J. Korean Math. Soc. **40** (2003), No. 1, pp. 1–28

HIGHEST WEIGHT VECTORS OF IRREDUCIBLE REPRESENTATIONS OF THE QUANTUM SUPERALGEBRA $\mathfrak{U}_q\,(gl(m,n))$

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ABSTRACT. The Iwahori-Hecke algebra $\mathcal{H}_k(q^2)$ of type A acts on the k-fold tensor product space of the natural representation of the quantum superalgebra $\mathfrak{U}_q(gl(m,n))$. We show the Hecke algebra $\mathcal{H}_k(q^2)$ and the quantum superalgebra $\mathfrak{U}_q(gl(m,n))$ have commuting actions on the tensor product space, and determine the centralizer of each other. Using this result together with Gyoja's q-analogue of the Young symmetrizers, we construct highest weight vectors of irreducible summands of the tensor product space.

0. Introduction

One of the main studies of finite dimensional representation theory of a semisimple Lie algebra $\mathfrak{g}$ is constructing all the irreducible $\mathfrak{g}$-modules and explicitly describing how $\mathfrak{g}$ acts on irreducible modules. Thanks to the highest weight theory of $\mathfrak{g}$-modules, we may attain this by constructing highest weight vectors of irreducible modules.

When $\mathfrak{g}$ is the *special linear Lie algebra* $sl(n)$ or the *general linear Lie algebra* $gl(n)$ over the complex field $\mathbb{C}$, this problem was successfully solved by I. Schur in [19] and [20]. Schur investigated tensor product spaces of the *natural representation*, which is the irreducible representation of $gl(n)$ with highest weight ϵ_1. He showed the action of $gl(n)$ on the k-fold tensor product space generates the full centralizer of the action of the symmetric group S_k on the same space. And then, from the

Received February 15, 2001. Revised May 11, 2002.
2000 Mathematics Subject Classification: Primary 17B37, 20C08; Secondary 05A17.
Key words and phrases: quantum superalgebra, maximal vector, Hecke algebra, Schur-Weyl duality.
Supported by grant # R01-2000-00022 from the Basic Research Program of the Korea Science & Engineering Foundation.

double centralizer theorem, we may show the associative algebra generated by actions commuting with the action of $gl(n)$, which is called the *centralizer algebra* of $gl(n)$, is a quotient of the group algebra $\mathbb{C}S_k$ of S_k. This result is often called *Schur-Weyl duality*, and it plays an important role in understanding the representation theory of $gl(n)$. Schur used the works on the symmetric group S_k by F. Frobenius [9] and by A. Young [22] to study the representation theory of $gl(n)$. For example, the irreducible decomposition of the $gl(n)$-module $V^{\otimes k}$ can be obtained from the decomposition of $\mathbb{C}S_k$ into minimal left ideals via minimal idempotents of $\mathbb{C}S_k$, the *Young Symmetrizers*.

The same approaches were made by A. Berele and A. Regev [3] and G. Benkart and C. Lee Shader [2] for the *general linear Lie superalgebra* $gl(m,n)$. When $\mathfrak{g} = gl(m,n)$, the centralizer algebra is again a homomorphic image of $\mathbb{C}S_k$, and we can also use Young symmetrizers to decompose the tensor product space.

In 1986, M. Jimbo [11] constructed the *Drinfel'd-Jimbo* quantized universal enveloping algebra $\mathfrak{U}_q(gl(n))$ of $gl(n)$. He also showed the action of the Iwahori-Hecke algebra of Type A, $\mathcal{H}_k(q^2)$, on the k-fold tensor product space of the natural representation of $\mathfrak{U}_q(gl(n))$ commutes with the action of $\mathfrak{U}_q(gl(n))$.

In this paper, we continue these preceding approaches. We show that the action of $\mathcal{H}_k(q^2)$ determines the commuting action of quantized universal enveloping algebra $\mathfrak{U}_q(gl(m,n))$ of $gl(m,n)$, and use the q-analogue of Young symmetrizers to construct irreducible representations of $\mathfrak{U}_q(gl(m,n))$ and highest weight vectors. Because conjugation is more complicated in the Hecke algebra than that in symmetric group, one has various constructions of the q-analogue of Young symmetrizers (see for example [15], [7], and [10]). In this paper, we use A. Gyoja's q-idempotents in [10].

The usual trick for proving that the action of the general linear Lie algebra $gl(n)$ generates the full centralizer of the symmetric group action uses the idempotent $\sum_{\sigma\in S_k} \sigma$ to construct a projection map onto the $gl(n)$-invariants. Unfortunately, this method is no longer useful for the quantum case because such a construction does not yield an idempotent in the Hecke algebra. To show the action of $\mathcal{H}_k(q^2)$ determines the full centralizer of $\mathfrak{U}_q(gl(n))$, R. Leduc and A. Ram used the path algebra approach to the centralizer algebra of $\mathfrak{U}_q(gl(n))$ in [13]. But their approach requires that the tensor product spaces are completely reducible $\mathfrak{U}_q(gl(n))$-modules. Recently in [1], G. Benkart, S.-J. Kang, and M. Kashiwara showed the completely reducibility of the tensor product spaces of the natural representation of $\mathfrak{U}_q(gl(m,n))$ using the crystal base theory of $\mathfrak{U}_q(gl(m,n))$. Now we use their result and obtain the full centralizer of $\mathfrak{U}_q(gl(m,n))$ in Section 3.

The other main result will appear in Section 5. We will decompose the tensor product space to obtain finite dimensional irreducible representations of $\mathfrak{U}_q(gl(m,n))$ and construct a highest weight vector of each finite dimensional irreducible representation in the tensor product space using Gyoja's q-analogue of the Young symmetrizers. The multiplication in the Hecke algebra is more complicated than that in symmetric group, and this is the main difficulty in obtaining highest weight vectors of irreducible representations of $\mathfrak{U}_q(gl(m,n))$. We will develop a technique to resolve this situation in Section 5.

1. The quantum superalgebra $\mathfrak{U}_q(gl(m,n))$

The *general linear Lie superalgebra* $gl(m,n) = gl(m,n)_{\bar{0}} \oplus gl(m,n)_{\bar{1}}$ is the set of all $(m+n)\times(m+n)$ matrices over $\mathbb{C}$, which is $\mathbb{Z}_2$-graded by

$$gl(m,n)_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| \, A \in \mathrm{M}_{m\times m}(\mathbb{C}), \quad D \in \mathrm{M}_{n\times n}(\mathbb{C}) \right\},$$
$$gl(m,n)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \middle| \, B \in \mathrm{M}_{m\times n}(\mathbb{C}), \quad C \in \mathrm{M}_{n\times m}(\mathbb{C}) \right\},$$

together with the super bracket

$$[x,y] = xy - (-1)^{ab}yx$$

for $x \in gl(n,n)_{\bar{a}}$, $y \in gl(n,n)_{\bar{b}}$, $a,b = 0,1$. We adopt the convention on parities that $p(x) = a$ if $0 \neq x \in gl(n)_{\bar{a}}$, $a = 0,1$.

We define the supertrace str on $gl(m,n)$ by,

$$str(x) = TrA - TrD,$$

for $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in gl(m,n)$, where Tr is the usual matrix trace. The special linear Lie superalgebra $sl(m,n)$ is the subalgebra,

$$sl(m,n) = \{x \in gl(m,n) | str(x) = 0\},$$

of $gl(m,n)$ of matrices of supertrace zero.

Let $E_{i,j} \in gl(m,n)$ denote the matrix unit which has 1 at (i,j)-position and 0's at other positions. We let I be a set such that $I := \{1,2,\dots,m+n-1\}$. The Cartan subalgebra $\mathfrak{h}$ of $gl(m,n)$ is the set of all diagonal matrices in $gl(m,n)$, which is the $\mathbb{C}$-span of $E_{i,i}$, $1 \le i \le m+n$. For $i \in I$, we let H_i be

$$H_i = \begin{cases} E_{i,i} - E_{i+1,i+1} & \text{if } i \neq m, \\ E_{m,m} + E_{m+1,m+1} & \text{if } i = m. \end{cases} \tag{1.1}$$

We denote the dual space of $\mathfrak{h}$ by $\mathfrak{h}^*$. Relative to the adjoint action of the Cartan subalgebra $\mathfrak{h}$, $gl(m,n)$ decomposes into root spaces

$$gl(m,n) = \mathfrak{h} \oplus \sum_{\alpha \in \mathfrak{h}^*} gl(m,n)_\alpha.$$

Let $\epsilon_1, \dots, \epsilon_m$ and $\delta_1, \dots, \delta_n$ be orthonormal bases of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. We will also use convention that $\epsilon_{m+1} := \delta_1, \dots, \epsilon_{m+n} := \delta_n$. The simple roots $\alpha_i \in \mathfrak{h}^*$ and the fundamental weights $\omega_i \in \mathfrak{h}^*$, for $i \in I$, are given by

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad 1 \le i \le m-1, \tag{1.2}$$

$$\omega_i = \epsilon_1 + \cdots + \epsilon_i. \tag{1.3}$$

A root α is *even* if $gl(m,n)_\alpha \cap gl(m,n)_{\bar{0}} \neq \{0\}$ or *odd* if $gl(m,n)_\alpha \cap gl(m,n)_{\bar{1}} \neq \{0\}$. Hence all the simple roots except α_m are even.

