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ON THE WEAK LAW FOR WEIGHTED SUMS INDEXED BY RANDOM VARIABLES UNDER NEGATIVELY ASSOCIATED ARRAYS

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ABSTRACT. Let $\{X_{nk} \mid 1 \le k \le n, n \ge 1\}$ be an array of row negatively associated (NA) random variables which satisfy $P(|X_{nk}| > x) \le P(|X| > x)$. For weighed sums $\sum_{k=1}^{T_n} a_k X_{nk}$ indexed by random variables $\{T_n | n \ge 1\}$, we establish a general weak law of large numbers ($WLLN$) of the form $\left(\sum_{k=1}^{T_n} a_k X_{nk} - \nu_{[nk]}\right)/b_{[\alpha_n]}$ under some suitable conditions, where $\{a_n | n \ge 1\}$, $\{b_n | n \ge 1\}$ are sequences of constants with $a_n > 0$, $0 < b_n \to \infty$, $n \ge 1$, and $\{\nu_{\alpha_n} | n \ge 1\}$ is an array of random variables, and the symbol $[x]$ denotes the greatest integer in x.

1. Introduction

A finite family $\{X_1, \cdots, X_n\}$is said to be negatively associated(abbreviated to(NA)) if for any disjoint subsets of and any two coordinatewise nonde creasing or nonincreasing functions f_1 and f_2

$$(1.1) \qquad Cov(f_1(X_i, i \in A), f_2(X_j, j \in B)) \le 0.$$

An Infinite family of random variables is NA if every finite subfamily is NA. Alam and Lal Saxena ([4]) and Joag-Dev and Proschan ([8]) introduced the notion of negatively associated random variables. Concepts of NA random variables are of considerable uses in multivariate statistical analysis and system reliability, the notion of NA has received more and more attention recently. There have been several new results on limiting properties for NA sequences. It was discovered that limiting properties

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of NA sequences are quite similar to those of independent sequences. One can refer to Newman ([14]) for the central limit theorem, Matula ([13]) for the three series theorem, Shao ([16]) for moment inequalities, Su, etc ([17]) for the negative associate arrays. However, the little work has been done for arrays of row NA random variables.

Let $\{X_{nk}|1 \leq k \leq n, n \geq 1\}$ be an array of row NA random variables X which satisfy $P(|X_{nk}| > x) \leq P(|X| > x)$ on the underlying probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and set $\mathcal{F}_{n,j} = \sigma(X_{nk}, 1 \leq k \leq j), n \geq 1, j \geq 1$, and $\mathcal{F}_{n,0} = \{\Phi, \Omega\}, n \geq 1$. Let $\{T_n|n \geq 1\}$ be a sequence of positive integer valued random variables and $1 \leq \alpha_n \to \infty$ is a sequence of constants such that $P(T_n > \lambda\alpha_n) = o(1)$ for some positive integer λ and let $\{a_n|n \geq 1\}$ and $\{b_n|n \geq 1\}$ be sequence of constants with $a_n > 0,\ 0 < b_n \to \infty,\ n \geq 1$, and $\{\nu_{\alpha_n}|n \geq 1\}$ is a suitable constants. In this paper, we establish a general weak law of large numbers $(WLLN)$ of the form

$$(1.2) \qquad \left(\sum_{k=1}^{T_n} a_k X_{nk} - \nu_{[\alpha_n]}\right) / b_{[\alpha_n]} \text{ converges in probability to zero}$$

as $n \to \infty$, where the symbol $[x]$ denotes the greatest integer in x. The $WLLN$s of the form (1.2) for an array of random variables have been established by Gut ([7]), Kowalski and Rychlik ([10]), and Rosalsky and Teicher ([15]), etc. Our result is to extend and have a general weak law of large numbers $(WLLN)$ for weighted sums indexed by random variables $\{T_n|n \geq 1\}$ under an array of row NA random variables which satisfy $P(|X_{nk}| > x) \leq P(|X| > x)$ in practice. Throught this paper, a sequence $\{c_n|n \geq 1\}$ is defined by $c_n = b_n/a_n, n \geq 1$ and the symbol C denotes a generic constant $(0 < C < \infty)$ which is not necessary the same one in each appearance.

