J. Korean Math. Soc. **40** (2003), No. 1, pp. 41–60

CONTINUITY FOR COMMUTATORS OF LITTLEWOOD-PALEY OPERATORS ON CERTAIN HARDY SPACES

Liu Lanzhe

Abstract. In this paper, the continuity for the commutators of Littlewood-Paley operators on certain Hardy and Herz-Hardy spaces are obtained.

1. Introduction and definitions

Let $b \in BMO(\mathbb{R}^n)$ and T be the Calderon-Zygmund operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

A classical result of Coifman, Rochberg and Weiss [3] proved that the commutator $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. However, it was observed that $[b,T]$ is not bounded, in general, from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ and from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ and from $H^{p,\infty}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ for $p \leq 1$. But, if $H^p(\mathbb{R}^n)$ is replaced by a suitable atomic space $H^p_b(\mathbb{R}^n)$ and $H^{p,\infty}(\mathbb{R}^n)$ by $H^{p,\infty}_b(\mathbb{R}^n)$ (see [1], [12]), then $[b,T]$ maps continuously $H^p_b(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ and $H^{p,\infty}_b(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$ for $p \in (n/(n+1), 1]$. In addition, we easily know that $H^p_b(\mathbb{R}^n) \subset H^p(\mathbb{R}^n)$ and $H^{p,\infty}_b(\mathbb{R}^n) \subset H^{p,\infty}(\mathbb{R}^n)$. The main purpose of this paper is to consider the continuity of the commutators related to the Littlewood-Paley operators and $BMO(\mathbb{R}^n)$ functions on certain Hardy and Herz-Hardy spaces. Let us first introduce some definitions (see [1], [4], [5], [6], [8], [9], [10], [11], [15]).

Received October 25, 2001. Revised April 10, 2002.
2000 Mathematics Subject Classification: 42B25, 42B20.
Key words and phrases: Littlewood-Paley operator, commutator, BMO($\mathbb{R}^n$), Hardy space, Herz-Hardy space.

DEFINITION 1. Let b be a locally integrable function and $0 < p \leq 1$. A bounded measurable function a on $\mathbb{R}^n$ is said to be a (p,b) atom if

i) $\operatorname{supp} a \subset B = B(x_0, r)$,

ii) $||a||_{L^\infty} \leq |B|^{-1/p}$,

iii) $\int a(y)dy = \int a(y)b(y)dy = 0$.

A temperate distribution f is said to belong to $H_b^p(\mathbb{R}^n)$, if, in the Schwartz distributional sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x),$$

where a_j's are (p,b) atoms, $\lambda_j \in C$ and $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. Moreover, we define

$$||f||_{H_b^p(\mathbb{R}^n)} = \inf(\sum_{j=1}^{\infty} |\lambda_j|^p)^{1/p},$$

where the infimum is taken over all the decompositions of f as above.

DEFINITION 2. Let b be a locally integrable function and $0 < p \leq 1$. A temperate distribution f is said to belong to the space $H_b^{p,\infty}(\mathbb{R}^n)$ if there exists a sequence of functions $\{f_k\}_{k=-\infty}^{\infty} \subset L^\infty(\mathbb{R}^n)$ such that

a) $f(x) = \sum_{k=-\infty}^{\infty} f_k(x)$ in the Schwartz distribution sense;

b) Each f_k can be decomposed into $f_k = \sum_{j=1}^{\infty} b_j^k$ in $L^\infty(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$, where b_j^k satisfies the following properties:

b)$_1$ Supp $b_j^k \subset B_j^k$, B_j^k's are the balls with

$$\sup_k \sum_{j=1}^{\infty} \chi_{B_j^k}(x) < \infty, \quad \sup_k 2^{kp} \sum_{j=1}^{\infty} |B_j^k| < \infty,$$

where, and in what follows, χ_E denotes the characteristic function of the set E,

b)$_2$ there exists a constant $C = C(n,p) > 0$ such that

$$||b_j^k||_{L^\infty} \leq C 2^k \quad \text{for every } k, j,$$

b)$_3$ $\int_{\mathbb{R}^n} b_j^k(x)dx = \int_{\mathbb{R}^n} b_j^k(x)b(x)dx = 0$.

The quasi-norm on the space $H_b^{p,\infty}(\mathbb{R}^n)$ is defined by

$$||f||^p_{H_b^{p,\infty}(\mathbb{R}^n)} = \inf_{\sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} b_j^k = f} \sup_{k \in \mathbb{Z}} 2^{kp} \sum_{j=1}^{\infty} |B_j^k|,$$

where the infimum is taken over all the decompositions of f as above.

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $A_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{A_k}, k \in \mathbb{Z}$.

DEFINITION 3. Let $0 < p,\ q < \infty$, $\alpha \in \mathbb{R}$

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p} = \{f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L_{loc}^q(\mathbb{R}^n) : ||f||_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}(\mathbb{R}^n)} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p + ||f\chi_{B_0}||_{L^q}^p\right]^{1/p}.$$

DEFINITION 4. Let $0 < p,\ q < \infty$, $\alpha \in \mathbb{R}$. For $k \in \mathbb{Z}$ and measurable function f on $\mathbb{R}^n$, let $m_k(\lambda, f) = |\{x \in A_k : |f(x)| > \lambda\}|$; for $k \in N$, let $\tilde{m}_k(\lambda, f) = m_k(\lambda, f)$ and $\tilde{m}_0(\lambda, f) = |\{x \in B_0 : |f(x)| > \lambda\}|$.