Let P be the $\mathbb{Z}$-span of $\{\epsilon_1, \dots, \epsilon_{m+n}\}$, which we call the *lattice of integral weights.* And the *dual weight lattice* $P^\vee \subset \mathfrak{h}$ is the free $\mathbb{Z}$-lattice spanned by $E_{i,i}$, $1 \le i \le m+n$. We may define the value $\lambda(h)$ for any $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$ by

$$\epsilon_i(E_{j,j}) = \delta_{ij}$$

and extending it by linearity. This allows us to define a natural pairing $\langle\, ,\, \rangle$ between $\mathfrak{h}$ and $\mathfrak{h}^*$ so that

$$\langle H_i, \alpha_j \rangle = \alpha_j(H_i).$$

The *Cartan* matrix $A = (a_{ij})_{1 \le i,j \le m+n-1}$, where $a_{ij} = \alpha_j(H_i)$, of $sl(m,n)$ (*or* $gl(m,n)$) satisfies

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \neq m, \\ 0 & \text{if } i = j = m, \\ -1 & j = i-1 \text{ or } j = i+1, i \neq m, \\ 1 & i = m,\ j = m+1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Note that the Cartan matrix A is symmetrizable, i.e., if we define d_i for $i \in I$ by

$$d_i = \begin{cases} 1 & \text{if } 1 \le i \le m, \\ -1 & \text{if } m+1 \le i \le m+n-1, \end{cases} \tag{1.5}$$

and if we let $D = \operatorname{diag}(d_1, \dots, d_{m+n-1})$ be the diagonal matrix with diagonal entries d_i for $i \in I$, then $A^{sym} = DA$ is a symmetric matrix.

Now we give a definition of the quantum superalgebra $\mathfrak{U}_q(gl(m,n))$. The Serre-type presentations of $sl(m,n)$ (or $gl(m,n)$) and the quantization of $sl(m,n)$ (or $gl(m,n)$) were obtained by various authors all roughly at about the same time (see for example [12], [8], [17] or [18]). Readers may refer those papers for the presentations of $sl(m,n)$. But in this paper we adopt a definition appeared in [1] to quote results there.

Let q be an indeterminate and let $\mathbb{C}(q)$ denote the field of rational functions in q. Let $q_i := q^{d_i}$, where d_i is defined in (1.5).

Definition 1.6 ([1]). The quantized universal enveloping algebra (or quantum superalgebra) $\mathfrak{U}_q(gl(m,n))$ is the unital associative algebra over $\mathbb{C}(q)$ with generators E_i, F_i $(i \in I)$, q^h $(h \in P^\vee)$, which satisfy the following defining relations:

- $q^h = 1$ for $h = 0$,
- $q^{h_1+h_2} = q^{h_1} q^{h_2}$ for $h_1, h_2 \in P^\vee$,
- $q^h E_i = q^{\langle h, \alpha_i \rangle} E_i q^h$ for $h \in P^\vee$ and $i \in I$,
- $q^h F_i = q^{-\langle h, \alpha_i \rangle} F_i q^h$ for $h \in P^\vee$ and $i \in I$,
- $E_i F_j - (-1)^{p(E_i)p(F_j)} F_j E_i = \delta_{i,j} \dfrac{q^{d_i H_i} - q^{-d_i H_i}}{q_i - q_i^{-1}}$ for $i, j \in I$,
- $E_i E_j - (-1)^{p(E_i)p(E_j)} E_j E_i = 0$ if $|i - j| \ge 2$,
- $F_i F_j - (-1)^{p(F_i)p(F_j)} F_j F_i = 0$ if $|i - j| \ge 2$,
- $E_i^2 E_j - (q_i + q_i^{-1}) E_i E_j E_i + E_j E_i^2 = 0$ if $|i - j| = 1$ and $i \ne m$,
- $F_i^2 F_j - (q_i + q_i^{-1}) F_i F_j F_i + F_j F_i^2 = 0$ if $|i - j| = 1$ and $i \ne m$,
- $E_m^2 = F_m^2 = 0$,
- $E_m E_{m-1} E_m E_{m+1} + E_m E_{m+1} E_m E_{m-1} + E_{m-1} E_m E_{m+1} E_m$
  $+ E_{m+1} E_m E_{m-1} E_m - (q + q^{-1}) E_m E_{m-1} E_{m+1} E_m = 0$,
- $F_m F_{m-1} F_m F_{m+1} + F_m F_{m+1} F_m F_{m-1} + F_{m-1} F_m F_{m+1} F_m$
  $+ F_{m+1} F_m F_{m-1} F_m - (q + q^{-1}) F_m F_{m-1} F_{m+1} F_m = 0.$

The parities are given as $p(q^h) = 0$ for all $h \in P^\vee$, $p(E_i) = p(F_i) = 0$ for $i \ne m$, and $p(E_m) = p(F_m) = 1$.

From now on, we set $k_i = q^{d_i H_i}$. A Hopf superalgebra structure of $\mathfrak{U}_q(gl(m,n))$ is given by the comultiplication $\Delta : \mathfrak{U}_q(gl(m,n)) \longrightarrow \mathfrak{U}_q(gl(m,n)) \otimes \mathfrak{U}_q(gl(m,n))$ such that

$$\begin{aligned} \Delta(E_i) &= E_i \otimes k_i^{-1} + 1 \otimes E_i, \\ \Delta(F_i) &= F_i \otimes 1 + k_i \otimes F_i, \\ \Delta(q^h) &= q^h \otimes q^h, \end{aligned} \tag{1.7}$$

the antipode $S : \mathfrak{U}_q(gl(m,n)) \longrightarrow \mathfrak{U}_q(gl(m,n))$ given by

$$S(E_i) = -E_i k_i, \quad S(F_i) = -k_i^{-1} F_i, \quad S(q^h) = q^{-h}, \tag{1.8}$$

and the counit $\varepsilon : \mathfrak{U}_q(gl(m,n)) \longrightarrow \mathbb{C}(q)$ by

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(q^h) = 1. \tag{1.9}$$

A $\mathfrak{U}_q(gl(m,n))$-module M is called a *weight module* if it admits a weight space decomposition

$$M = \bigoplus_{\lambda \in P} M_\lambda, \tag{1.10}$$

where $M_\lambda = \{u \in M \mid q^h u = q^{\langle h, \lambda \rangle} u \text{ for any } h \in P^\vee\}$. A weight module M is a *highest weight module* with *highest weight* $\lambda \in P$, if there exists a unique nonzero vector $v_\lambda \in M$, which is called a *highest weight vector*, up to constant multiples such that

- $M = \mathfrak{U}_q(gl(m,n)) v_\lambda$,
- $E_i v_\lambda = 0$ for all $i \in I$ and
- $q^h v_\lambda = q^{\lambda(h)} v_\lambda$ for all $h \in P^\vee$.

We denote the irreducible highest weight module with highest weight λ by $V(\lambda)$.

The set of dominant integral weights is given as

$$\Gamma = \left\{ \lambda = \sum_{i=1}^{m+n} \lambda_i \epsilon_i \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m, \lambda_{m+1} \geq \cdots \geq \lambda_{m+n} \right\}. \tag{1.11}$$

Now we define the *fundamental representation* of $\mathfrak{U}_q(gl(m,n))$. Let $V = V_{\bar{0}} \oplus V_{\bar{1}} = \mathbb{C}(q)^m \oplus \mathbb{C}(q)^n$ be a $\mathbb{Z}_2$-graded vector space of dimension $(m+n)$ over $\mathbb{C}(q)$. Let $T = \{t_1, \ldots, t_m\}$ be a basis of $V_{\bar{0}}$ and $U = \{u_1, \ldots, u_n\}$ be a basis of $V_{\bar{1}}$ so that the parities of the basis vectors are given by $p(t_i) = 0$ and $p(u_i) = 1$. Sometimes it is convenient to write $b_1 := t_1, \ldots, b_m := t_m$, and $b_{m+1} := u_1, \ldots, b_{m+n} := u_n$. The fundamental (super) representation (ρ, V), $\rho : \mathfrak{U}_q(gl(m,n)) \longrightarrow \mathrm{End}(V)$, of $\mathfrak{U}_q(gl(m,n))$ is given by setting

$$\begin{aligned} \rho(E_i) &= E_{i,i+1}, \\ \rho(F_i) &= E_{i+1,i}, \\ \rho(q^h) &= \sum_{i=1}^{m+n} q^{\epsilon_i(h)} E_{i,i}. \end{aligned} \tag{1.12}$$

It is easy to see that this representation is in fact an irreducible highest weight module $V(\epsilon_1)$ with highest weight ϵ_1.

Also we let $W = W_{\bar{0}} \oplus W_{\bar{1}} = \mathbb{C}^m \oplus \mathbb{C}^n$ be a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$. Then we may identify $V_{\bar{0}} = \mathbb{C}^m \otimes_{\mathbb{C}} \mathbb{C}(q)$ and $V_{\bar{1}} = \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}(q)$. Moreover we also regard $V^{\otimes k} = W^{\otimes k} \otimes_{\mathbb{C}} \mathbb{C}(q)$, which has a $\mathbb{Z}_2$-grading $(V^{\otimes k})_{\bar{0}} = (W^{\otimes k})_{\bar{0}} \otimes_{\mathbb{C}} \mathbb{C}(q)$ and $(V^{\otimes k})_{\bar{1}} = (W^{\otimes k})_{\bar{1}} \otimes_{\mathbb{C}} \mathbb{C}(q)$. Also we regard $\mathrm{End}(V^{\otimes k}) = \mathrm{End}(W^{\otimes k}) \otimes_{\mathbb{C}} \mathbb{C}(q)$, which has a $\mathbb{Z}_2$-grading similarly. Since $\mathfrak{U}_q(gl(m,n))$ is a Hopf superalgebra, the tensor product representation $(\rho^{\otimes k}, V^{\otimes k})$ of ρ is a well-defined super representation for each $k \geq 1$.

2. The universal R-matrix, the Hecke algebra, and Gyoja's q-analogue of the Young symmetrizers

In this section we recall the universal R-matrix of $\mathfrak{U}_q(gl(m,n))$ which appeared in [12] and show that there is an action of a certain Hecke algebra $\mathcal{H}_k(q^2)$ on $V^{\otimes k}$ coming from the universal R-matrix, which commutes with the action of $\mathfrak{U}_q(gl(m,n))$ on $V^{\otimes k}$.

Let

$$\theta : \mathfrak{U}_q(gl(m,n)) \otimes \mathfrak{U}_q(gl(m,n)) \longrightarrow \mathfrak{U}_q(gl(m,n)) \otimes \mathfrak{U}_q(gl(m,n))$$

be given by $\theta(x \otimes y) = (-1)^{p(x)p(y)} y \otimes x$. We define opposite comultiplication Δ' by $\Delta' = \theta\Delta$.