2. Preliminaries

LEMMA 2.1. (a) *Let* $\{X_{nk}|1 \leq k \leq n, n \geq 1\}$ *be an array of row* NA *random variables satisfying* $P(|X_{nk}| > x) \leq P(|X| > x)$. (b) *Let* $\{T_n|n \geq 1\}$ *be a sequence of positive integer valued random variables and* $1 \leq \alpha_n \to \infty$ *is a sequence of constants such that* $P(T_n > \lambda\alpha_n) = o(1)$ *and* $b_{[\lambda\alpha_n]} = O(b_{[\alpha_n]})$ *for some positive integer* λ, (c) *let* $\{a_n|n \geq 1\}$ *and* $\{b_n|n \geq 1\}$ *be a sequence of constants with* $a_n > 0,\ 0 < b_n \to \infty, n \geq 1$,

and suppose that one of the following conditions holds.
(2.1)
$$c_n = \frac{b_n}{a_n} \uparrow, \ \frac{b_n}{na_n} \downarrow, \ \sum_{k=1}^{n} a_k^2 = o(b_n^2), \ \text{and} \sum_{k=1}^{n} \frac{b_k^2}{k^2 a_k^2} = O\left(\frac{b_n^2}{\sum_{k=1}^{n} a_k^2}\right)$$

or

$$\frac{b_n}{a_n} \uparrow, \ \frac{b_n}{na_n} \to \infty,$$

$$\sum_{k=1}^{n} a_k^2 = O(na_n^2), \ \text{and} \sum_{k=1}^{n} \frac{b_k^2}{k^2 {a_k}^2} = O\left(\frac{b_n^2}{\sum_{k=1}^{n} a_k^2}\right) \tag{2.2}$$

or

$$\frac{b_n}{na_n} \uparrow, \ \text{and} \sum_{k=1}^{n} a_k^2 = O(na_n^2). \tag{2.3}$$

If

$$nP\{|X| > c_n\} = o(1) \tag{2.4}$$

then,

$$\sum_{k=1}^{n} a_k^2 P\{|X_{nk}| > c_n\} = o(a_n^2) \tag{2.5}$$

and

$$\sum_{k=1}^{n} a_k^2 E|X_{nk}|^2 I(|X_{nk}| \le c_n) = o(b_n^2). \tag{2.6}$$

Also, if (2.4) holds then,

$$\frac{\sum_{k=1}^{T_n} a_k (X_{nk} - X'_{nk})}{b_{[\alpha_n]}} \to 0 \text{ in probability as } n \to \infty, \tag{2.7}$$

where $X'_{nk} = -c_{[\alpha_n]} I(X_{nk} < c_{[\alpha_n]}) + X_{nk} I(|X_{nk}| \le c_{[\alpha_n]}) + c_{[\alpha_n]} I(X_{nk} > c_{[\alpha_n]})$.

Proof. First, We will use the idea of the proof of Theorem in Adler et al.([3]). To prove (2.5), observe that under (2.1)

$$\begin{aligned}&\frac{1}{a_n^2}\sum_{k=1}^{n} a_k^2 P\{|X_{nk}| > c_n\}\\ &\le \frac{Cb_n^2 P\{|X| > c_n\}}{a_n^2 \sum_{k=1} n(c_k^2/k^2)}\\ &\le \frac{Cb_n^2 P\{|X| > c_n\}}{n(c_n^2/n^2)} = CnP\{|X| > c_n\} = o(1)\end{aligned}$$

by (2.4). On the other hand, under (2.2) or (2.3)

$$\frac{1}{a_n^2}\sum_{k=1}^{n} a_k^2 P\{|X_{nk}| > c_n\} \le CnP\{|X|\} > c_n\} = o(1)$$

again by (2.4) and so (2.5) obtains. To prove (2.6), note that $c_n \uparrow$ under (2.1), (2.2) or (2.3) and that (2.2) and (2.3) individually ensure

$$\sum_{k=1}^{n} a_k^2 = o(b_n^2). \tag{2.8}$$

Thus (2.8) holds under (2.1), (2.2) or (2.3). Let $c_0 = 0$ and $d_n = c_n/n, n \ge 1$. Define an array $\{B_{nk},\ 0 \le k \le n,\ n \ge 1\}$ by

$$B_{nk} = \begin{cases} \left(\frac{1}{b_n^2}\sum_{i=1}^{n} a_i^2\right)\left(\frac{c_{k+1}^2 - c_k^2}{k}\right), & \text{for } 1 \le k \le n-1,\ n \ge 2\\ 0, & \text{for } k = 0,\ n,\ n \ge 1.\end{cases}$$

