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CHARACTERIZATIONS OF THE PARETO DISTRIBUTION BY CONDITIONAL EXPECTATIONS OF RECORD VALUES

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ABSTRACT. Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables with continuous cumulative distribution function $F(x)$. X_j is an upper record value of this sequence if $X_j > \max\{X_1, X_2, \cdots, X_{j-1}\}$. We define $u(n) = \min\{j | j > u(n-1), X_j > X_{u(n-1)}, n \geq 2\}$ with $u(1) = 1$. Then $F(x) = 1 - x^\theta$, $x > 1$, $\theta < -1$ if and only if $(\theta+1)E[X_{u(n+1)}|X_{u(m)} = y] = \theta E[X_{u(n)}|X_{u(m)} = y]$, $(\theta+1)^2 E[X_{u(n+2)}|X_{u(m)} = y] = \theta^2 E[X_{u(n)}|X_{u(m)} = y]$, or $(\theta+1)^3 E[X_{u(n+3)}|X_{u(m)} = y] = \theta^3 E[X_{u(n)}|X_{u(m)} = y]$, $n \geq m+1$.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed(i.i.d) random variables with a common continuous distribution function $F(x)$ and probability density function $f(x)$. Suppose $Y_n = \max\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of this sequence if $Y_j > Y_{j-1}, j > 1$. By convention X_1 is an upper as well as a lower record value. We can transform from upper records to lower records by replacing the original sequence of random variables by $\{-X_j, j \geq 1\}$. We define the record times $u(n)$ by $u(1) = 1$ and

$$u(n) = \min\{j | j > u(n-1), X_j > X_{u(n-1)}, n \geq 2\}.$$

The record times of the sequence $\{X_n, n \geq 1\}$ are random variables and are the same as those for the sequence $\{F(X_n); n \geq 1\}$. We know that

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the distribution of $u(n)$ does not depend on $F(x)$. Hence, the distribution of $u(n)$ can be determined by considering the uniform distribution $F(x) = x$. Also we will denote $L(n)$ as the indices where the lower record value occur. We will call the random variable $X \in \mathrm{PAR}(\theta_1, \theta_2)$ if the corresponding probability cumulative function $F(x)$ of x is of the form

$$F(x;\theta_1,\theta_2) = \begin{cases} 1 - \left(\dfrac{\theta_1}{x}\right)^{\theta_2}, & \theta_1 < x, \quad \theta_1 > 0, \quad \theta_2 > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Characterizations of the pareto distribution have been rarely studied in the literature. Nagaraja(1977) characterized the pareto distribution that if $E[h(X_{L_1})|X_{L_0} = y] = K(y)$ almost surely with respect to the distribution of X_{L_0} where $K(y)$ is a nondecreasing function on $[c, d]$, then $F(x)$ is uniquely determined.

In this paper we will give a characterization of the pareto distribution by considering conditional expectations of record values.

2. Results

THEOREM 1. $F(x) = 1 - x^{\theta},\ x > 1,\ \theta < -1$ *if and only if*

$$(1)\qquad \begin{aligned}&(\theta+1)E[X_{u(n+1)}|X_{u(m)} = y] \\ &= \theta E[X_{u(n)}|X_{u(m)} = y], \quad n \geq m+1.\end{aligned}$$

THEOREM 2. $F(x) = 1 - x^{\theta},\ x > 1,\ \theta < -1$ *if and only if*

$$(2)\qquad \begin{aligned}&(\theta+1)^2E[X_{u(n+2)}|X_{u(m)} = y] \\ &= \theta^2 E[X_{u(n)}|X_{u(m)} = y], \quad n \geq m+1.\end{aligned}$$

THEOREM 3. $F(x) = 1 - x^{\theta},\ x > 1,\ \theta < -1$ *if and only if*

$$(3)\qquad \begin{aligned}&(\theta+1)^3E[X_{u(n+3)}|X_{u(m)} = y] \\ &= \theta^3 E[X_{u(n)}|X_{u(m)} = y], \quad n \geq m+1.\end{aligned}$$

3. Proof

PROOF OF THEOREM 1. If $F(x) = 1-x^{\theta}$, then $E[X_{u(n)}|X_{u(m)} = y] = \left(\frac{\theta}{\theta+1}\right)^{n-m} y$ [see Ahsanullah(1995)]. Hence (1) holds. Conversely, suppose (1) holds. From Ahsanullah formula (1995) we can obtain the following equation.