THEOREM 2.1 (See [12]). *There is a unique invertible solution of parity* 0

$$\mathcal{R} = \sum_i x_i \otimes y_i \in \mathfrak{U}_q(gl(m,n)) \widehat{\otimes} \mathfrak{U}_q(gl(m,n))$$

(the completion of $\mathfrak{U}_q(gl(m,n)) \otimes \mathfrak{U}_q(gl(m,n))$*) of the equations*

$$\begin{aligned} &\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1} \quad \text{for all } x \in \mathfrak{U}_q(gl(m,n)), \\ &(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{23} \quad \text{and} \\ &(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{12}, \end{aligned} \tag{2.2}$$

where $\mathcal{R}^{12} = \sum_i x_i \otimes y_i \otimes 1$, $\mathcal{R}^{23} = \sum_i 1 \otimes x_i \otimes y_i$, *and* $\mathcal{R}^{13} = \sum_i x_i \otimes 1 \otimes y_i$.

The element $\mathcal{R}$ which satisfies (2.2) is called *the universal R-matrix.* The universal R-matrix $\mathcal{R}$ of $\mathfrak{U}_q(gl(m,n))$ is given explicitly in [12]. Let $R \in \mathrm{End}(V \otimes V)$ be the transformation induced by the action of $\mathcal{R}$ on $V \otimes V$. Applying $\mathcal{R}$ to $V \otimes V$ relative to the basis $\{b_i \otimes b_j \mid i,j = 1,\dots,m+n\}$, we may compute the matrix of R in $\mathrm{End}(V \otimes V)$ as

(2.3)

$$\begin{aligned} R = &\sum_{i=1}^{m} q^2 E_{i,i} \otimes E_{i,i} + \sum_{i=m+1}^{m+n} E_{i,i} \otimes E_{i,i} \\ &+ \sum_{\substack{i \neq j \\ 1 \leq i,j \leq m+n}} qE_{i,i} \otimes E_{j,j} + \sum_{\substack{i<j \\ 1 \leq i,j \leq m+n}} (-1)^{p(b_i)}(q^2-1)E_{j,i} \otimes E_{i,j}. \end{aligned}$$

Let $\check{R} = \sigma R$, where $\sigma : V \otimes V \longrightarrow V \otimes V$ is given by $\sigma(v \otimes w) = (-1)^{p(v)p(w)} w \otimes v$. Then

(2.4)

$$\begin{aligned} \check{R} = &\sum_{i=1}^{m} q^2 E_{i,i} \otimes E_{i,i} - \sum_{i=m+1}^{m+n} E_{i,i} \otimes E_{i,i} \\ &+ \sum_{\substack{i \neq j \\ 1 \leq i,j \leq m+n}} (-1)^{p(b_i)} qE_{j,i} \otimes E_{i,j} + \sum_{\substack{i<j \\ 1 \leq i,j \leq m+n}} (q^2-1)E_{i,i} \otimes E_{j,j}. \end{aligned}$$

Note that the action of $\mathrm{End}(V \otimes V)$ on $V \otimes V$ is $\mathbb{Z}_2$-graded, i.e., for homogeneous elements $X \otimes Y \in \mathrm{End}(V \otimes V) = \mathrm{End}(V) \otimes \mathrm{End}(V)$ and $v \otimes w \in V \otimes V$, we have $(X \otimes Y)(v \otimes w) = (-1)^{p(Y)p(v)} Xv \otimes Yw$. And also, the product of tensors is given as $(X_1 \otimes X_2)(Y_1 \otimes Y_2) = (-1)^{p(X_2)p(Y_1)} X_1Y_1 \otimes X_2Y_2$ for homogeneous $X_1 \otimes X_2, Y_1 \otimes Y_2 \in \mathrm{End}(V \otimes V)$. By direct calculation, it is easy to see that

$$\check{R}^2 + (1-q^2)\check{R} = q^2 \mathrm{id}_{V \otimes V}. \tag{2.5}$$

For each $j = 1,\dots,k-1$, we let

$$r_j = \mathrm{id}_V^{\otimes j-1} \otimes \check{R} \otimes \mathrm{id}_V^{\otimes k-j-1} \in \mathrm{End}(V^{\otimes k}),$$

where $\check{R}$ operates on the jth and the $(j+1)$st tensor slots. From (2.5) we know

$$(r_i + \mathrm{id})(r_i - q^2\mathrm{id}) = 0. \tag{2.6}$$

Furthermore using arguments similar to those in [13, Proposition 2.18], we have

PROPOSITION 2.7. *Each r_j commutes with the action of $\mathfrak{U}_q(gl(m,n))$ on $(V^{\otimes k})$. In other words each r_j is in $\mathrm{End}_{\mathfrak{U}_q(gl(m,n))}(V^{\otimes k})$. Furthermore, the following braid relations are satisfied:*

$$\begin{aligned} r_i r_j &= r_j r_i && \text{for } |i-j| \geq 2, \\ r_i r_{i+1} r_i &= r_{i+1} r_i r_{i+1} && \text{for } 1 \leq i \leq m-2. \end{aligned}$$

DEFINITION 2.8. The Iwahori-Hecke algebra of type A, denoted by $\mathcal{H}_k(q^2)$, is the associative algebra over $\mathbb{C}(q)$ generated by $1, h_1, \ldots, h_{k-1}$ subject to the relations

- (B1) $h_i h_j = h_j h_i$ if $|i-j| \leq 2$,
- (B2) $h_i h_{i+1} h_i = h_{i+1} h_i h_{i+1}$ for $1 \leq i \leq k-2$,
- (IH) $(h_i + 1)(h_i - q^2) = 0$.

Notice that $\mathcal{H}_k(q^2)$ is a q-analogue of the group algebra $\mathbb{C}S_k$ of the symmetric group S_k in the sense that, when q is specialized to 1, $\mathcal{H}_k(q^2)$ is isomorphic to $\mathbb{C}S_k$. Let $\sigma = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ be a reduced expression for $\sigma \in S_k$, where s_j is the transposition $(j\,j+1)$, $1 \leq j \leq k-1$. Then we let that $h(\sigma) = h_{i_1} \cdots h_{i_\ell}$. It is well-known that *the definition of $h(\sigma)$ does not depend on the reduced expression chosen for σ*, i.e. $h(\sigma_1\sigma_2) = h(\sigma_1)h(\sigma_2)$ if and only if $\ell(\sigma_1\sigma_2) = \ell(\sigma_1) + \ell(\sigma_2)$, where $\ell(\sigma)$ is the usual length of permutation σ. From (2.6), Proposition 2.7 and Definition 2.8, we have the following:

PROPOSITION 2.9. *There is a representation*

$$\Psi_q : \mathcal{H}_k(q^2) \longrightarrow \mathrm{End}_{\mathfrak{U}_q(gl(m,n))}(V^{\otimes k})$$

of the Iwahori-Hecke algebra $\mathcal{H}_k(q^2)$ given by $h_i \mapsto r_i$.

There is also a representation $\Psi : \mathbb{C}S_k \longrightarrow \mathrm{End}(W^{\otimes k})$ of the group algebra $\mathbb{C}S_k$, which commute with the action of $gl(m,n)$ on $W^{\otimes k}$, given by $\mathbb{Z}_2$-graded place permutation on simple tensors such that, for $w = w_1 \otimes$

$\cdots \otimes w_k \in W^{\otimes k}$ and $\sigma = (i\ j) \in S_k$ with $i < j$, $\sigma w = \mathrm{sgn}(\sigma, w) w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(k)}$, where

$$\mathrm{sgn}(\sigma, w) = \begin{cases} 1 & \text{if } p(w_i) = p(w_j) = 0, \\ -1 & \text{if } p(w_i) = p(w_j) = 1, \\ 1 & \text{if } p(w_i) \neq p(w_j) \text{ and number of } w_k \text{ in } W_{\bar{1}}, \\ & \quad i < k < j, \text{ is even}, \\ -1 & \text{if } p(w_i) \neq p(w_j) \text{ and number of } w_k \text{ in } W_{\bar{1}}, \\ & \quad i < k < j, \text{ is odd}. \end{cases}$$

Note that Ψ_q is a q-deformation of Ψ.

Write $\lambda \vdash k$ to denote that λ is a partition of k. Then we denote by $\ell(\lambda)$ the length of λ, which is the number of nonzero parts of λ. Corresponding to $\lambda \vdash k$ is its *Young frame* having k boxes with λ_i boxes in the ith row and with the boxes in each row left justified. We let λ^* be the *conjugate partition* of λ whose frame is obtained by reflecting that of λ about the main diagonal. Then λ_j^* is just the number of boxes in the jth column of λ.

A partition λ is said to be of (m, n)-*hook shape* if $\lambda_{m+1} \leq n$. We let $H(m, n; k)$ denote the set of all partitions of k which are of (m, n)-hook shape. Note that a partition $\lambda \in H(m, n; k)$ may be identified with a dominant integral weight

$$\lambda_1 \epsilon_1 + \cdots + \lambda_m \epsilon_m + \lambda_1' \delta_1 + \cdots + \lambda_n' \delta_n \in \Gamma,$$

where

$$\lambda_j' = \max\{\lambda_j^* - m, 0\} \quad \text{for } j = 1, \ldots, n.$$

The irreducible representations of S_k over any field $\mathbb{F}$ of characteristic 0 are indexed by the partitions $\lambda \vdash k$. Thus we have

$$\mathbb{C}S_k = \bigoplus_{\lambda \vdash k} I_\lambda, \tag{2.10}$$

where I_λ is a simple ideal of $\mathbb{C}S_k$ which is isomorphic to a matrix algebra $M_{d_\lambda}(\mathbb{C})$. Here d_λ is the dimension of the irreducible S_k-module labelled by λ.

The Iwahori-Hecke algebra $\mathcal{H}_k(q)$ is the associative algebra over $\mathbb{C}(q)$ generated by $1, \tilde{h}_1, \ldots, \tilde{h}_{k-1}$ subject to the relations

- (B1) $\tilde{h}_i \tilde{h}_j = \tilde{h}_j \tilde{h}_i$ if $|i - j| \leq 2$,
- (B2) $\tilde{h}_i \tilde{h}_{i+1} \tilde{h}_i = \tilde{h}_{i+1} \tilde{h}_i \tilde{h}_{i+1}$ for $1 \leq i \leq k - 2$,
- (IH') $(\tilde{h}_i + 1)(\tilde{h}_i - q) = 0$.