(1) The homogeneous weak Herz space is defined by

$$W\dot{K}_q^{\alpha,p} = \{f \in L_{loc}^q(\mathbb{R}^n) : ||f||_{W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$||f||_{W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} m_k(\lambda, f)^{p/q}\right]^{1/p}.$$

(2) The nonhomogeneous weak Herz space is defined by

$$WK_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L_{loc}^q(\mathbb{R}^n) : ||f||_{WK_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$||f||_{WK_q^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, f)^{p/q}\right]^{1/p}.$$

DEFINITION 5. Let $\alpha \in R$, $1 < q \le \infty$, $b \in L_{loc}(\mathbb{R}^n)$. A function $a(x)$ on $\mathbb{R}^n$ is called a central (α, q, b)-atom (or a central (α, q, b)-atom of restrict type), if

1) Supp $a \subset B(0, r)$ for some $r > 0$ (or for some $r \ge 1$),
2) $||a||_{L^q} \le |B(0, r)|^{-\alpha/n}$,
3) $\int a(x)dx = \int a(x)b(x)dx = 0$.

A temperate distribution f is said to belong to $H\dot{K}^{\alpha,p}_{q,b}(\mathbb{R}^n)$ (or $HK^{\alpha,p}_{q,b}$ $(\mathbb{R}^n)$), if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(\mathbb{R}^n)$ sense, where a_j is a central (α, q, b)-atom (or a central (α, q, b)-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$ (or $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$). Moreover

$$||f||_{H\dot{K}^{\alpha,p}_{q,b}} (or ||f||_{HK^{\alpha,p}_{q,b}}) = \inf(\sum_j |\lambda_j|^p)^{1/p},$$

where the infimum is taken over all the decompositions of f as above.

DEFINITION 6. Let $\varepsilon > 0$ and fix a function ψ satisfying the following properties:

(1) $\int \psi(x)dx = 0$,
(2) $|\psi(x)| \le C(1 + |x|)^{-(n+\varepsilon)}$,
(3) $|\psi(x + y) - \psi(x)| \le C|y|^{\varepsilon}(1 + |x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

Let b be a locally integrable function and $\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$. The commutator of the Littlewood-Paley operator is defined by

$$S_{\psi,b}(f)(x) = \left[\int_{\Gamma(x)} |F_{b,t}(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

where

$$F_{b,t}(f)(x, y) = \int_{\mathbb{R}^n} \psi_t(y - z) f(z)(b(x) - b(z))dz,$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. We also define that

$$S_{\psi}(f)(x) = \left(\int_{\Gamma(x)} |f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

which is the Littlewood-Paley operator (see [15]).

2. Theorems and proofs

THEOREM 1. *Let $b \in BMO(\mathbb{R}^n)$ and $1 \geq p > n/(n+\varepsilon)$. Then the commutator $S_{\psi,b}$ is bounded from $H_b^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

Proof. It suffices to show that there exists a constant $C > 0$ such that for every (p,b) atom a,

$$||S_{\psi,b}(a)||_{L^p} \leq C.$$

Let a be a (p,b) atom supported on a ball $B = B(x_0, r)$. We write

$$\begin{aligned}
&\int_{\mathbb{R}^n} [S_{\psi,b}(a)(x)]^p dx \\
&= \int_{|x-x_0|\leq 2r} [S_{\psi,b}(a)(x)]^p dx + \int_{|x-x_0|>2r} [S_{\psi,b}(a)(x)]^p dx \\
&\equiv I + II.
\end{aligned}$$

For I, taking $q > 1$, by Hölder's inequality and the L^q-boundedness of $S_{\psi,b}$ (see [2]), we see that

$$\begin{aligned}
I &\leq C||S_{\psi,b}(a)||_{L^q}^p \cdot |B(x_0, 2r)|^{1-p/q} \\
&\leq C||a||_{L^q}^p |B|^{1-p/q} \leq C.
\end{aligned}$$

For II, by Hölder's inequality, we have

$$\begin{aligned}
II &\leq C \sum_{k=1}^{\infty} \int_{2^{k+1}r \geq |x-x_0| > 2^k r} [S_{\psi,b}(a)(x)]^p dx \\
&\leq C \sum_{k=0}^{\infty} |B(x_0, 2^{k+1}r)|^{1-p} \left[\int_{2^{k+1}r \geq |x-x_0| > 2^k r} S_{\psi,b}(a)(x) dx \right]^p .
\end{aligned}$$

Since, by Definition 6(3) and the vanishing moment of a,

$$\begin{aligned}
&S_{\psi,b}(a)(x) \\
&\leq \left[\int_{\Gamma(x)} \left(\int_B |\psi_t(y-z) - \psi_t(y-x_0)||a(z)||b(x) - b(z)| dz \right)^2 \frac{dydt}{t^{n+1}} \right]^{1/2}
\end{aligned}$$

$$\leq C\left[\int_{\Gamma(x)}\left(\int_B t^{-n}|a(z)||b(x)-b(z)|\frac{(|x_0-z|/t)^\varepsilon}{(1+|x_0-y|/t)^{n+1+\varepsilon}}dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$

$$= C\left[\int_{\Gamma(x)}\left(\int_B \frac{|B|^{\varepsilon/n}|B|^{-1/p}t}{(t+|x_0-y|)^{n+1+\varepsilon}}|b(x)-b(z)|dz\right)^2\frac{dt}{t^{n+1}}\right]^{1/2}$$

$$\leq C|B|^{\varepsilon/n-1/p}$$

$$\times\left[\int_{\Gamma(x)}\left(\frac{t^{1-n}2^{2(n+1+\varepsilon)}}{(2t+2|x_0-y|)^{2(n+1+\varepsilon)}}\int_B|b(x)-b(z)|dz\right)^2 dydt\right]^{1/2}.$$

