_i],$$
$$[X_i, W_i] = \xi_3 = [Y_i, V_i], \ [W_i, X_i] = -\xi_3 = [V_i, Y_i],$$

and all other brackets are zero. Define on $\mathcal{N}$ the inner product by giving that

$$\{X_i, Y_i, V_i, W_i, \xi_1, \xi_2, \xi_3 | 1 \leq i \leq n\}$$

forms an orthonormal basis. The simply connected 2-step nilpotent group of Heisenberg type, N which is determined by $\mathcal{N}$ and equipped with a left-invariant metric induced by the inner product in $\mathcal{N}$ is called the quaternionic Heisenberg group.

We have another definition of the quaternionic Heisenberg group. Let $\mathbb{H}$ be the quaternions, $\mathcal{Z}^{\perp} = \mathbb{H}^n$ and $\mathcal{Z}$ be the three dimensional vector space spanned by $\{i, j, k\}$. For two vectors $X = (x_1, x_2, \cdots, x_n)$ and $Y = (y_1, y_2, \cdots, y_n)$ in $\mathbb{H}^n$, define the Lie bracket of X and Y by

$$[X, Y] = Im(X \cdot Y) = Im(\sum_{i=1}^{n} x_i \bar{y_i}),$$

where Im is the imaginary part and $\bar{y_i}$ is the conjugate of y_i and all other brackets are zero. Denote 1_m the vector in $\mathbb{H}^n$, the m-the component of which is 1 and the others are 0 and denote i_m, j_m and k_m similarly. Define on $\mathcal{N} = \mathcal{Z}^{\perp} \oplus \mathcal{Z}$ the inner product by giving that

$$\{1_m, i_m, j_m, k_m, i, j, k | 1 \leq m \leq n\}$$

forms an orthonormal basis. Then, the simply connected 2-step nilpotent group N which is determined by $\mathcal{N}$ and equipped with a left-invariant metric induced by the inner product in $\mathcal{N}$ is the quaternionic Heisenberg group.

In [1], Berndt et al. obtained some conjugate points on the group of Heisenberg type with a left-invariant metric and characterized all the conjugate points on the $(2n+1)$-dimensional Heisenberg group.

THEOREM [1]. *Let N be a 2-step nilpotent group of Heisenberg type with a left invariant metric and $\mathcal{N}$ its Lie algebra. Let $\gamma(t)$ be an unit speed geodesic in N with $\gamma(0) = e$ (the identity element of N) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$.*

(1) *If $Z_0 = 0$, then there are no conjugate points along γ.*
(2) *If $X_0 = 0$, then the conjugate points along γ are at $t \in 2\pi\mathbb{Z}^*$.*
(3) *If $X_0 \neq 0$ and $Z_0 \neq 0$, then every $t \in \frac{2\pi}{|Z_0|}\mathbb{Z}^* \cup \mathbb{A}$ determines a conjugate point along γ where*

$$\mathbb{Z}^* = \{\pm 1, \pm 2, \cdots\}$$

and

$$\mathbb{A} = \left\{ t \in \mathbb{R} - \{0\} | (1 - |Z_0|^2)\frac{|Z_0|t}{2} = \tan\frac{|Z_0|t}{2} \right\}.$$

COROLLARY [1]. *Let N be the $(2n+1)$-dimensional Heisenberg group and $\mathcal{N}$ its Lie algebra. Let $\gamma(t)$ be an unit speed geodesic in N with $\gamma(0) = e$(the identity element of N) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$. If $X_0 \neq 0$ and $Z_0 \neq 0$, then all the conjugate points along γ are at $t \in \frac{2\pi}{|Z_0|}\mathbb{Z}^* \cup \mathbb{A}$ where*

$$\mathbb{Z}^* = \{\pm 1, \pm 2, \cdots\}$$

and

$$\mathbb{A} = \left\{ t \in \mathbb{R} - \{0\} | (1 - |Z_0|^2)\frac{|Z_0|t}{2} = \tan\frac{|Z_0|t}{2} \right\}.$$

In this paper, we characterize all the conjugate points on the quaternionic Heisenberg group. In fact, we show that some conjugate points in (3) of Theorem above are all the conjugate points in case of the quaternionic Heisenberg group.