The following theorem is well-known to experts. (See, for example, a remark in Section 1 of [16], Introduction of [21], Section 4 of [6] and Theorem 3.1 of [14].)

THEOREM 2.11. *The Hecke algebra $\mathcal{H}_k(q)$ and the group algebra $\mathbb{C}(q)S_k$ of the symmetric group S_k over the field $\mathbb{C}(q)$ are isomorphic as associative algebras.*

We also have the following lemma:

LEMMA 2.12. *There is an isomorphism between $\mathcal{H}_k(q)$ and $\mathcal{H}_k(q^2)$ as associative algebras over $\mathbb{C}(q)$.*

Proof. Let $f : \mathcal{H}_k(q) \longrightarrow \mathcal{H}_k(q^2)$ be given by $f(1) = 1$ and

$$f(\tilde{h}_i) = \frac{1}{1+q^2}\left(q(1-q) + (1+q)h_i\right)$$

for $i = 1, 2, \ldots, k-1$. We extend f by linearity and the property that $f(\tilde{h}_{i_1} \cdots \tilde{h}_{i_l}) = f(\tilde{h}_{i_1}) \cdots f(\tilde{h}_{i_l})$. Then f is an algebra isomorphism. □

COROLLARY 2.13. *The Hecke algebra $\mathcal{H}_k(q^2)$ and the group algebra $\mathbb{C}(q)S_k$ of the symmetric group S_k over the field $\mathbb{C}(q)$ are isomorphic as associative algebras.*

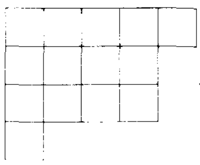
Therefore, the irreducible representations of $\mathcal{H}_k(q^2)$ are also indexed by the partitions $\lambda \vdash k$, and we also have that

$$\mathcal{H}_k(q^2) = \bigoplus_{\lambda \vdash k} I^q_\lambda, \tag{2.14}$$

where I^q_λ is a simple ideal of $\mathcal{H}_k(q^2)$ which is isomorphic to a matrix algebra $M_{d_\lambda}(\mathbb{C}(q))$.

A q-analogue of the Young symmetrizers is obtained by A. Gyoja in [10]. A *standard tableau* T of shape $\lambda \vdash k$ is obtained by filling in the frame of λ with elements of $\{1, \ldots, k\}$, so that the entries increase across the rows from left to right and down the columns. By a *λ-tableau* we mean a standard tableau T of shape λ. Associated to λ are two standard tableaux $S_+ = S^+_\lambda$ and $S_- = S^-_\lambda$, which we illustrate by the following example:

EXAMPLE 2.15. If

$\lambda =$  , then

$$S_+ = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & \\ 10 & 11 & 12 & 13 & \\ 14 & & & & \end{array}, \quad \text{and } S_- = \begin{array}{ccccc} 1 & 5 & 8 & 11 & 14 \\ 2 & 6 & 9 & 12 & \\ 3 & 7 & 10 & 13 & \\ 4 & & & & \end{array}.$$

Note from the example that the entries of S_+ increase by one across the rows from left to right, and the entries of S_- increase by one down the columns.

Let T be a standard tableau. Let $R(T)$ be the row group of elements of S_k which permute the entries within each row and $C(T)$ be the column group of T of permutations which permute the entries within each column. Now, for $\lambda \vdash k$, let

$$e_+ = e_\lambda^+ := \sum_{\sigma \in R(S_+)} h(\sigma),$$
$$e_- = e_\lambda^- := \sum_{\sigma \in C(S_-)} (-q^2)^{-\ell(w)} h(\sigma).$$

Then e_+ and e_- have the following important properties (see [10]) :

$$h(\sigma)e_+ = e_+h(\sigma) = q^{2\ell(\sigma)}e_+ \qquad \text{for } \sigma \in R(S_+), \tag{2.16}$$

$$h(\sigma)e_- = e_-h(\sigma) = (-1)^{\ell(\sigma)}e_- \qquad \text{for } \sigma \in C(S_-). \tag{2.17}$$

Let S and T be two standard tableaux of shape $\lambda \vdash k$. We let σ_S^T denote the permutation which transforms S to T. We also write $\sigma_\pm^T$ (respectively $\sigma_T^\pm$, $\sigma_\mp^\pm$) for $\sigma_{S_\pm}^T$ (respectively $\sigma_T^{S_\pm}$, $\sigma_{S_\mp}^{S_\pm}$). For example, if

$$\lambda = \begin{array}{ccccc} \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \\ \square & \square & \square & \square & \\ \square & & & & \end{array}, \quad \text{and } T = \begin{array}{ccccc} 1 & 2 & 4 & 7 & 14 \\ 3 & 5 & 6 & 8 & \\ 9 & 10 & 11 & 13 & \\ 12 & & & & \end{array},$$

then

$$\sigma_+^T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 2 & 4 & 7 & 14 & 3 & 5 & 6 & 8 & 9 & 10 & 11 & 13 & 12 \end{pmatrix},$$

and

$$\sigma_-^T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 3 & 9 & 12 & 2 & 5 & 10 & 4 & 6 & 11 & 7 & 8 & 13 & 14 \end{pmatrix}.$$

Let T be a standard tableau of shape $\lambda \vdash k$. Define $x_T(q) \in \mathcal{H}_k(q^2)$ as

$$x_T(q) = h\left(\sigma_-^T\right) e_\lambda^- \left(h\left(\sigma_-^T\right)\right)^{-1} h\left(\sigma_+^T\right) e_\lambda^+ \left(h\left(\sigma_+^T\right)\right)^{-1}. \tag{2.18}$$

Then there exists a $\xi \in \mathbb{C}(q)$ depending on the shape λ of T that

$$x_T(q)x_T(q) = \xi x_T(q).$$

Now Gyoja's q-analogue of the Young symmetrizer is defined as

$$y_T(q) := \frac{1}{\xi} x_T(q).$$

PROPOSITION 2.19 (See [10]). *The set of all $y_T(q)$'s is a set of primitive idempotents in the Iwahori-Hecke algebra $\mathcal{H}_k(q^2)$.*

Note when $q \to 1$, then $y_T(q)$ specializes to the Young symmetrizer y_T corresponding to T in the standard case.

3. Centralizer theorem

In this section we show that the action of $\mathcal{H}_k(q^2)$ on $V^{\otimes k}$ determines the full centralizer of $\mathfrak{U}_q(gl(m,n))$. First we recall the work by Berele and Regev.

THEOREM 3.1 (See [3]). *The image $\Psi(\mathbb{C}S_k)$ is given by*

$$\Psi(\mathbb{C}S_k) \cong \mathbb{C}S_k \Big/ \Big(\bigoplus_{\substack{\lambda \vdash k \\ \lambda \notin H(m,n;k)}} I_\lambda \Big) \cong \bigoplus_{\lambda \in H(m,n;k)} I_\lambda.$$

The representation Ψ_q of $\mathcal{H}_k(q^2)$ is completely reducible because $\mathcal{H}_k(q^2)$ is semisimple by Corollary 2.13 (see, for example, Section 25.7 of [4]). Moreover, we have

COROLLARY 3.2. *The image $\Psi_q(\mathcal{H}_k(q^2))$ is given by*

$$\Psi_q(\mathcal{H}_k(q^2)) \cong \mathcal{H}_k(q^2) \Big/ \Big(\bigoplus_{\substack{\lambda \vdash k \\ \lambda \notin H(m,n;k)}} I_\lambda^q \Big) \cong \bigoplus_{\lambda \in H(m,n;k)} I_\lambda^q.$$

Proof. First we prove

$$\dim_{\mathbb{C}(q)}(y_T(q)(V^{\otimes k})) \geq \dim_{\mathbb{C}}(y_T(W^{\otimes k})) \tag{3.3}$$

for each standard tableau T. Let $v_1, \ldots, v_l \in y_T(W^{\otimes k})$ be linearly independent vectors over $\mathbb{C}$. For each $i = 1, 2, \ldots, l$, write $v_i = y_T w_i$

for some $w_i \in W^{\otimes k}$. By identifying $V^{\otimes k} = W^{\otimes k} \otimes_{\mathbb{C}} \mathbb{C}(q)$, we have $v_1(q) := y_T(q)w_1, \ldots, v_l(q) := y_T(q)w_l$ are vectors in $y_T(q)(V^{\otimes k})$ such that $v_i(1) = v_i$ for all i.

We assume $v_1(q), \ldots, v_l(q)$ are linearly dependent over $\mathbb{C}(q)$. Then there exist $h_1(q), \ldots, h_l(q) \in \mathbb{C}(q)$ such that

$$h_1(q)v_1(q) + \cdots + h_l(q)v_l(q) = 0. \tag{3.4}$$

By clearing denominators, we may assume $h_i(q) \in \mathbb{C}[q]$ for all $1 \le i \le l$. And let $r \ge 0$ be the largest integer such that $(q-1)^r$ divides $h_i(q)$ in $\mathbb{C}[q]$ for all $1 \le i \le l$. Dividing both sides of (3.4) by $(q-1)^r$, we may also assume that $(q-1)$ does not divide $h_j(q)$ in $\mathbb{C}[q]$ for some $1 \le j \le l$. Then $0 \ne h_j(1) \in \mathbb{C}$, and by putting $q = 1$ in (3.4)

$$h_1(1)v_1(1) + \cdots + h_l(1)v_l(1) = 0.$$

Hence we have $v_1(1), \ldots, v_l(1)$ are linearly dependent over $\mathbb{C}$, and which is a contradiction. Therefore, $v_1(q), \ldots, v_l(q)$ are linearly independent, and we obtain (3.3).