Let $b_0 = |B(x_0,r)|^{-1}\int_{B(x_0,r)} b(y)dy$. Notice that $2t+|x_0-y| > 2t+|x_0-x|-|x-y| > t+|x_0-x|$ when $|x-y|<t$, and

$$\int_0^\infty \frac{tdt}{(t+|x-x_0|^{2(n+1+\varepsilon)})} = C|x-x_0|^{-2(n+\varepsilon)},$$

then, we deduce

$$S_{\psi,b}(a)(x) \leq C|B|^{\varepsilon/n-1/p+1}$$

$$\Big[\int_{\Gamma(x)}\frac{t^{1-n}}{(2t+|x_0-y|)^{2(n+1+\varepsilon)}}$$

$$\times(|B|^{-1}\int_B(|b(x)-b_0|+|b_0-b(z)|)dz)^2\Big]dydt)^{1/2}$$

$$\leq C|B|^{\varepsilon/n-1/p+1}$$

$$\times\left[\int_{\Gamma(x)}(\int_B\frac{t^{1-n}}{(t+|x_0-x|)^{2(n+1+\varepsilon)}}(|b(x)-b_0|+||b||_{BMO}))^2)dydt\right]^{1/2}$$

$$\leq C|B|^{\varepsilon/n-1/p+1}(|b(x)-b_0|+||b||_{BMO})$$

$$\times\left(\int_0^\infty\frac{tdt}{(t+|x-x_0|)^{2(n+1+\varepsilon)}}\right)^{1/2}$$

$$= C|B|^{\varepsilon/n-1/p+1}(|b(x)-b_0|+||b||_{BMO})|x-x_0|^{-(n+\varepsilon)}.$$

Thus

$$II \le C\sum_{k=0}^{\infty}|B(x_0,2^{k+1}r)|^{1-p}|B|^{(\varepsilon/n-1/p+1)p}||b||^p_{BMO}$$
$$\left[\int_{2^{k+1}r\ge|x-x_0|>2^k r}|x-x_0|^{-(n+\varepsilon)}dx\right]^p$$
$$+C\sum_{k=0}^{\infty}|B(x_0,2^{k+1}r)|^{1-p}|B|^{(\varepsilon/n-1/p+1)p}$$
$$\left[\int_{2^{k+1}r\ge|x-x_0|>2^k r}|x-x_0|^{-(n+\varepsilon)}|b(x)-b_0|dx\right]^p$$
$$\equiv II_1+II_2.$$

For II_2, using the properties of $BMO(\mathbb{R}^n)$ (see [13]), we obtain

$$\begin{aligned}II_2 &\le C\sum_{k=0}^{\infty}|B(x_0,2^{k+1}r)|^{1-\frac{n+\varepsilon}{n}p}k^p||b||^p_{BMO}|B|^{(1+\varepsilon/n-1/p)p}\\ &\le C||b||^p_{BMO}\sum_{k=0}^{\infty}k^p2^{kn(1-(n+\varepsilon)p/n)}\\ &\le C||b||^p_{BMO}.\end{aligned}$$

For II_1, similar to the estimate of II_2, we obtain $II_1 \le C||b||^p_{BMO}$. Combining the estimates of II_1 with II_2, we gain

$$II \le C||b||^p_{BMO} \le C.$$

This finishes the proof of Theorem 1. □

THEOREM 2. *Let $b\in L^\infty(\mathbb{R}^n)$ and $p=n/(n+\varepsilon)$. Then $S_{\psi,b}$ is bounded from $H^p_b(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$.* To prove the theorem, we recall the following lemma (see [14]):

LEMMA. *Let $\{f_k\}$ be a sequence of measurable functions and $p\in(0,1)$, and assume that*

$$|\{x\in\mathbb{R}^n : |f_k(x)|>\lambda\}| \le c\lambda^{-p} \text{ for any } k \text{ and } \lambda>0.$$

Then, for every p-summable numerical sequence $\{C_k\}$ we have

$$\left|\left\{x\in\mathbb{R}^n : \left|\sum_k C_kf_k(x)\right|>\lambda\right\}\right| \le \frac{2-p}{1-p}\frac{C}{\lambda^p}\sum_k|C_k|^p.$$

Proof of Theorem 2. It suffices to show that there exists a constant $C > 0$ such that for each (p, b) atom a and any $\lambda > 0$, we have

$$\lambda^p |\{x \in \mathbb{R}^n : S_{\psi,b}(a)(x) > \lambda\}| \leq c||b||_{L^\infty}^p.$$

We write

$$\begin{aligned}
&\lambda^p |\{x \in \mathbb{R}^n : S_{\psi,b}(a)(x) > \lambda\}| \\
&\leq \lambda^p |\{x \in \mathbb{R}^n : |b(x)| S_\psi(a)(x) > \lambda/2\}| + \lambda^p |\{x \in \mathbb{R}^n : S_\psi(ab)(x) > \lambda/2\}| \\
&\equiv I + II.
\end{aligned}$$

For I, by $a \in H^p(\mathbb{R}^n)$ and the boundedness of S_ψ from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ $(0 < p \leq 1)$, we have

$$\begin{aligned}
I &\leq C\lambda^p |\{x \in \mathbb{R}^n : S_\psi(a)(x) > \lambda/(2||b||_{L^\infty})\}| \\
&\leq C||b||_{L^\infty}^p.
\end{aligned}$$

For II, noting that $ab/||b||_\infty$ is also a (p, ∞) atom in the space $H^p(\mathbb{R}^n)$, thus, we gain

$$II \leq \lambda^p |\{x \in \mathbb{R}^n : S_\psi(ab/||b||_{L^\infty})(x) > \lambda/(2||b||_{L^\infty})\}| \leq C||b||_{L^\infty}^p.$$