MAIN THEOREM. *Let N be the quaternionic Heisenberg group and $\mathcal{N}$ its Lie algebra. Let $\gamma(t)$ be a unit speed geodesic in N with $\gamma(0) = e$(the identity element of N) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$.*

If $X_0 \neq 0$ and $Z_0 \neq 0$, then all the conjugate points along γ are at $t \in \frac{2\pi}{|Z_0|}\mathbb{Z}^* \cup \mathbb{A}$, where

$$\mathbb{Z}^* = \{\pm 1, \pm 2, \cdots\}$$

and

$$\mathbb{A} = \left\{ t \in \mathbb{R} - \{0\} | (1 - |Z_0|^2)\frac{|Z_0|t}{2} = \tan\frac{|Z_0|t}{2} \right\}.$$

2. Preliminaries

Let $\gamma(t)$ be a curve in N such that $\gamma(0) = e$ (identity element in N) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^\perp$ and $Z_0 \in \mathcal{Z}$. Since $\exp : \mathcal{N} \to N$ is a diffeomorphism ([5]), the curve $\gamma(t)$ can be expressed uniquely by $\gamma(t) = \exp(X(t) + Z(t))$ with

$$\begin{aligned} X(t) \in \mathcal{Z}^\perp, \quad X'(0) = X_0, \quad X(0) = 0, \\ Z(t) \in \mathcal{Z}, \quad Z'(0) = Z_0, \quad Z(0) = 0. \end{aligned}$$

A. Kaplan [3, 4] shows that the curve $\gamma(t)$ is a geodesic in N if and only if

$$\begin{aligned} X''(t) &= j(Z_0)X'(t), \\ Z'(t) + \frac{1}{2}[X'(t), X(t)] &\equiv Z_0. \end{aligned}$$

LEMMA 2.1 [2]. *Let N be a simply connected 2-step nilpotent Lie group with a left invariant metric, and let $\gamma(t)$ be a geodesic of N with $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^\perp$ and $Z_0 \in \mathcal{Z}$. Then*

$$\gamma'(t) = dl_{\gamma(t)}(X'(t) + Z_0), t \in R,$$

where $X'(t) = e^{tj(Z_0)}X_0$ and $l_{\gamma(t)}$ is the left translation by $\gamma(t)$.

Throughout this paper, different tangent spaces will be identified with $\mathcal{N}$ via left translation. So, in above lemma, we can consider $\gamma'(t)$ as

$$\gamma'(t) = X'(t) + Z_0 = e^{tj(Z_0)}X_0 + Z_0.$$

Let ∇ be unique Riemannian connection on N.

LEMMA 2.2 [2]. *For a 2-step nilpotent Lie group N with a left invariant metric, the following hold.*

(1) $\nabla_X Y = \frac{1}{2}[X,Y]$ *for* $X, Y \in \mathcal{Z}^\perp$.
(2) $\nabla_X Z = \nabla_Z X = -\frac{1}{2}j(Z)X$ *for* $X \in \mathcal{Z}^\perp$ *and* $Z \in \mathcal{Z}$.
(3) $\nabla_Z Z^* = 0$ *for* $Z, Z^* \in \mathcal{Z}$.

The curvature tensor R on $\mathcal{N}$ is defined by

$$R(\xi_1,\xi_2)\xi_3 = \nabla_{\xi_2}(\nabla_{\xi_1}\xi_3) - \nabla_{\xi_1}(\nabla_{\xi_2}\xi_3) + \nabla_{[\xi_1,\xi_2]}\xi_3$$

for all $\xi_1, \xi_2, \xi_3 \in \mathcal{N}$. And recall that the Jacobi operator along a geodesic γ is defined by

$$R_{\gamma'(t)}(\cdot) := R(\gamma'(t),\cdot)\gamma'(t).$$

Using Lemma 2.2, it is easy to show the following lemma.

LEMMA 2.3 [1]. *If N is a group of Heisenberg type, then the Jacobi operator $R_{\gamma'(t)}$ is given by*

$$\begin{aligned} &R_{\gamma'(t)}(X+Z) \\ =&\frac{3}{4}j([X,X'(t)])X'(t) + \frac{3}{4}j(Z)j(Z_0)X'(t) \\ &+ \frac{1}{4}|Z_0|^2 X + \frac{1}{2}\langle Z, Z_0\rangle X'(t) \\ &- \frac{3}{4}[X, j(Z_0)X'(t)] + \frac{1}{4}|X'(t)|^2 Z + \frac{1}{2}\langle X, X'(t)\rangle Z_0, \end{aligned}$$

where $X \in \mathcal{Z}^\perp$ *and* $Z \in \mathcal{Z}$.