Next we have the following inequalities in dimensions;

$$\begin{aligned}(m+n)^k = \dim_{\mathbb{C}(q)}(V^{\otimes k}) &\ge \sum_{\substack{T:\ \lambda\text{-tableau} \\ \lambda \in H(m,n;k)}} \dim_{\mathbb{C}(q)}(y_T(q)(V^{\otimes k})) \\ &\ge \sum_{\substack{T:\ \lambda\text{-tableau} \\ \lambda \in H(m,n;k)}} \dim_{\mathbb{C}}(y_T(W^{\otimes k})) \\ &= \dim_{\mathbb{C}}(W^{\otimes k}) = (m+n)^k.\end{aligned} \tag{3.5}$$

Therefore

$$\dim_{\mathbb{C}(q)}(V^{\otimes k}) = \sum_{\substack{T:\ \lambda\text{-tableau} \\ \lambda \in H(m,n;k)}} \dim_{\mathbb{C}(q)}(y_T(q)(V^{\otimes k})),$$

and

$$V^{\otimes k} = \bigoplus_{\substack{T:\ \lambda\text{-tableau} \\ \lambda \in H(m,n;k)}} y_T(q)(V^{\otimes k}).$$

Now the corollary follows. $\square$

As a consequence, we have

$$\dim_{\mathbb{C}}\big(\Psi(\mathbb{C}S_k)\big) = \dim_{\mathbb{C}(q)}\big(\Psi_q(\mathcal{H}_k(q^2))\big), \tag{3.6}$$

because $\dim_{\mathbb{C}} I_\lambda = \dim_{\mathbb{C}(q)} I^q_\lambda = d^2_\lambda$.

Our next goal is to prove that the image $\Psi_q(\mathcal{H}_k(q^2))$ of $\mathcal{H}_k(q^2)$ is in fact the full centralizer $\mathrm{End}_{\mathfrak{U}_q(gl(m,n))}(V^{\otimes k})$ of $\mathfrak{U}_q(gl(m,n))$ on $V^{\otimes k}$. The completely reducibility of $\mathfrak{U}_q(gl(m,n))$-module $V^{\otimes k}$ and the branching rule of $\mathfrak{U}_q(gl(m,n))$-modules for tensoring by $V = V(\epsilon_1)$ are obtained by G. Benkart, S. Kang and M. Kashiwara (see [1]) using the crystal basis theory of $\mathfrak{U}_q(gl(m,n))$:

PROPOSITION 3.7 (See Proposition 3.1 in [1]). *The $\mathfrak{U}_q(gl(m,n))$-module $V^{\otimes k}$ is completely reducible for all $k \geq 1$.*

THEOREM 3.8 (See Theorem 4.13 in [1]). *Let $\lambda_0 \vdash k$ be an (m,n)-hook shape. Then the tensor product $V(\lambda_0) \otimes V(\epsilon_1)$ has the following decomposition into irreducible $\mathfrak{U}_q(gl(m,n))$-modules:*

$$V(\lambda_0) \otimes V(\epsilon_1) = \bigoplus_{\lambda \in \Lambda} V(\lambda), \tag{3.9}$$

where λ runs over the set Λ of all (m,n)-hook shape Young diagrams obtained from λ_0 by adding a box to λ_0.

Let $\mathfrak{g}$ be a Lie superalgebra. A $\mathfrak{g}$-module V is *irreducible* if V does not have $\mathfrak{g}$ invariant $\mathbb{Z}_2$-graded subspace. Note that irreducible modules appearing in (3.9) do not have any $\mathbb{Z}_2$-graded or non-graded subspace which is $\mathfrak{U}_q(gl(m,n))$-invariant. Therefore, the Schur's lemma is still true in our case:

LEMMA 3.10. *Let $V(\lambda)$ and $V(\mu)$ be any two irreducible $\mathfrak{U}_q(gl(m,n))$-module appearing in the branching rule (3.9). Then*

$$\mathrm{End}_{\mathfrak{U}_q(gl(m,n))}(V(\lambda), V(\mu)) = \begin{cases} \mathbb{C}(q) & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

The branching rule of $gl(m,n)$-modules for tensoring by $W = \mathbb{C}^m \oplus \mathbb{C}^n$ was obtained by Berele and Regev [3], and it is same as (3.9). The centralizer theorem

$$\mathrm{End}_{\mathfrak{U}(gl(m,n))}(W^{\otimes k}) = \Psi(\mathbb{C}S_k) \tag{3.11}$$

for the non-quantum case was also obtained by Berele and Regev [3]. Because the branching rules for quantum and non-quantum cases are the same, we have from Lemma 3.10 and (3.6) and (3.11) that

$$\begin{aligned} \dim_{\mathbb{C}(q)} \mathrm{End}_{\mathfrak{U}_q(gl(m,n))}(V^{\otimes k}) &= \dim_{\mathbb{C}} \mathrm{End}_{\mathfrak{U}(gl(m,n))}(W^{\otimes k}) \\ &= \dim_{\mathbb{C}} \Psi(\mathbb{C}S_k) \\ &= \dim_{\mathbb{C}(q)} \Psi_q(\mathcal{H}_k(q^2)). \end{aligned} \tag{3.12}$$

Thus, we have arrived at the main result of this section:

THEOREM 3.13. *The centralizer algebra of the action of* $\mathfrak{U}_q(gl(m,n))$ *on* $V^{\otimes k}$ *is the image of Iwahori-Hecke algebra* $\mathcal{H}_k(q^2)$ *under* Ψ_q, *i.e.*

$$\mathrm{End}_{\mathfrak{U}_q(gl(m,n))}(V^{\otimes k}) = \Psi_q(\mathcal{H}_k(q^2)).$$

Moreover the *double centralizer theory* (see [5, Theorem 3.54] for this standard result) gives

COROLLARY 3.14. *The centralizer algebra of the action of* $\mathcal{H}_k(q^2)$ *on* $V^{\otimes k}$ *is the image of* $\mathfrak{U}_q(gl(m,n))$ *under* $\rho^{\otimes k}$, *i.e.*

$$\mathrm{End}_{\Psi(\mathcal{H}_k(q^2))}(V^{\otimes k}) = \rho^{\otimes k}\left(\mathfrak{U}_q(gl(m,n))\right).$$

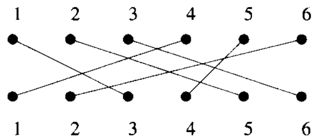
4. Symmetric groups and k-diagrams

It is helpful to represent permutations of the symmetric group S_k by k-diagrams in the rest of this paper. We give a little explanation of k-diagrams in this section.

Consider a graph with two rows of k vertices each, one above the other, and k edges such that each vertex in the top row is incident to precisely one vertex in the bottom row. There is a natural one-to-one correspondence between such k-diagrams and elements of the symmetric group S_k, which is illustrated by the following example:

EXAMPLE 4.1.

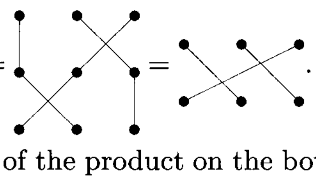
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 4 & 2 \end{pmatrix} =$$



Notice that the ith vertex in top row is incident to the $\sigma(i)$th vertex in bottom row.

Let d_1 and d_2 be the diagrams corresponding to permutations σ_1 and σ_2 respectively. Place d_1 below d_2 and identify the vertices in the bottom row of d_2 with the corresponding vertices in the top row of d_1. The resulting diagram corresponds to the product $\sigma_1\sigma_2$. For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (12)(23) =$$



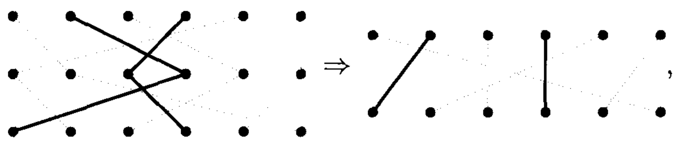
Note that we stack the left element of the product on the bottom of the diagram and the right element on the top.

With our identification of a permutation $\sigma \in S_k$ with its k-diagram, the *length* $\ell(\sigma)$ is the number of crossings of edges in the k-diagram identified with σ. And an expression $\sigma = s_{i_1} \cdots s_{i_j}$ of $\sigma \in S_k$ is reduced if $j = \ell(\sigma)$. For example, the k-diagram shown in Example 4.1 has 9 edge crossings, and so $\ell(\sigma) = 9$, and

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 4 & 2 \end{pmatrix} = s_3 s_4 s_2 s_3 s_4 s_5 s_1 s_2 s_3,$$

where the product on the right is a reduced expression for σ.

Now we may explain the situation when we have $\ell(\sigma_1\sigma_2) < \ell(\sigma_1) + \ell(\sigma_2)$ using k-diagrams as follows: In the following example,

(4.2) 

the crossings given by the darkened edges disappear in the product, so we have $\ell(\sigma_1\sigma_2) < \ell(\sigma_1) + \ell(\sigma_2)$.

5. Maximal vectors of $\mathfrak{U}_q(gl(m,n))$ modules

In this section we construct highest weight vectors of $\mathfrak{U}_q(gl(m,n))$ in $V^{\otimes k}$ explicitly using Gyoja's q-analogue of the Young symmetrizers. First we note the following lemma from [10].

LEMMA 5.1 (See [10]). *Let T be a standard tableau of shape $\lambda \vdash k$. Then there exists a $\gamma \in \mathbb{C}(q)$ such that*

$$e_\lambda^- \left(h\left(\sigma_-^T\right)\right)^{-1} h\left(\sigma_+^T\right) e_\lambda^+ = \gamma e_\lambda^- h\left(\sigma_+^-\right) e_\lambda^+.$$

Now let $\widehat{\Pi}(m,n;k)$ be

$$(5.2)\quad \widehat{\Pi}(m,n;k) := \left\{ (\mu,\nu) \;\middle|\; \begin{array}{c} \mu \vdash s, \nu \vdash t, \quad s+t=k, \\ \ell(\mu) \le m, \ell(\nu) \le n, \text{ and } \mu_m \ge \ell(\nu) \end{array} \right\}.$$

LEMMA 5.3 (See [2]). *There is a bijection between $H(m,n;k)$ and $\widehat{\Pi}(m,n;k)$ given by $\lambda \mapsto (\lambda^1, \lambda^2)$, where*

$$\lambda^1 = (\lambda_1, \ldots, \lambda_m), \quad \textit{and } \lambda^2 = (\lambda_1^2, \ldots, \lambda_n^2),$$

such that $\lambda_j^2 = \max\{\lambda_j^* - m, 0\}$ for $j = 1, \ldots, n$.