This finishes the proof of Theorem 2. □

THEOREM 3. *Let $b \in BMO(\mathbb{R}^n)$ and $1 \geq p > n/(n + \varepsilon)$. Then $S_{\psi,b}$ is bounded from $H_b^{p,\infty}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$.*

Proof. Given $f \in H_b^{p,\infty}(\mathbb{R}^n)$, let $f = \sum_{k=-\infty}^{\infty} f_k = \sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} b_j^k$ be an atomic decomposition of f as in Definition 2. By a limiting argument, it suffices to show that

$$\sup_{\lambda > 0} \lambda^p |\{x \in \mathbb{R}^n : S_{\psi,b}\left(\sum_{k=-N}^{N} f_k\right)(x) > \lambda\}| \leq CC_1$$

for every $N = 0, 1, 2, \cdots$, where $C_1 = \sup_{k \in \mathbb{Z}} 2^{kp} \sum_{j=1}^{\infty} |B_j^k|$. Given $\lambda > 0$, we take $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leq \lambda < 2^{k_0+1}$. Let

$$\sum_{k=-N}^{N} f_k = \sum_{k=-N}^{k_0} f_k + \sum_{k=k_0+1}^{N} f_k \equiv F_1 + F_2.$$

Note that

$$|F_1(x)| \le C \sum_{k=-N}^{k_0} 2^k \sum_{j=1}^{\infty} |b_j^k(x)| \le C \sum_{k=-N}^{k_0} 2^k \sum_{j=1}^{\infty} \chi_{B_j^k}(x)$$
$$\le C \sum_{k=-N}^{k_0} 2^k \chi_{\cup_{j=1}^{\infty} B_j^k}(x),$$

thus

$$||F_1||_{L^q} \le C \sum_{k=-N}^{k_0} 2^k \left| \bigcup_{j=1}^{\infty} B_j^k \right|^{1/q} \le C \sum_{k=-N}^{k_0} 2^k \left(\sum_{j=1}^{\infty} |B_j^k| \right)^{1/q}$$
$$\le C C_1^{1/q} \sum_{k=-N}^{k_0} 2^{k-kp/q} \le C C_1^{1/q} 2^{k_0-k_0p/q}$$
$$\le C C_1^{1/q} \lambda^{1-p/q} \quad \text{for any } 1 < q < \infty.$$

Since $S_{\psi,b}$ is bounded on $L^q(\mathbb{R}^n)$ for $1 < q < \infty$ (see [2]), we have

$$\lambda^p \left|\{x \in \mathbb{R}^n : S_{\psi,b}(F_1)(x) > \lambda/2\}\right| \le C\lambda^{p-q} ||F_1||_{L^q}^q \le CC_1.$$

For F_2, let $A_k B_j^k$ be the ball with the same center as B_j^k and A_k times the radius of B_j^k, where $A_k \ge 2^{n+1}$ is a positive number to be determined later. For brevity, let

$$B_{k_0,N} = \bigcup_{k_0+1 \le k \le N, j \ge 1} A_k B_j^k.$$

We write

$$\begin{aligned}
&\lambda^p|\{x \in \mathbb{R}^n : S_{\psi,b}(F_2)(x) > \lambda/2\}| \\
&= \lambda^p|\{x \in B_{k_0,N} : S_{\psi,b}(F_2)(x) > \lambda\}| + \lambda^p|\{x \notin B_{k_0,N} : S_{\psi,b}(F_2)(x) > \lambda\}| \\
&\equiv I + II.
\end{aligned}$$

Let $C_j^k = |B_j^k|^{-1} \int_{B_j^k} b(y)dy$; first, we have

$$\begin{aligned}
II &\le \lambda^p \left|\left\{x \notin B_{k_0,N} : |b(x) - C_j^k| S_\psi(F_2)(x) > \lambda/2\right\}\right| \\
&\quad + \lambda^p \left|\left\{x \notin B_{k_0,N} : S_\psi((b - C_j^k)F_2)(x) > \lambda/2\right\}\right| \\
&\equiv II_1 + II_2
\end{aligned}$$

and

$$II_1 \le C \int_{(B_{k_0,N})^c} |b(x) - C_j^k|^p (S_\psi(F_2)(x))^p dx$$
$$\le C \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} \int_{(A_k B_j^k)^c} |b(x) - C_j^k| (S_\psi(b_j^k)(x))^p dx.$$

Let us now fix j and k. Then

$$\int_{(A_k B_j^k)^c} |b(x) - C_j^k| (S_\psi(b_j^k)(x))^p dx$$
$$= \sum_{l=0}^{\infty} \int_{2^{l+1} A_k B_j^k \setminus 2^l A_k B_j^k} |b(x) - C_j^k|^p (S_\psi(b_j^k)(x))^p dx$$
$$\le \sum_{l=0}^{\infty} \left(\int_{2^{l+1} A_k B_j^k \setminus 2^l A_k B_j^k} (S_\psi(b_j^k)(x))^{p/(1-p)} dx \right)^{1-p}$$
$$\times \left(\int_{2^{l+1} A_k B_j^k \setminus 2^l A_k B_j^k} |b(x) - C_j^k| dx \right)^p$$

(where usual modification is made when $p = 1$)

$$\equiv L;$$

let $B_l = |2^{l+1} A_k B_j^k|^{-1} \int_{2^{l+1} A_k B_j^k} |b(x) - C_j^k| dx$, then

$$L \le \sum_{l=0}^{\infty} B_l^p |2^{l+1} A_k B_j^k|^p \left(\int_{2^{l+1} A_k B_j^k \setminus 2^l A_k B_j^k} (S_\psi(b_j^k)(x))^{p/(1-p)} dx \right)^{1-p}$$