The following Lemma is the elementary but useful properties of groups of Heisenberg type.

LEMMA 2.4 ([2]). *If N is a group of Heisenberg type with a left-invariant metric, then the following hold.*

(1) $\langle j(Z)X, j(Z^*)X\rangle = \langle Z, Z^*\rangle |X|^2$ *for all* $Z, Z^* \in \mathcal{Z}$ *and all* $X \in \mathcal{Z}^\perp$.
(2) $\langle j(Z)X, j(Z)Y\rangle = |Z|^2\langle X, Y\rangle$ *for all* $Z \in \mathcal{Z}$ *and all* $X, Y \in \mathcal{Z}^\perp$.
(3) $j(Z)j(Z^*) + j(Z^*)j(Z) = -2\langle Z, Z^*\rangle\,\mathrm{id}$ *for all* $Z, Z^* \in \mathcal{Z}$.
(4) $[X, j(Z)X] = |X|^2 Z$ *for all* $X \in \mathcal{Z}^\perp$ *and all* $Z \in \mathcal{Z}$.

3. Proof of theorem

Assume that $X_0 \neq 0$ and $Z_0 \neq 0$ and we start with the following Lemma:

LEMMA 3.1. *If Z_1 is orthogonal to Z_0 in $\mathcal{Z}$, then there is an orthogonal basis $\{Z_0, Z_1, Z_2\}$ of $\mathcal{Z}$ such that $j(Z_2) = j(Z_1)j(Z_0)$.*

Proof. We may assume that Z_0 and Z_1 are orthonormal. From the definition of the quaternionic Heisenberg groups, it is easy to show that

$$j(\xi_1)j(\xi_2) = j(\xi_3), \ j(\xi_2)j(\xi_3) = j(\xi_1) \text{ and } j(\xi_3)j(\xi_1) = j(\xi_2).$$

Let $Z_2 = Z_1 \times Z_0$ and $A \in O(3)$ such that

$$A \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} Z_1 \\ Z_0 \\ Z_2 \end{pmatrix}.$$

Then, it is just a calculation to show that $j(Z_2) = j(Z_1)j(Z_0)$. $\square$

By Lemma 3.1, we choose an orthogonal basis $\{Z_0, Z_1, Z_2\}$ such that $j(Z_2) = j(Z_1)j(Z_0)$ and denote

$$\mathcal{A}_0 = \left\{X_0 + Z_0, X_0 - \frac{|X_0|^2}{|Z_0|^2} Z_0, j(Z_0)X_0\right\}$$

and

$$\mathcal{A}_1 = \{Z_1, Z_2, j(Z_1)X_0, j(Z_2)X_0\}.$$

Then, the orthogonal complement of

$$Span\{X_0, j(Z_0)X_0, j(Z_1)X_0, j(Z_2)X_0\}$$

in $\mathcal{Z}^\perp$ has an orthogonal basis of the form

$$\mathcal{A}_2 = \cup_{k=1}^{2n-2}\{Y_k, j(Z_0)Y_k\}.$$

Hence, we have a basis

$$\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$$

of $\mathcal{N} = \mathcal{Z} \oplus \mathcal{Z}^\perp$.

Next, replacing again X_0 and Y_k's by $e^{tj(Z_0)}X_0 = X'(t)$ and $e^{tj(Z_0)}Y_k$'s (which will be denoted by $Y_k(t)$'s) respectively, we have a frame along $\gamma(t)$.

LEMMA 3.2. *Let*

$$\mathcal{A}_0(t) = \left\{X'(t) + Z_0, X'(t) - \frac{|X_0|^2}{|Z_0|^2}Z_0, j(Z_0)X'(t)\right\},$$
$$\mathcal{A}_1(t) = \{Z_1, Z_2, j(Z_1)X'(t), j(Z_2)X'(t)\},$$
$$\mathcal{A}_2(t) = \cup_{k=1}^{2n-2}\{Y_k(t), j(Z_0)Y_k(t)\}.$$

Then, $\mathcal{A}_0(t) \cup \mathcal{A}_1(t) \cup \mathcal{A}_2(t)$ is a frame along $\gamma(t)$.

The calculations of the covariant derivatives of this frame along γ are easy.

LEMMA 3.3. *Let $\mathcal{A}_0(t)$, $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ be stated in Lemma 3.2. Then, the followings hold.*

(1) $\nabla_{\gamma'(t)}Z = -\frac{1}{2}j(Z)X'(t)$ *for each* $Z \in \mathcal{Z}$.