It will now be shown that $\{B_{nk},\ 0 \le k \le n,\ n \ge 1\}$ is a Toeplitz array, that is,

$$\sum_{k=0}^{n} |B_{nk}| = O(1) \tag{2.9}$$

and

$$B_{nk} \to 0 \text{ as } n \to \infty \text{ for all fixed } k \ge 0. \tag{2.10}$$

Clearly (2.8) implies (2.10). To verify (2.9), note that $B_{nk} \ge 0,\ 0 \le k \le n,\ n \ge 1$, since $c_n \uparrow$ and that $k \ge 1$,

$$\frac{c_{k+1}^2 - c_k^2}{k} = \frac{(k+1)^2 d_{k+1}^2 - j^2 d_k^2}{k} \le (k+3)d_{k+1}^2 - kd_k^2 \tag{2.11}$$

Then under (2.1), since $d_n \downarrow$, it follows from (2.11) that

$$\frac{c_{k+1}^2 - c_k^2}{k^2} \le 3d_k^2 = \frac{3c_k^2}{k^2},\ k \ge 1.$$

Hence, for $n \ge 2$,

$$\sum_{k=0}^{n} B_{nk} \le \left(\frac{3}{b_n^2}\sum_{i=1}^{n} a_i^2\right)\left(\sum_{k=1}^{n-1}\frac{c_k^2}{k^2}\right) = O(1)$$

and so (2.10) holds. Now under (2.5) or (2.6), for $n \ge 2$,

$$\begin{aligned}
\sum_{k=0}^{n} B_{nk} &\le \left(\frac{1}{b_n^2}\sum_{i=1}^{n} a_i^2\right)\left(\sum_{k=1}^{n-1}((k+3)d_{k+1}^2 - kd_k^2)\right) \text{ by (2.11)}\\
&\le \left(\frac{1}{b_n^2}\sum_{i=1}^{n} a_i^2\right)\left(\sum_{k=1}^{n-1}((k+3)d_{k+1}^2 - kd_k^2)\right)\\
&\quad + \left(\frac{3}{b_n^2}\sum_{i=1}^{n} a_i^2\right)\left(\sum_{k=1}^{n-1} d_{k+1}^2\right) \qquad (2.12)\\
&\le \frac{Cn}{c_n^2} nd_n^2 + \left(\frac{3}{b_n^2}\sum_{i=1}^{n} a_i^2\right)\left(\sum_{k=1}^{n-1} d_{k+1}^2\right)\\
&= C + \left(\frac{3}{b_n^2}\sum_{i=1}^{n} a_i^2\right)\left(\sum_{k=1}^{n-1} d_{k+1}^2\right).
\end{aligned}$$

Under (2.2), for $n \ge 2$,

$$\left(\frac{3}{b_n^2}\sum_{i=1}^{n} a_i^2\right)\left(\sum_{k=1}^{n-1} d_{k+1}^2\right) \le \left(\frac{C}{b_n^2}\sum_{i=1}^{n} a_i^2\right)\left(\sum_{k=1}^{n}\frac{c_n^2}{k^2}\right) = O(1).$$

Under (2.3), for $n \ge 2$,

$$\begin{aligned}
\left(\frac{3}{b_n^2}\sum_{i=1}^{n} a_i^2\right)\left(\sum_{k=1}^{n-1} d_{k+1}^2\right) &\le \frac{3d_n^2}{b_n^2}\left(\sum_{i=1}^{n} a_i^2\right)(n-1) \text{ since } d_n \uparrow\\
&= O(1).
\end{aligned}$$