Similar to the proof of Theorem 1, we have

$$S_\psi(b_j^k)(x) \le C|x|^{-(n+\varepsilon)} \int_{B_j^k} |y|^\varepsilon |b_j^k(y)| dy \le C 2^k |x|^{-(n+\varepsilon)} |B_j^k|^{1+\varepsilon/n}.$$

Thus

$$L \le C \sum_{l=0}^{\infty} B_l^p |2^{l+1} A_k B_j^k|^p 2^{kp} |B_j^k|^{(1+\varepsilon/n)p} |2^{l+1} A_k B_j^k|^{1-p(n+\varepsilon)/n-p}$$
$$= C 2^{kp} A_k^{n(1-(n+\varepsilon)p/n)} |B_j^k| \sum_{l=0}^{\infty} B_l^p 2^{(l+1)(n-(n+\varepsilon)p)}$$
$$\le C ||b||_{BMO}^p 2^{kp} A_k^{n-p(n+\varepsilon)} |B_j^k|.$$

Now we take $A_k = A2^{(k-k_0)/(n+\varepsilon)}$, where A is fixed and large enough; then

$$\begin{aligned}
II_1 &\le C\|b\|_{BMO}^p \sum_{j=1}^{\infty} \sum_{k=k_0+1}^{N} 2^{kp}|B_j^k| A_k^{n-p(n+\varepsilon)} \\
&\le CC_1 \sum_{k=k_0+1}^{N} A_k^{n-p(n+\varepsilon)} \\
&\le CC_1, \text{ since } p > n/(n+\varepsilon).
\end{aligned}$$

Now, let us estimate II_2. By the estimate of S_ψ, similar to above, we deduce

$$\begin{aligned}
II_2 &\le \lambda^p \left| \left\{ x \notin B_{k_0,N} : \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} S_\psi((b - C_j^k) b_j^k)(x) > \lambda/2 \right\} \right| \\
&\le \lambda^p \Bigg| \Bigg\{ x \notin B_{k_0,N} : \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} C|x|^{-(n+\varepsilon)} 2^k |B_j^k|^{1+\varepsilon/n} \\
&\qquad \times \left(|B_j^k|^{-1} \int_{B_j^k} |b(y) - c_j^k| dy \right) > \lambda/2 \Bigg\} \Bigg| \\
&\le C\lambda^p \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} \left[2^k |B_j^k|^{1+\varepsilon/k} \|b\|_{BMO} \right]^{n/(n+\varepsilon)} \lambda^{-n/(n+\varepsilon)} \\
&\le C\lambda^{p-n/(n+\varepsilon)} \|b\|_{BMO}^{n/(n+\varepsilon)} \cdot C_1 \sum_{k=k_0+1}^{N} 2^{k(n/(n+\varepsilon)-p)} \\
&\le CC_1 \lambda^{p-n/(n+\varepsilon)} 2^{k_0(n/(n+\varepsilon)-p)} \\
&\le CC_1, \text{ since } \lambda \le 2^{k_0+1}.
\end{aligned}$$

Combining the estimates of II_1 with II_2, we obtain

$$II \le CC_1.$$

Finally, let us estimate I. We have

$$\begin{aligned}
I &\le \lambda^p \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} A_k^n |B_j^k| \le C\lambda^p C_1 \sum_{k=k_0+1}^{N} 2^{n(k-k_0)/(n+\varepsilon)} 2^{-kp} \\
&\le CC_1 \lambda^p 2^{-k_0 p} \sum_{k=k_0+1}^{N} 2^{(k-k_0)(n/(n+\varepsilon)-p)} \le CC_1.
\end{aligned}$$

Now, let us combine the estimates of I with II, and let $N \to \infty$, then, we have obtained the conclusion of Theorem 3. □

THEOREM 4. *Let $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \le \alpha < n(1-1/q) + \varepsilon$ and $b \in BMO(\mathbb{R}^n)$. Then $S_{\psi,b}$ is bounded from $H\dot{K}_{q,b}^{\alpha,p}(\mathbb{R}^n)$ to $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.*

Proof. Let $f \in H\dot{K}_{q,b}^{\alpha,p}(\mathbb{R}^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x))$ be the atomic decomposition for f as in Definition 5. We write

$$\|S_{\psi,b}(f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \le C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3} |\lambda_j| \|S_{\psi,b}(a_j)\chi_k\|_{L^q}\right)^p\right]^{1/p}$$
$$+ C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j| \|S_{\psi,b}(a_j)\chi_k\|_{L^q}\right)^p\right]^{1/p}$$
$$\equiv I + II.$$

For II, by the boundedness of $S_{\psi,b}$ on $L^q(\mathbb{R}^n)$(see [2]), we have

$$II \le C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|\ \|a_j\|_{L^q}\right)^p\right]^{1/p}$$
$$\le C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j| 2^{-j\alpha}\right)^p\right]^{1/p}$$
$$\le C\begin{cases} \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=k-2}^{\infty} |\lambda_j|^p 2^{-j\alpha p}\right]^{1/p}, & 0 < p \le 1 \\ \Big[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}(\sum_{j=k-2}^{\infty} |\lambda_j|^p 2^{-j\alpha p/2}) & \\ \qquad \times \left(\sum_{j=k-2}^{\infty} 2^{-j\alpha p'/2}\right)^{p/p'}\Big]^{1/p}, & p > 1 \end{cases}$$
$$\le C\begin{cases} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p}\right) |\lambda_j|^p 2^{-j\alpha p}\right]^{1/p}, & 0 < p \le 1 \\ \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2}\right)\right]^{1/p}, & p > 1 \end{cases}$$
$$\le C\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p} \le C\|f\|_{H\dot{K}_{q,b}^{\alpha,p}(\mathbb{R}^n)}.$$