(2) $\nabla_{\gamma'(t)}j(Z_1)X'(t) = \frac{3}{2}j(Z_2)X'(t) + \frac{1}{2}|X_0|^2 Z_1,$

$\nabla_{\gamma'(t)}j(Z_2)X'(t) = -\frac{3}{2}|Z_0|^2 j(Z_1)X'(t) + \frac{1}{2}|X_0|^2 Z_2.$

(3) $\nabla_{\gamma'(t)}Y_k(t) = \frac{1}{2}j(Z_0)Y_k(t),$

$\nabla_{\gamma'(t)}j(Z_0)Y_k(t) = -\frac{1}{2}|Z_0|^2 Y_k(t)$ *for each* $k = 1, 2, \cdots, 2n-2$.

(4) $\nabla_{\gamma'(t)}(X'(t) + Z_0) = 0,$

$\nabla_{\gamma'(t)}(X'(t) - \frac{|X_0|^2}{|Z_0|^2}Z_0) = \frac{1}{2|Z_0|^2}j(Z_0)X'(t),$

$\nabla_{\gamma'(t)}j(Z_0)X'(t) = -\frac{1}{2}|Z_0|^2(X'(t) - \frac{|X_0|^2}{|Z_0|^2}Z_0).$

Recall that a vector field $J(t)$ along $\gamma(t)$ is said to be a Jacobi field if

$$(\nabla^2_{\gamma'(t)} + R_{\gamma'(t)})J(t) = 0$$

and a point $\gamma(t), t \neq 0$ is said to be a conjugate point to $\gamma(0)$ if there exists a nonzero Jacobi field J along γ such that $J(t) = J(0) = 0$. For each $k = 0, 1, 2$, a point $\gamma(t), t \neq 0$ will be called $\mathcal{A}_k(t)$-conjugate point to $\gamma(0)$ if there exists a nonzero Jacobi field $J \in Span\mathcal{A}_k(t)$ along γ such that $J(t) = J(0) = 0$. Then, by (1) of Lemma 3.4, Lemma 3.5 and

Lemma 3.6 below, we have that $\gamma(t), t \neq 0$ is a conjugate point to $\gamma(0)$ along γ if and only if $\gamma(t)$ is a $\mathcal{A}_k(t)$-conjugate point for some k. Hence by (2) of Lemma 3.4, Lemma 3.5 and Lemma 3.6 below, conjugate points to $\gamma(0)$ are at

$$t \in \frac{2\pi}{|Z_0|}\mathbb{Z}^* \cup \mathbb{A},$$

where

$$\mathbb{Z}^* = \{\pm 1, \pm 2, \cdots\}$$

and

$$\mathbb{A} = \{t \in \mathbb{R} - \{0\} | (1 - |Z_0|^2)\frac{|Z_0|t}{2} = \tan\frac{|Z_0|t}{2}\}.$$

To complete the proof, it is sufficient to prove Lemma 3.4, Lemma 3.5 and Lemma 3.6. We give only the proof of Lemma 3.5 because Berndt et. al. proved already Lemma 3.4 and Lemma 3.6 in [1].

Throughout this section, we mean ($\mathcal{A}_k(t)$-)*conjugate point* by ($\mathcal{A}_k(t)$)-*conjugate point to* $\gamma(0)$ and

$$\cos|Z_0|t, \ \sin|Z_0|t \text{ and } \frac{(1 + |Z_0|^2)|Z_0|}{|X_0|^2}t$$

are denoted by

$$C(t), \ S(t) \text{ and } P(t),$$

respectively.

LEMMA 3.4.

(1) *If $J(t) \in Span\mathcal{A}_0(t)$ is a vector field along γ, then*

$$(\nabla^2_{\gamma'(t)} + R_{\gamma'(t)})J(t) \in Span\mathcal{A}_0(t).$$

(2) *The $\mathcal{A}_0(t)$-conjugate points along γ are at $t \in \frac{2\pi}{|Z_0|}\mathbb{Z}^* \cup \mathbb{A}$.*

LEMMA 3.5.