For a standard tableau T of shape $\lambda = (\lambda^1, \lambda^2) \in H(m,n;k)$, we let T_{λ^1} be the subtableau of T of shape λ^1 and let T_{λ^2} be the conjugate of the skew tableau $T/\ T_{\lambda^1}$. Then we associate to T a simple tensor $w_T = v_1 \otimes \cdots \otimes v_k$ in $V^{\otimes k}$ which is defined by

$$v_l = \begin{cases} t_i & \text{if } l \text{ is in the } i\text{th row of } T_{\lambda^1}, \\ u_j & \text{if } l \text{ is in the } j\text{th row of } T_{\lambda^2}. \end{cases}$$

Note that the weight of w_T is

$$\lambda = \lambda_1 \epsilon_1 + \cdots + \lambda_m \epsilon_m + \lambda_1^2 \delta_1 + \cdots + \lambda_n^2 \delta_n.$$

For a partition $\lambda \in H(m,n;k)$, we denote $w_\lambda^+ := w_{S_\lambda^+}$ and $w_\lambda^- := w_{S_\lambda^-}$.

EXAMPLE 5.4. Suppose $m = 2$ and $n = 3$. Let $\lambda = (4,2,2,1,1)$. Then $\lambda \in H(2,3;10)$.

$$\lambda = \ydiagram{4,2,2,1,1}$$

The corresponding pair (λ^1, λ^2) is given by $\lambda^1 = (4,2)$ and $\lambda^2 = (3,1)$ so that

$$\lambda \mapsto \left(\ydiagram{4,2}\, , \ydiagram{3,1} \right).$$

Let T be a standard tableau of shape λ such that

$$T = \begin{ytableau} 1 & 3 & 5 & 9 \\ 2 & 6 \\ 4 & 8 \\ 7 \\ 10 \end{ytableau} .$$

Then

$$T_{\lambda^1} = \begin{ytableau} 1 & 3 & 5 & 9 \\ 2 & 6 \end{ytableau}, \qquad T_{\lambda^2} = \begin{ytableau} 4 & 7 & 10 \\ 8 \end{ytableau}.$$

Then the simple tensor $w_T \in V^{\otimes 10}$ is given as

$$w_T = t_1 \otimes t_2 \otimes t_1 \otimes u_1 \otimes t_1 \otimes t_2 \otimes u_1 \otimes u_2 \otimes t_1 \otimes u_1.$$

THEOREM 5.5. *Let λ be a partition in $H(m,n;k)$ and T be a standard tableau of shape $\lambda=(\lambda_1,\lambda_2)$. Let $v_+=y_T(q)h\left(\sigma_+^T\right)w_\lambda^+$. Then v_+ is a maximal vector in $y_T(q)(V^{\otimes k})$ of weight $\lambda=\lambda_1\epsilon_1+\cdots+\lambda_m\epsilon_m+\lambda_1^2\delta_1+\cdots+\lambda_n^2\delta_n$. Hence $y_T(q)(V^{\otimes k})\cong V(\lambda)$, the irreducible $\mathfrak{U}_q(gl(m,n))$-module with highest weight λ.*

Proof. First note that when q goes to 1, $v_+=y_T(q)h\left(\sigma_+^T\right)w_\lambda^+$ goes to a maximal vector $y_T\sigma_+^Tw_\lambda^+$ of $y_T(W^{\otimes k})$ in the classical case (see [2]). Therefore, we know that v_+ is a nonzero vector.

Next observe that the weight of $y_T(q)h\left(\sigma_+^T\right)w_\lambda^+$ is same as the weight of w_λ^+, which is $\lambda=\lambda_1\epsilon_1+\cdots+\lambda_m\epsilon_m+\lambda_1^2\delta_1+\cdots+\lambda_n^2\delta_n$ by construction, where $\lambda_j^2=\max\{\lambda_j^*-m,0\}$.

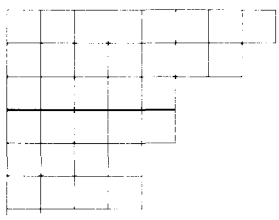
Now let us prove that v_+ is annihilated by $\rho^{\otimes k}(E_p)$ for $p\in I$. First we note

$$
\begin{aligned}
(5.6)\qquad v_+ &= y_T(q)h\left(\sigma_+^T\right)w_\lambda^+\\
&= \frac{1}{\xi}h\left(\sigma_-^T\right)e_\lambda^-\left(h\left(\sigma_-^T\right)\right)^{-1}h\left(\sigma_+^T\right)e_\lambda^+\left(h\left(\sigma_+^T\right)\right)^{-1}h\left(\sigma_+^T\right)w_\lambda^+\\
&= \frac{1}{\xi}h\left(\sigma_-^T\right)e_\lambda^-\left(h\left(\sigma_-^T\right)\right)^{-1}h\left(\sigma_+^T\right)e_\lambda^+w_\lambda^+.
\end{aligned}
$$

(1) $1\le p\le m$.

When we apply $\rho^{\otimes k}(E_p)$ on a simple tensor $w=b_{j_1}\otimes\cdots\otimes b_{j_k}$, we obtain a sum of simple tensors each of which has one tensor slot changed from w. In our case of $1\le p\le m$, either t_{p+1} changed to t_p $(1\le p\le m)$ or u_1 changed to t_m, because $\rho(E_p)=E_{p,p+1}$.

Recall S_λ^+ and S_λ^- are fixed λ-tableaux as given in Example 2.15. We fix another λ-tableau S_λ°. We give an example below and do not bother to give the precise definition of S_λ°. If

$\lambda =$  $\in H(3,5)$, then

$$S^\circ_\lambda = \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & \\ 16 & 17 & 18 & 19 & 20 & & & \\ 21 & 25 & 28 & 31 & 34 & & & \\ 22 & 26 & 29 & 32 & & & & \\ 23 & 27 & 30 & 33 & & & & \\ 24 & & & & & & & \end{array}.$$

Note that the entries in S°_λ increase across rows from left to right for the first $m(=3)$ rows, then entries increase down the columns.

Applying Lemma 5.1 for λ-tableau T, we have

$$e^-_\lambda \left(h\left(\sigma^T_-\right)\right)^{-1} h\left(\sigma^T_+\right) e^+_\lambda = c_1 e^-_\lambda h\left(\sigma^-_+\right) e^+_\lambda \tag{5.7}$$

for some $c_1 \in \mathbb{C}(q)$, and for λ-tableau S°_λ, we have

$$e^-_\lambda \left(h\left(\sigma^{S^\circ_\lambda}_-\right)\right)^{-1} h\left(\sigma^{S^\circ_\lambda}_+\right) e^+_\lambda = c_2 e^-_\lambda h\left(\sigma^-_+\right) e^+_\lambda \tag{5.8}$$

for some $c_2 \in \mathbb{C}(q)$. Note we know c_1 and c_2 are nonzero because the left sides of (5.7) and (5.8) are nonzero when we specialize $q \mapsto 1$. Combining (5.7) and (5.8) we have

$$e^-_\lambda \left(h\left(\sigma^T_-\right)\right)^{-1} h\left(\sigma^T_+\right) e^+_\lambda = c e^-_\lambda \left(h\left(\sigma^{S^\circ_\lambda}_-\right)\right)^{-1} h\left(\sigma^{S^\circ_\lambda}_+\right) e^+_\lambda \tag{5.9}$$

for some nonzero $c \in \mathbb{C}(q)$. Now from (5.6) and (5.9)

$$\begin{aligned} v_+ &= \frac{1}{\xi} h\left(\sigma^T_-\right) e^-_\lambda \left(h\left(\sigma^T_-\right)\right)^{-1} h\left(\sigma^T_+\right) e^+_\lambda w^+_\lambda \\ &= c' h\left(\sigma^T_-\right) \underbrace{e^-_\lambda \left(h\left(\sigma^{S^\circ_\lambda}_-\right)\right)^{-1} h\left(\sigma^{S^\circ_\lambda}_+\right) e^+_\lambda w^+_\lambda}_{(*)} \end{aligned}$$

for some $c' \in \mathbb{C}(q)$. We will show that $(*)$ is a linear combination of simple tensors which are killed by $\rho^{\otimes k}(E_p)$ for $1 \le p \le m$.

For brevity we reduce the notation and write i for b_i, $1 \le i \le m+n$. And we also write $\bar{l} := l + m$ for $l = 1, \ldots, n$. Then we may write the simple tensor w^+_λ as

(5.10)

$$\begin{aligned} w^+_\lambda = &\underbrace{1 \otimes \cdots \otimes 1}_{\lambda_1} \otimes \underbrace{2 \otimes \cdots \otimes 2}_{\lambda_2} \otimes \cdots \otimes \underbrace{m \otimes \cdots \otimes m}_{\lambda_m} \\ &\otimes \bar{1} \otimes \bar{2} \otimes \cdots \otimes \bar{\lambda}_{m+1} \otimes \bar{1} \otimes \cdots \otimes \bar{\lambda}_{m+2} \otimes \cdots \otimes \bar{1} \otimes \cdots \otimes \bar{\lambda}_{\ell(\lambda)}. \end{aligned}$$

Recall that $e_\lambda^+ = \sum_{\sigma\in R(S_+)} h(\sigma)$. For a fixed $\sigma \in R(S_+)$, we have $h(\sigma)w_\lambda^+$ is a linear combination of simple tensors which are the same as w_λ^+ up to scalar multiplications except the orders of tensor slots where vectors are from $\{u_1,\ldots,u_n\}$ are changed. This is because if σ moves entries in the lth row of S_+ and $l \le m$, then the action of $h(\sigma)$ on w_λ^+ does not create any new simple tensors because $\check{R}$ maps $t_i \otimes t_i$ to $q^2 t_i \otimes t_i$. And if σ moves entries in the lth row of S_+ and $l \ge m+1$, then $h(\sigma)w_\lambda^+$ is a linear combination of simple tensors which are the same as w_λ^+ up to scalar multiplications except the orders of tensor slots where vectors are from $\{u_1,\ldots,u_n\}$ are changed because

$$\check{R}(u_i\otimes u_j) = \begin{cases} -qu_j\otimes u_i + (q^2-1)u_i\otimes u_j & \text{if } i<j,\\ -qu_j\otimes u_i & \text{if } i>j.\end{cases}$$

Thus e_λ^+ maps (5.10) to a linear combination of simple tensors such that

$$(5.11)\quad \underbrace{1\otimes\cdots\otimes 1}_{\lambda_1}\otimes\underbrace{2\otimes\cdots\otimes 2}_{\lambda_2}\otimes\cdots\otimes\underbrace{m\otimes\cdots\otimes m}_{\lambda_m}\otimes\overline{*}\otimes\overline{*}\otimes\cdots\cdots\otimes\overline{*},$$

whose leftmost $(\lambda_1+\cdots+\lambda_m)$ tensor slots are the same as those of w_λ^+ and vectors in the rest tensor slots are from $\{u_1,\ldots,u_n\}$.