For I, let $b_j = |B_j|^{-1}\int_{B_j} b(x)dx$; by the vanishing moment of a_j and the properties of $BMO(\mathbb{R}^n)$(see [13]), similar to the proof of Theorem 1, we have, for $x \in A_k$, $j \le k-3$,

$$
\begin{aligned}
&S_{\psi,b}(a_j)(x)\\
&\le \left[\int_{\Gamma(x)}\left(\int_{B_j}|\psi_t(y-z)-\psi_t(y-x_0)||a_j(z)||b(x)-b(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}\\
&\le C2^{-k(n+\varepsilon)}[|b(x)-b_j|2^{j(\varepsilon+n(1-1/q)-\alpha)}+2^{j(\varepsilon-\alpha)}(\int_{B_j}|b(y)-b_j|^{q'}dy)^{1/q'}]\\
&\le C2^{-k(n+\varepsilon)}[2^{j(\varepsilon+n(1-1/q)-\alpha)}+(k-j)2^{j(\varepsilon+n(1-1/q)-\alpha)}||b||_{BMO}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
I \le\ & C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|2^{-k(n+\varepsilon)+j(\varepsilon+n(1-1/q)-\alpha)}\right.\right.\\
&\qquad\left.\left.\times\left(\int_{B_k}|b(x)-b_k|^q dx\right)^{1/q}\right)^p\right]^{1/p}\\
&+C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|(k-j)2^{-k(n+\varepsilon)+j(\varepsilon+n(1-1/q)-\alpha)}\right.\right.\\
&\qquad\left.\left.\times 2^{kn/q}||b||_{BMO}\right)^p\right]^{1/p}\equiv I_1+I_2.
\end{aligned}
$$

To estimate I_1 and I_2, we consider two cases.

Case 1. $0 < p \le 1$. In this case, we have

$$
\begin{aligned}
I_1 \le\ & C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\right.\\
&\left.\times\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p 2^{[-k(n+\varepsilon)+j(\varepsilon+n(1-1/q)-\alpha)]p}2^{knp/q}||b||_{BMO}\right)^p\right]^{1/p}\\
\le\ & C||b||_{BMO}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p}\right]^{1/p}
\end{aligned}
$$

$$\begin{aligned}
&\leq C\|b\|_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\\
&\leq C\|f\|_{H\dot{K}_{q,b}^{\alpha,p}(\mathbb{R}^n)}
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\leq C\|b\|_{BMO}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}(k-j)^p2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p}\right]^{1/p}\\
&\leq C\|b\|_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\\
&\leq C\|f\|_{H\dot{K}_{q,b}^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned}$$

Case 2. $p>1$. By Hölder's inequality, we deduce that

$$\begin{aligned}
I_1 &\leq C\|b\|_{BMO}\Bigg[\sum_{j=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p2^{(j-k)p(\varepsilon+n(1-1/q)-\alpha)/2}\right)\\
&\quad\times\left(\sum_{j=-\infty}^{k-3}2^{(j-k)p'(\varepsilon+n(1-1/q)-\alpha)/2}\right)^{p/p'}\Bigg]^{1/p}\\
&\leq C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\\
&\leq C\|f\|_{H\dot{K}_{q,b}^{\alpha,p}(\mathbb{R}^n)}
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\leq C\|b\|_{BMO}\Bigg[\sum_{j=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p(k-j)^p2^{(j-k)p(\varepsilon+n(1-1/q)-\alpha)/2}\right)\\
&\quad\times\left(\sum_{j=-\infty}^{k-3}2^{(j-k)p'(\varepsilon+n(1-1/q)-\alpha)/2}\right)^{p/p'}\Bigg]^{1/p}
\end{aligned}$$

$$\leq C\|b\|_{BMO}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}(k-j)^p 2^{(j-k)p(\varepsilon+n(1-1/q)-\alpha)/2}\right]^{1/p}$$

$$\leq C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}$$

$$\leq C\|f\|_{H\dot{K}_{q,b}^{\alpha,p}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 4. □

THEOREM 5. *Let $0 < p \leq 1 \leq q < \infty$, $\alpha = n(1-1/q)+\varepsilon$ and $b \in BMO(\mathbb{R}^n)$. Then, for any $\lambda > 0$ and $f \in HK_{q,b}^{\alpha,p}(\mathbb{R}^n)$, we have*

$$\left[\sum_{k=0}^{\infty}2^{k\alpha p}\tilde{m}_k(\lambda, S_{\psi,b}(f))^{p/q}\right]^{1/p}$$
$$\leq C\lambda^{-1}\|f\|_{HK_{q,b}^{\alpha,p}(\mathbb{R}^n)}\left(1+\log^+(\lambda^{-1}\|f\|_{HK_{q,b}^{\alpha,p}(\mathbb{R}^n)})\right).$$