(1) *If $J(t) \in Span\mathcal{A}_1(t)$ is a vector field along γ, then*

$$(\nabla^2_{\gamma'(t)} + R_{\gamma'(t)})J(t) \in Span\mathcal{A}_1(t).$$

(2) *The $\mathcal{A}_1(t)$-conjugate points along γ are at $t \in \frac{2\pi}{|Z_0|}\mathbb{Z}^*$.*

Proof. Let

$$J(t) = \alpha_1(t)Z_1 + \alpha_2(t)Z_2 + \beta_1(t)j(Z_1)X'(t) + \beta_2(t)j(Z_2)X'(t)$$

be a vector field in $Span\mathcal{A}_1(t)$. Then, using Lemma 2.3, Lemma 2.4 and Lemma 3.3 we have that

$$\begin{aligned}
&(\nabla^2_{\gamma'(t)} + R_{\gamma'(t)})J(t) \\
&= \{\alpha_1''(t) + |X_0|^2\beta_1'(t)\}Z_1 + \{\alpha_2''(t) + |X_0|^2\beta_2'(t)\}Z_2 \\
&\quad + \{\beta_1''(t) - \alpha_1'(t) - 3|Z_0|^2\beta_2'(t) - (1+|Z_0|^2)\beta_1(t)\}j(Z_1)X'(t) \\
&\qquad + \{\beta_2''(t) - \alpha_2'(t) + 3\beta_1'(t) - (1+|Z_0|^2)\beta_2(t)\}j(Z_2)X'(t),
\end{aligned}$$

which shows (1).

If $J(t)$ is a Jacobi field with $J(0) = 0$, then we have the following differential equations:

$$\begin{aligned}
&\alpha_1''(t) + |X_0|^2\beta_1'(t) = 0, \\
&\alpha_2''(t) + |X_0|^2\beta_2'(t) = 0, \\
&\beta_1''(t) - 3|Z_0|^2\beta_2'(t) - 2|Z_0|^2\beta_1(t) - (\alpha_1'(t) + |X_0|^2\beta_1(t)) = 0, \\
&\beta_2''(t) + 3\beta_1'(t) - 2|Z_0|^2\beta_2(t) - (\alpha_2'(t) + |X_0|^2\beta_2(t)) = 0 \\
&\qquad\qquad \text{with} \quad \alpha_1(0) = \alpha_2(0) = \beta_1(0) = \beta_2(0) = 0.
\end{aligned}$$

Denote

$$\mathbf{x}(t) = {}^t(\alpha_1(t), \alpha_2(t)), \mathbf{y}(t) = {}^t(\beta_1(t), \beta_2(t))$$

and

$$D = \frac{d}{dt}.$$

Then, these differential equations are written as follows:

$$\begin{aligned}
&D(D\mathbf{x} + |X_0|^2\mathbf{y}) = 0, \\
&(D-A)(D-2A)\mathbf{y} = D\mathbf{x} + |X_0|^2\mathbf{y} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{y}(0) = 0,
\end{aligned}$$

where

$$A = \begin{pmatrix} 0 & |Z_0|^2 \\ -1 & 0 \end{pmatrix}.$$

Letting

$$D\mathbf{x} + |X_0|^2\mathbf{y} = -2|Z_0|^2\mathbf{c}_3$$

with

$$\mathbf{c}_3 \in \mathbb{R}^2,$$

we see that

$$\mathbf{y}(t) = e^{tA}\mathbf{c}_1 + e^{2tA}\mathbf{c}_2 + \mathbf{c}_3.$$

The initial condition that $\mathbf{y}(0) = 0$ gives that $\mathbf{c}_3 = -\mathbf{c}_1 - \mathbf{c}_2$, and so, we have that

$$\mathbf{y}(t) = (e^{tA} - I)\mathbf{c}_1 + (e^{2tA} - I)\mathbf{c}_2,$$

where $\mathbf{c}_1$ and $\mathbf{c}_2$ are arbitrary constant vectors in $\mathbb{R}^2$.

And also, from

$$D\mathbf{x} + |X_0|^2\mathbf{y} = -2|Z_0|^2\mathbf{c}_3 = 2|Z_0|^2(\mathbf{c}_1 + \mathbf{c}_2),$$

we have that

$$D\mathbf{x} = (-|X_0|^2 e^{tA} + (1 + |Z_0|^2)I)\mathbf{c}_1 + (-|X_0|^2 e^{2tA} + (1 + |Z_0|^2)I)\mathbf{c}_2.$$

So, this equation with the initial condition $\mathbf{x}(0) = 0$ gives that

$$\begin{aligned} \mathbf{x}(t) = &\left\{ \frac{|X_0|^2}{|Z_0|^2} A(e^{tA} - I) + (1 + |Z_0|^2)tI \right\} \mathbf{c}_1 \\ &+ \left\{ \frac{|X_0|^2}{2|Z_0|^2} A(e^{2tA} - I) + (1 + |Z_0|^2)tI \right\} \mathbf{c}_2. \end{aligned}$$