Now the next action is by $h\left(\sigma_+^{S_\lambda^\circ}\right)$. Note that $\sigma_+^{S_\lambda^\circ}$ is the permutation which transforms S_+ to S_λ°. Thus if the permutation $\sigma_+^{S_\lambda^\circ}$ acts on the k-fold tensor space by $\mathbb{Z}_2$-graded place permutation, then it maps w_λ^+ to $\pm w_{S_\lambda^\circ}$, i.e., it does not change the leftmost $(\lambda_1+\cdots+\lambda_m)$ tensor slots and it permutes only the elements $\overline{1},\ldots,\overline{n}$ in the tensor w_λ^+. Therefore the action of the Hecke element $h\left(\sigma_+^{S_\lambda^\circ}\right)$ on simple tensors of the form (5.11) produces a linear combination of simple tensors whose leftmost $(\lambda_1+\cdots+\lambda_m)$ tensor slots are the same as those of $w_{S_\lambda^\circ}$ and vectors in the rest tensor slots are from $\{u_1,\ldots,u_n\}$. Therefore we may also write $h\left(\sigma_+^{S_\lambda^\circ}\right)e_\lambda^+ w_\lambda^+$ as a linear combination of simple tensors such that

$$(5.12)\quad \underbrace{1\otimes\cdots\otimes 1}_{\lambda_1}\otimes\underbrace{2\otimes\cdots\otimes 2}_{\lambda_2}\otimes\cdots\otimes\underbrace{m\otimes\cdots\otimes m}_{\lambda_m}\otimes\overline{*}\otimes\overline{*}\otimes\cdots\cdots\otimes\overline{*}$$

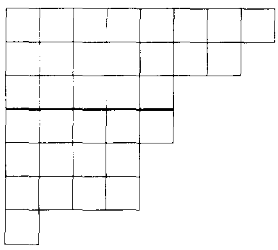
just like (5.11).

Our next goal is to show $\left(h\left(\sigma_-^{S_\lambda^\circ}\right)\right)^{-1}$ maps simple tensors of (5.12) to scalar multiples of simple tensors which are the same as w_λ^- except the order of tensor slots where vectors are from $\{u_1, \dots, u_n\}$ are changed. Note the simple tensor w_λ^- is of the form

$$
\begin{aligned}
w_\lambda^- =& 1 \otimes 2 \otimes \cdots \otimes m \otimes \underbrace{\bar{1} \otimes \cdots \otimes \bar{1}}_{(\lambda^2)_1} \otimes 1 \otimes 2 \otimes \cdots \otimes m \\
& \otimes \underbrace{\bar{2} \otimes \cdots \otimes \bar{2}}_{(\lambda^2)_2} \otimes \cdots \otimes \underbrace{1 \otimes 2 \otimes \cdots \otimes 1 \otimes 2}_{2(\lambda_2 - \lambda_3)} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{\lambda_1 - \lambda_2}.
\end{aligned}
\tag{5.13}
$$

We note that $\sigma_{S_\lambda^\circ}^-$ maps $w_{S_\lambda^\circ}$ to w_λ^- by $\mathbb{Z}_2$-graded place permutation. Decompose $\sigma_{S_\lambda^\circ}^-$ into a product of permutations $\sigma_1, \dots, \sigma_{\lambda_2}$ so that $\sigma_{S_\lambda^\circ}^- = \sigma_{\lambda_2} \cdots \sigma_1$ and $\ell\left(\sigma_{S_\lambda^\circ}^-\right) = \ell(\sigma_{\lambda_2}) + \cdots + \ell(\sigma_1)$ as in the followings ways: First we define a sequence $T^0 = S_\lambda^\circ$, T^1, $T^2, \dots,$ $T^{\lambda_2} = S_\lambda^-$ of λ-tableaux such that entries of T^l increase by one down the first l columns, and then other entries increase by one just like they are in the S_λ° for the rest of columns, i.e. entries increase by one across rows for the first m rows then entries increase by on down the columns. Then, we let $\sigma_l := \sigma_{T^{l-1}}^{T^l}$ for $l = 1, \dots, \lambda_2$. We give an example to explain our idea.

Example 5.14. If

$\lambda = (\lambda^1, \lambda^2) =$  $\in H(3,5)$,

then $T^0 = S_\lambda^\circ$, T^1, $\dots$, T^6 and $T^7 = S_\lambda^-$ are

$$
T^0 = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\cline{1-7}
16 & 17 & 18 & 19 & 20 \\
\cline{1-5}
21 & 25 & 28 & 31 & 34 \\
\cline{1-5}
22 & 26 & 29 & 32 \\
\cline{1-4}
23 & 27 & 30 & 33 \\
\cline{1-4}
24 \\
\cline{1-1}
\end{array}, \qquad
T^1 = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
1 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\hline
2 & 15 & 16 & 17 & 18 & 19 & 20 \\
\cline{1-7}
3 & 21 & 22 & 23 & 24 \\
\cline{1-5}
4 & 25 & 28 & 31 & 34 \\
\cline{1-5}
5 & 26 & 29 & 32 \\
\cline{1-4}
6 & 27 & 30 & 33 \\
\cline{1-4}
7 \\
\cline{1-1}
\end{array},
$$

$$T^2 = \begin{array}{|c|c|c|c|c|c|c|c}
1 & 8 & 14 & 15 & 16 & 17 & 18 & 19 \\
2 & 9 & 20 & 21 & 22 & 23 & 24 & \\
3 & 10 & 25 & 26 & 27 & & & \\
4 & 11 & 28 & 31 & 34 & & & \\
5 & 12 & 29 & 32 & & & & \\
6 & 13 & 30 & 33 & & & & \\
7 & & & & & & &
\end{array}, \qquad
T^3 = \begin{array}{|c|c|c|c|c|c|c|c}
1 & 8 & 14 & 20 & 21 & 22 & 23 & 24 \\
2 & 9 & 15 & 25 & 26 & 27 & 28 & \\
3 & 10 & 16 & 29 & 30 & & & \\
4 & 11 & 17 & 31 & 34 & & & \\
5 & 12 & 18 & 32 & & & & \\
6 & 13 & 19 & 33 & & & & \\
7 & & & & & & &
\end{array},$$

$$\vdots$$

$$T^6 = \begin{array}{|c|c|c|c|c|c|c|c}
1 & 8 & 14 & 20 & 26 & 30 & 32 & 33 \\
2 & 9 & 15 & 21 & 27 & 31 & 34 & \\
3 & 10 & 16 & 22 & 28 & & & \\
4 & 11 & 17 & 23 & 29 & & & \\
5 & 12 & 18 & 24 & & & & \\
6 & 13 & 19 & 25 & & & & \\
7 & & & & & & &
\end{array}, \qquad
T^7 = \begin{array}{|c|c|c|c|c|c|c|c}
1 & 8 & 14 & 20 & 26 & 30 & 32 & 34 \\
2 & 9 & 15 & 21 & 27 & 31 & 33 & \\
3 & 10 & 16 & 22 & 28 & & & \\
4 & 11 & 17 & 23 & 29 & & & \\
5 & 12 & 18 & 24 & & & & \\
6 & 13 & 19 & 25 & & & & \\
7 & & & & & & &
\end{array},$$

and $\sigma^{-}_{S^{\circ}_{\lambda}} = \sigma_7 \cdots \sigma_1$.

Next we argue $\ell(\sigma^{-}_{S^{\circ}_{\lambda}}) = \ell(\sigma_{\lambda_2}) + \cdots + \ell(\sigma_1)$. First see Figure 1 which illustrates the decomposition $\sigma^{-}_{S^{\circ}_{\lambda}} = \sigma_{\lambda_2} \cdots \sigma_1$ of Example 5.14 using k-diagrams ($k = 34$). Note, from the illustration, that the situation explained in (4.2) does not happen in the decomposition $\sigma^{-}_{S^{\circ}_{\lambda}} = \sigma_7 \cdots \sigma_1$, because once an entry moves to the left, then it is fixed by the following series of transformations σ_l so that it goes straight down, and it does not produce any further crossings of edges. This is clear from Figure 1. In general with our construction of σ_l's,

$$\ell(\sigma^{-}_{S^{\circ}_{\lambda}}) = \ell(\sigma_{\lambda_2} \cdots \sigma_1) = \ell(\sigma_{\lambda_2}) + \cdots + \ell(\sigma_1),$$

and so,

$$h\left(\sigma^{-}_{S^{\circ}_{\lambda}}\right) = h\left(\sigma_{\lambda_2} \cdots \sigma_1\right) = h\left(\sigma_{\lambda_2}\right) \cdots h\left(\sigma_1\right).$$

Next, we decompose each σ_l into a product of transpositions. For example, a decomposition for σ_4 in Figure 1 is explained in Figure 2. If each σ_l is expressed as a product of transpositions as shown in Figure 2, then the expression is reduced for the same reason as we have seen in Figure 1. So Hecke element $h\left(\sigma_l\right)$ is also a product $h_{j_1} h_{j_2} \cdots h_{j_t}$ which corresponds to the reduced expression $\sigma_l = s_{j_1} s_{j_2} \cdots s_{j_t}$, where $s_j = (j\ j+1)$. Note that we only exchange i and $i-1$ or i and $\overline{*}$ during the process of applying those transpositions.