Proof. Let $f \in HK_{q,b}^{\alpha,p}(\mathbb{R}^n)$ and $f(x) = \sum_{j=0}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 5. We write

$$\left[\sum_{k=0}^{\infty}2^{k\alpha p}\tilde{m}_k(\lambda, S_{\psi,b}(f))^{p/q}\right]^{1/p}$$
$$\leq C\left[\sum_{k=0}^{3}2^{k\alpha p}\tilde{m}_k(\lambda, S_{\psi,b}(f))^{p/q}\right]^{1/p}$$
$$+C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/2, \sum_{j=0}^{k-3}|\lambda_j|S_{\psi,b}(a_j)\right)^{p/q}\right]^{1/p}$$
$$+C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/2, S_{\psi,b}\left(\sum_{j=k-2}^{\infty}\lambda_j a_j\right)\right)^{p/q}\right]^{1/p}$$
$$\equiv I_1 + I_2 + I_3.$$

For I_1 and I_3, by the weak (q,q) type boundedness of $S_{\psi,b}$ (see [2]) and

$0 < p \leq 1$, we have

$$I_1 \leq C\lambda^{-1}\left[\sum_{k=0}^{3} 2^{k\alpha p}||f||_{L^q}^p\right]^{1/p} \leq C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p||a_j||_{L^q}^p\right)^{1/p}$$
$$\leq C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p 2^{-j\alpha p}\right)^{1/p} \leq C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}$$
$$\leq C\lambda^{-1}||f||_{HK_{q,b}^{\alpha,p}(\mathbb{R}^n)}$$

and

$$I_3 \leq C\lambda^{-1}\left[\sum_{k=4}^{\infty} 2^{k\alpha p}\left|\left|\sum_{j=k-2}^{\infty}\lambda_j a_j\right|\right|_{L^q}^p\right]^{1/p}$$
$$\leq C\lambda^{-1}\left[\sum_{k=4}^{\infty} 2^{k\alpha p}\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{-j\alpha p}\right]^{1/p}$$
$$\leq C\lambda^{-1}\left[\sum_{j=0}^{\infty}|\lambda_j|^p\sum_{k=0}^{j+2} 2^{(k-j)\alpha p}\right]^{1/p}$$
$$\leq C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}$$
$$\leq C\lambda^{-1}||f||_{HK_{q,b}^{\alpha,p}(\mathbb{R}^n)}.$$

For I_2, by the same argument as that of Theorems 1 and 4,

$$S_{\psi,b}(a_j)(x) \leq C2^{-k(n+\varepsilon)}(|b(x)-b_k| + k||b||_{BMO});$$

therefore

$$I_2 \leq C\left[\sum_{k=4}^{\infty} 2^{k\alpha p}\tilde{m}_k\left(\lambda/4, C2^{-k(n+\varepsilon)}|b(x)-b_k|\sum_{j=0}^{\infty}|\lambda_j|\right)^{p/q}\right]^{1/p}$$
$$+ C\left[\sum_{k=4}^{\infty} 2^{k\alpha p}\tilde{m}_k\left(\lambda/4, C2^{-k(n+\varepsilon)}k||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|\right)^{p/q}\right]^{1/p}$$
$$\equiv I_2^{(1)} + I_2^{(2)}.$$

For $I_2^{(1)}$, by using John-Nirenberg inequality (see [15]), we gain

$$
\begin{aligned}
I_2^{(1)} &\leq C\left[\sum_{k=4}^{\infty} 2^{k\alpha p}\left(\exp\left(-\frac{C2^{k(n+\varepsilon)}\lambda}{||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)2^{kn}\right)^{p/q}\right]^{1/p} \\
&\leq C\left[\sum_{k=0}^{\infty} 2^{k(n+\varepsilon)p}\exp\left(-\frac{C\lambda 2^{k(n+\varepsilon)}}{||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)\right]^{1/p} \\
&\leq C\left[\int_0^{\infty} x^{p-1}\exp\left(-\frac{c\lambda x}{||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)dx\right]^{1/p} \\
&\leq C\lambda^{-1}||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|\left(\int_0^{\infty} t^{p-1}e^{-t}dt\right)^{1/p} \\
&\leq C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p} \\
&\leq C\lambda^{-1}||f||_{HK_{q,b}^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned}
$$

For $I_2^{(2)}$, by using the following fact: If there exists $u > 1$ such that $2^x/x \leq u$ for $x \geq 3$, then $2^x \leq cu\log^+ u$. We have, if $\left|\left\{x \in A_k : C2^{-k(n+\varepsilon)}k||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j| > \lambda/4\right\}\right| \neq 0$, then

$$
1 < 2^{k(n+\varepsilon)} < C\lambda^{-1}||b||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|;
$$

thus

$$
2^{k(n+\varepsilon)} \leq C\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\log^+\left(\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\right).
$$

Let K_λ be the maximal integer k which satisfies the above inequality;

then

$$\begin{aligned}
I_2^{(2)} &\le C\left(\sum_{k=4}^{K_\lambda} 2^{k\alpha p}2^{knp/q}\right)^{1/p} \le C2^{K_\lambda(n+\varepsilon)}\\
&\le C\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\log^+\left(\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\right)\\
&\le C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}\log^+\left(\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}\right)\\
&\le C\lambda^{-1}\|f\|_{HK_{q,b}^{\alpha,p}(\mathbb{R}^n)}\log^+\left(\lambda^{-1}\|f\|_{HK_{q,b}^{\alpha,p}(\mathbb{R}^n)}\right).
\end{aligned}$$

Now, summing up the above estimates, we have

$$\begin{aligned}
(\sum_{k=0}^{\infty} 2^{k\alpha p}\tilde{m}_k(\lambda, S_{\psi,b}(f))^{p/q}]^{1/p} &\le C\,\lambda^{-1}\|f\|_{HK_{q,b}^{\alpha,p}(\mathbb{R}^n)}\\
&\quad\times\left(1+\log^+(\lambda^{-1}\|f\|_{HK_{q,b}^{\alpha,p}(\mathbb{R}^n)})\right).
\end{aligned}$$

This completes the proof of Theorem 5. □

In Theorem 5, if we relax the condition of b, then we get the following conclusion.