Hence, the solutions of the differential equations are given as follows:

$$\begin{pmatrix} \mathbf{y}(t) \\ \mathbf{x}(t) \end{pmatrix} = G \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix},$$

where

$$G = \begin{pmatrix} e^{tA} - I & e^{2tA} - I \\ \frac{|X_0|^2}{|Z_0|^2} A(e^{tA} - I) + (1 + |Z_0|^2)tI & \frac{|X_0|^2}{2|Z_0|^2} A(e^{2tA} - I) + (1 + |Z_0|^2)tI \end{pmatrix}.$$

Note that $\gamma(t), t \neq 0$ is an $\mathcal{A}_1(t)$-conjugate point if and only if $\det G = 0$ at t. So, to compute $\det G$, denote

$$B = \frac{|X_0|^2}{|Z_0|^2} A(e^{tA} - I) + (1 + |Z_0|^2)tI$$

and

$$C = \frac{|X_0|^2}{2|Z_0|^2} A(e^{2tA} - I) + (1 + |Z_0|^2)tI.$$

Then

$$G = \begin{pmatrix} e^{tA} - I & e^{2tA} - I \\ B & C \end{pmatrix} = \begin{pmatrix} e^{tA} - I & 0 \\ B & I \end{pmatrix} \begin{pmatrix} I & e^{tA} + I \\ 0 & C - B(e^{tA} + I) \end{pmatrix}.$$

Thus we have that

$$\begin{aligned} \det G &= \det(e^{tA} - I) \det(C - B(e^{tA} + I)) \\ &= \det(e^{tA} - I) \det(e^{tA}) \det(\frac{|X_0|^2}{2|Z_0|^2} A(e^{tA} - e^{-tA}) + (1 + |Z_0|^2)tI). \end{aligned}$$

Using that

$$e^{tA} = C(t)I + \frac{S(t)}{|Z_0|} A = \begin{pmatrix} C(t) & |Z_0|S(t) \\ -\frac{S(t)}{|Z_0|} & C(t) \end{pmatrix},$$

it is easy to show that

$$\begin{aligned} &\det(e^{tA} - I) \det(e^{tA}) \det(\frac{|X_0|^2}{2|Z_0|^2} A(e^{tA} - e^{-tA}) + (1 + |Z_0|^2)tI) \\ &= \frac{2|X_0|^2}{|Z_0|}(1 - C(t))(P(t) - S(t)). \end{aligned}$$

Therefore, $\gamma(t), t \neq 0$ is an $\mathcal{A}_1(t)$-conjugate point if and only if

$$(1 - C(t))(P(t) - S(t)) = 0,$$

or equivalently, $t \in \frac{2\pi}{|Z_0|}\mathbb{Z}^*$ since $P(t) - S(t) = 0$ if and only if $t = 0$. This completes the proof of Lemma 3.5. □

LEMMA 3.6.

(1) *If $J(t) \in Span\,\mathcal{A}_2(t)$ is a vector field along γ, then*

$$(\nabla^2_{\gamma'(t)} + R_{\gamma'(t)})J(t) \in Span\,\mathcal{A}_2(t).$$

(2) *The $\mathcal{A}_2(t)$-conjugate points along γ are at $t \in \frac{2\pi}{|Z_0|}\mathbb{Z}^*$.*

References

[1] J. Berndt, F. Tricerri and L. Vanhecke, *Generalized Heisenberg groups Damek-Ricci harmonic extensions*, Lecture Notes in Mathematics **1598** (1995), 51–68.

[2] P. Eberlein, *Geometry of 2-step nilpotent groups with a left invariant metric*, Ann. Sci. Ecole Norm. Sup. **27** (1994), no. 5, 611–660.

[3] A. Kaplan, *Riemannian nilmanifolds attached to Clifford modules*, Geom. Dedicata **11** (1981), no. 2, 127–136.

[4] ———, *On the geometry of groups of Heisenberg type*, Bull. London Math. Soc. **15** (1983), no. 1, 35–42.

[5] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete **68**, Springer-Verlag, 1972.

Changrim Jang, Yeonwook Kim, and Keun Park
Department of Mathematics
University of Ulsan
Ulsan 680-749, Korea
E-mail: kpark@mail.ulsan.ac.kr

Junkon Kim
Hwamyung High School
Pusan 616-120, Korea