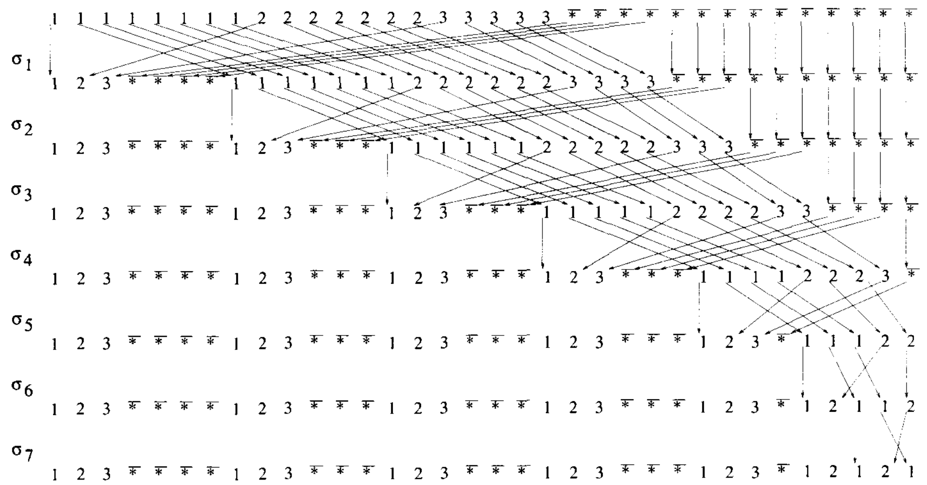


FIGURE 1. $\sigma^-_{S^\circ_\lambda} = \sigma_7 \cdots \sigma_1$

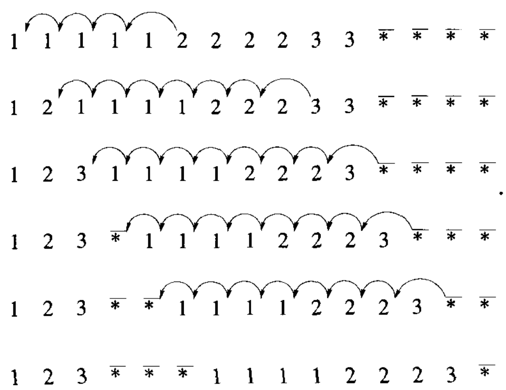


FIGURE 2. Decomposition of σ_4 into a product of transpositions

Because $\check{R}$ maps $t_j \otimes t_i$ to $qt_i \otimes t_j$ for $i < j$ and $u_j \otimes t_i$ to $qt_i \otimes u_j$, we have

$$\begin{aligned} \check{R}^{-1}: \quad & t_i \otimes t_j \mapsto q^{-1} t_j \otimes t_i && \text{for } i < j, \\ & t_i \otimes u_j \mapsto q^{-1} u_j \otimes t_i && \text{for all } i, j. \end{aligned}$$

Thus each $(h_j)^{-1}$ in the decomposition of $\left(h\left(\sigma_-^{S^\circ_\lambda}\right)\right)^{-1}$ acts on a simple tensor of the form (5.12) just like a $\mathbb{Z}_2$-graded place permutation except for scalar multiplication. Thus we have $\left(h\left(\sigma_-^{S_\lambda^\circ}\right)\right)^{-1}$ maps simple tensors of the form (5.12) to scalar multiples of simple tensors which are the same as w_λ^- except the order of tensor slots where vectors are from $\{u_1,\dots,u_n\}$ are changed. Hence $\left(h\left(\sigma_-^{S_\lambda^\circ}\right)\right)^{-1} h\left(\sigma_+^{S_\lambda^\circ}\right) e_\lambda^+ w_\lambda^+$ is a linear combination of simple tensors of the form

$$\begin{aligned}(5.15)\qquad &1\otimes 2\otimes\cdots\otimes m\otimes\underbrace{\bar{*}\otimes\cdots\bar{*}}_{(\lambda^2)_1}\otimes 1\otimes\cdots\otimes m\otimes\underbrace{\bar{*}\otimes\cdots\otimes\bar{*}}_{(\lambda^2)_2}\\ &\otimes\cdots\otimes\underbrace{1\otimes 2\otimes\cdots\otimes 1\otimes 2}_{2(\lambda_2-\lambda_3)}\otimes\underbrace{1\otimes\cdots\otimes 1}_{\lambda_1-\lambda_2}.\end{aligned}$$

Let w'_{λ_-} be a simple tensor of the form (5.15). Now we will show that $e_\lambda^- w'_{\lambda_-}$ is killed by $\rho^{\otimes k}(E_p)$ for $1\le p\le m$. The action of $\mathfrak{U}_q(gl(m,n))$ on $V^{\otimes k}$ commutes with the action of $\mathcal{H}_k(q^2)$, so that $\rho^{\otimes k}(E_p)e_\lambda^- w'_{\lambda_-} = e_\lambda^-\rho^{\otimes k}(E_p)w'_{\lambda_-}$. Note $\rho^{\otimes k}(E_p)w'_{\lambda_-}$ is a linear combination of simple tensors θ_α such that each θ_α has a tensor slot where t_p has been changed to t_{p-1} or u_1 has been changed to t_m. If the first case happens, then there is a $(j\,j+1)\in C(S_-)$ such that $h_j\theta_\alpha = q^2\theta_\alpha$ (note $\check{R}(t_i\otimes t_i) = q^2 t_i\otimes t_i$). So

$$\begin{aligned}e_\lambda^-\theta_\alpha &= q^{-2}e_\lambda^- h_j\theta_\alpha\\ &= q^{-2}(-1)e_\lambda^-\theta_\alpha.\end{aligned}$$

Hence, we obtain $e_\lambda^-\theta_\alpha = 0$ as expected.

If the second case happens, then for some $1\le a,b\le k$, where a,b are in the same column of S_-, the vectors in the ath and bth tensor slots are both t_m. Note the vectors between the ath and bth tensor slots are from $\{u_1,\dots,u_n\}$. Consider a Hecke element

$$h_a h_{a+1}\cdots h_{b-2}h_{b-1}(h_{b-2})^{-1}\cdots(h_a)^{-1}\in\mathcal{H}_k(q^2).$$

Without loss of generality we assume $a=1$. Then

$$\begin{aligned}h_a h_{a+1}&\cdots h_{b-2}h_{b-1}(h_{b-2})^{-1}\cdots(h_a)^{-1}(t_m\otimes u_i\otimes\cdots\otimes u_j\otimes t_m)\\ &= q^{-(b-a-2)}h_a h_{a+1}\cdots h_{b-2}h_{b-1}(u_i\otimes\cdots\otimes u_j\otimes t_m\otimes t_m)\\ &= q^{-(b-a-2)}q^2 h_a h_{a+1}\cdots h_{b-2}(u_i\otimes\cdots\otimes u_j\otimes t_m\otimes t_m)\\ &= q^{-(b-a-2)}q^2 q^{(b-a-2)}(t_m\otimes u_i\otimes\cdots\otimes u_j\otimes t_m)\\ &= q^2(t_m\otimes u_i\otimes\cdots\otimes u_j\otimes t_m),\end{aligned}$$

which explains that

$$h_a h_{a+1} \cdots h_{b-2} h_{b-1} (h_{b-2})^{-1} \cdots (h_a)^{-1} \theta_\alpha = q^2 \theta_\alpha.$$

Note $e_\lambda^- h_j = -e_\lambda^-$ and $e_\lambda^- (h_j)^{-1} = -e_\lambda^-$ if $(j\; j+1) \in C(S_-)$, so that

$$\begin{aligned} e_\lambda^- \theta_\alpha &= q^{-2} e_\lambda^- h_a h_{a+1} \cdots h_{b-2} h_{b-1} (h_{b-2})^{-1} \cdots (h_a)^{-1} \theta_\alpha \\ &= -q^{-2} e_\lambda^- \theta_\alpha. \end{aligned}$$

Therefore we have $e_\lambda^- \theta_\alpha = 0$ again in this case.

As a result we have shown here that $\rho^{\otimes k}(E_p) e_\lambda^- w'_\lambda = 0$, and consequently $\rho^{\otimes k}(E_p) v_+ = 0$ for $1 \le p \le m$.

(2) $m < p \le m + n - 1$.

This case is somewhat easier than the other case. Note $\rho^{\otimes k}(E_p) w_\lambda^+$ is a linear combination of simple tensors θ_α, such that one of u_{p+1} in tensor slots of w_λ^+ is changed to u_p. Then for some $(j\; j+1) \in R(S_+)$, $h_j \theta_\alpha = -\theta_\alpha$ because $\check{R}(u_i \otimes u_i) = -u_i \otimes u_i$. Thus,

$$\begin{aligned} e_\lambda^+ \theta_\alpha &= -e_\lambda^+ h_j \theta_\alpha \\ &= -q^2 e_\lambda^+ \theta_\alpha. \end{aligned}$$

Thus, we have $e_\lambda^+ \theta_\alpha = 0$, and $\rho^{\otimes k}(E_p) v_+ = 0$ for $p > m$ as expected. □

Now from Proposition 2.19, Theorem 3.13, Corollary 3.14, and Theorem 5.5, we have

THEOREM 5.16. *Let $\lambda \vdash k$ be a partition in $H(m,n)$. Let T be a standard tableau of shape λ. Then $\mathfrak{U}_q(gl(m,n))$-submodule $y_T(q)(V^{\otimes k})$ is isomorphic to the irreducible $\mathfrak{U}_q(gl(m,n))$-module $V(\lambda)$. Moreover as an $\mathcal{H}_k(q^2) \times \mathfrak{U}_q(gl(m,n))$ bimodule*

$$V^{\otimes k} \simeq \bigoplus_{\substack{\lambda \vdash k \\ \lambda \in H(m,n)}} H^\lambda \otimes V(\lambda),$$

where H^λ is the irreducible $\mathcal{H}_k(q^2)$-module labelled by λ.

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