THEOREM 6. *Let $0 < p \le 1 \le q < \infty$, $\alpha = n(1-1/q)+\varepsilon$ and $b \in L^\infty(\mathbb{R}^n)$. Then $S_{\psi,b}$ is bounded from $H\dot{K}_{q,b}^{\alpha,p}(\mathbb{R}^n)$ to $W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.*

Proof. Let $f \in H\dot{K}_{q,b}^{\alpha,p}(\mathbb{R}^n)$ and $f(x) = \sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 5. Write

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} 2^{k\alpha p} m_k(\lambda, S_{\psi,b}(f))^{p/q}\\
&\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p} m_k\left(\lambda/2, \sum_{j=-\infty}^{k-3}|\lambda_j|S_{\psi,b}(a_j)\right)^{p/q}\\
&+\sum_{k=-\infty}^{\infty} 2^{k\alpha p} m_k\left(\lambda/2, S_{\psi,b}(\sum_{j=k-2}^{\infty}\lambda_j a_j)\right)^{p/q}\\
&\equiv I + II.
\end{aligned}$$

For II, by the weak (q,q) type boundedness of $S_{\psi,b}$ and $0<p\le 1$, we have

$$\begin{aligned} II &\le C\lambda^{-p}\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\sum_{j=k-2}^{\infty}|\lambda_j|^p||a_j||_{L^q}^p \\ &\le C\lambda^{-p}\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p} \\ &\le C\lambda^{-p}||f||^p_{H\dot{K}_{q,b}^{\alpha,p}(\mathbb{R}^n)}. \end{aligned}$$

For I, since, similar to the proof of Theorems 1 and 4,

$$S_{\psi,b}(a_j)(x)\le C2^{-k(n+\varepsilon)}||b||_{L^\infty},$$

if $|x\in A_k : C2^{-k(n+\varepsilon)}||b||_{L^\infty}\sum_{j==-\infty}^{k-3}|\lambda_j|>\lambda/2|\neq 0$, then

$$2^{k(n+\varepsilon)}\le C\lambda^{-1}||b||_{L^\infty}\sum_{j=-\infty}^{\infty}|\lambda_j|\le C\lambda^{-1}(\sum_{j=-\infty}^{\infty}|\lambda_j|^p)^{1/p}.$$

Let K_λ be the maximal integer k which satisfies the above inequality; then

$$\begin{aligned} I &\le C\sum_{k=-\infty}^{K_\lambda}2^{k\alpha p}2^{knp/q}\le C2^{K_\lambda(n+\varepsilon)p} \\ &\le C\lambda^{-p}\sum_{j=-\infty}^{\infty}|\lambda_j|^p\le C\lambda^{-p}||f||^p_{H\dot{K}_{q,b}^{\alpha,p}(\mathbb{R}^n)}. \end{aligned}$$

This completes the proof of Theorem 6. □

REMARK. Theorems 4 and 6 also hold for nonhomogeneous Herz-type spaces; we omit the details.

ACKNOWLEDGEMENT. The author would like to express his deep gratitude to the referee for his very valuable comments and suggestions.

References

[1] J. Alvarez, *Continuity properties for linear commutators of Calderon-Zygmund operators*, Collect. Math. **49** (1998), 17–31.

[2] J. Alvarez, R. J. Babgy, D. S. Kurtz and C. Perez, *Weighted estimates for commutators of linear operators*, Studia Math. **104** (1993), 195–209.

[3] R. Coifman, R. Rochberg and G. Weiss, *Factorization theorem for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611–635.

[4] J. Garcia-Cuerva and M. J. L. Herrero, *A theory of Hardy spaces associated to Herz spaces*, Proc. London Math. Soc. **69** (1994), 605–628.

[5] G. Hu, S. Lu and D. Yang, *The weak Herz spaces*, J. of Beijing Normal University (Natural Sci.) **31** (1997), 27–36.

[6] ———, *The applications of weak Herz spaces*, Adv. Math. (China) **26** (1997), 417–428.

[7] H. Liu, *The weak H^p spaces on homogeneous groups*, Lecture Notes in Math. **1494**, Springer, Berlin, 1991, 113–118.

[8] S. Lu and D. Yang, *The local versions of $H^p(\mathbb{R}^n)$ spaces at the origin*, Studia Math. **116** (1995), 103–131.

[9] ———, *The decomposition of the weighted Herz spaces and its applications*, Sci. China Ser. A **38** (1995), 147–158.

[10] ———, *The weighted Herz spaces type Hardy spaces and its applications*, Sci. China Ser. A **38** (1995), 662–673.

[11] ———, *The continuity of commutators on Herz type spaces*, Michigan Math. J. **44** (1997), 255–281.

[12] C. Perez, *Endpoint estimates for commutators of singular integral operators*, J. Funct. Anal. **128** (1995), 163–185.

[13] E. M. Stein, *Harmonic Analysis: real variable methods orthogonality and oscillation integrals*, Princeton Mathematical Series, **43**, Princeton University Press, Princeton, NJ, 1993.

[14] E. M. Stein, M. Taibleson and G. Weiss, *Weak type estimates for maximal operators on certain H^p spaces*, Rend. Circ. Mat. Palermo, suppl. **1** (1981), 81–97.

[15] A. Torchinsky, *The real variable methods in harmonic analysis*, Pure and Applied Math. **123**, Academic Press, New York, 1986.

Department of Applied Mathematics
Hunan University
Changsha, 410082, P. R. China
E-mail: lanzheliu@263.net