$$\frac{\theta+1}{(n-m)!}\int_y^{\infty}\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m} xf(x)dx$$

$$(4) \qquad = \frac{\theta}{(n-m-1)!}\int_y^{\infty}\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m-1} xf(x)dx.$$

Since $F(x)$ is absolutely continuous, we can differentiate $(n-m+1)$ times both sides of (4) with respect to y and simplify, then we obtain the following equation.

$$(5) \qquad -yf(y) = \theta(1-F(y)) \text{ i.e. } -\frac{f(y)}{1-F(y)} = \frac{\theta}{y}.$$

Integrating both sides of (5) with respect to y, we get $F(y) = 1-y^{\theta}$. This completes the proof. □

PROOF OF THEOREM 2. If $F(x) = 1-x^{\theta}$, then $E[X_{u(n)}|X_{u(m)} = y] = \left(\frac{\theta}{\theta+1}\right)^{n-m} y$. Hence (2) holds. Conversely, suppose (2) holds. From Ahsanullah formula we can obtain the following equation.

$$\frac{(\theta+1)^2}{(n-m+1)!}\int_y^{\infty}\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m+1} xf(x)dx$$

$$(6) \qquad = \frac{(\theta)^2}{(n-m-1)!}\int_y^{\infty}\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m-1} xf(x)dx.$$

Since $F(x)$ is absolutely continuous, we can differentiate $(n-m+2)$ times both sides of (6) with respect to y and simplify, then we obtain the following differential equation.

$$(\theta+1)^2(-yf(y))$$
$$= \theta^2\left[(1-F(y)) - yf(y) - \frac{2(1-F(y))(-f^2(y)) - f'(y)(1-F(y))^2}{f^2(y)}\right]$$

i.e.

$$(2\theta+1)yf(y) + 3\theta^2(1-F(y)) + \theta^2\frac{f'(y)(1-F(y))^2}{f^2(y)} = 0. \tag{7}$$

Therefore, there exists a unique solution of the differential equation (7) that satisfies the prescribed initial conditions $F(1)=0\,, F'(1)=-\theta$. By the existence and uniqueness Theorem, we get $F(y)=1-y^\theta$ from (7).

This completes the proof. □

PROOF OF THEOREM 3. If $F(x)=1-x^\theta$, then $E[X_{u(n)}|X_{u(m)}=y]=\left(\frac{\theta}{\theta+1}\right)^{n-m}y$. Hence (3) holds. Conversely, suppose (3) holds. From Ahsanullah formula we can obtain the following equation.

$$\begin{aligned}&\frac{(\theta+1)^3}{(n-m+2)!}\int_y^\infty\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m+2}xf(x)dx\\ &=\frac{(\theta)^3}{(n-m-1)!}\int_y^\infty\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m-1}xf(x)dx.\end{aligned} \tag{8}$$

Since $F(x)$ is absolutely continuous, we can differentiate $(n-m+3)$ times both sides of (8) with respect to y and simplify, then we obtain the following differential equation.

$$\begin{aligned}&(\theta+1)^3(-yf(y))\\ &=\theta^3\Bigg[-\frac{6(1-F(y))(-f^2(y))-3f'(y)(1-F(y))^2}{f^2(y)}\\ &\quad+(1-F(y))-yf(y)-\frac{(f''(y)(1-F(y))^3}{f^6(y)}\\ &\quad+\frac{3(1-F(y))^2(-f(y))f(y))f^3(y)-3f^2(y)(f'(y))^2(1-F(y))^3}{f^6(y)}\Bigg]\end{aligned}$$

i.e.

$$(9) \quad (3\theta^2+3\theta+1)yf(y)+7\theta^3(1-F(y))+\theta^3\frac{6f'(y)(1-F(y))^2}{f^2(y)}$$

$$-\theta^3\frac{f''(y)(1-F(y))^3}{f^3(y)}+\theta^3\frac{3(f'(y))^2(1-F(y))^3}{f^4(y)}=0.$$

Therefore, there exists a unique solution of the differential equation (9) that satisfies the prescribed initial conditions $F(1) = 0$, $F'(1) = -\theta\,, F''(1) = -\theta(\theta-1)$. By the existence and uniqueness Theorem, we get $F(y) = 1 - y^{\theta}$ from (9).

This completes the proof. □

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