Thus, under (2.2) or (2.3), recalling (2.12)

$$\sum_{k=0}^{n} B_{nk} = O(1)$$

and again (2.9) holds, there by proving that $\{B_{nk}, 0 \le k \le n, n \ge 1\}$ is a Toeplitz array. By (2.1) and the Toeplitz lemma (see, e.g., [9] or [12])

$$\sum_{k=0}^{n} B_{nk} k P\{|X| > c_k\} = o(1). \tag{2.13}$$

Next, note that

$$\begin{aligned}
& \frac{1}{b_n^2}\sum_{i=1}^{n} a_i^2 E|X|^2 I(|X| \le c_n) \\
= & \frac{1}{b_n^2}\sum_{i=1}^{n} a_i^2 \sum_{k=1}^{n} E|X|^2 I(c_{k-1} < |X| \le c_k) \\
\le & \frac{1}{b_n^2}\sum_{i=1}^{n} a_i^2 \sum_{k=1}^{n} c_k^2 P\{c_{k-1} \le |X| \le c_k\} \\
= & \frac{1}{b_n^2}\sum_{i=1}^{n} a_i^2 \sum_{k=1}^{n} c_k^2 (P\{|X| > c_{k-1}\} - P\{|X| > c_k\}) \\
= & \frac{1}{b_n^2}\sum_{i=1}^{n} a_i^2 (c_1^2 P\{|X| > 0\} - c_n^2 P\{|X| > c_n\} \\
& + \sum_{k=1}^{n} (c_{k+1}^2 - c_k^2) P\{|X| > c_k\}) \\
\le & \frac{1}{b_n^2}\sum_{i=1}^{n} a_i^2 \sum_{k=1}^{n} \frac{c_{k+1}^2 - c_k^2}{k} P\{|X| > c_k\} + o(1) \\
= & \sum_{k=0}^{n} B_{nk} k P\{|X| > c_k\} + o(1) \\
= & o(1) \text{ by (2.13)}.
\end{aligned}$$

Therefore (2.6) holds. Finally, since $X'_{nk} = -c_{[\alpha_n]} I(X_{nk} < c_{[\alpha_n]}) + X_{nk} I(|X_{nk}| \le c_{[\alpha_n]}) + c_{[\alpha_n]} I(X_{nk} > c_{[\alpha_n]})$, $\{X'_{nk}|1 \le k \le n, n \ge 1\}$ is still an array of row NA random variables.

To prove (2.7), for an arbitrary $\epsilon > 0$,

$$\begin{aligned}
& P\left\{\frac{|\sum_{k=1}^{T_n} a_k(X_{nk} - X'_{nk})|}{b_{[\alpha_n]}} > \epsilon\right\} \\
\leq\ & P\left\{\frac{|\sum_{k=1}^{T_n} a_k(X_{nk} - X'_{nk})|}{b_{[\alpha_n]}} > \epsilon, T_n \leq \lambda\alpha_n\right\} + P(T_n > \lambda\alpha_n) \\
\leq\ & P(\bigcup_{l=1}^{[\lambda\alpha_n]} \{|\sum_{k=1}^{l} a_k(X_{nk} - X'_{nk})| > \epsilon b_{[\alpha_n]}\}) + o(1) \text{ by } P(T_n > \lambda\alpha_n) \\
=\ & o(1) \\
=\ & P(max_{1\leq l\leq[\lambda\alpha_n]}|\sum_{k=1}^{l} a_k(X_{nk} - X'_{nk})| > \epsilon b_{[\alpha_n]}) + o(1) \\
\leq\ & \frac{1}{\epsilon^2 b^2_{[\alpha_n]}} E\left(\sum_{k=1}^{[\lambda\alpha_n]} a_k(X_{nk} - X'_{nk})\right)^2 + o(1) \\
\leq\ & \frac{1}{\epsilon^2 b^2_{[\alpha_n]}} \sum_{k=1}^{[\lambda\alpha_n]} a_k^2 c^2_{[\alpha_n]} P(|X_{nk}| \geq c_{[\alpha_n]}) \\
& + \frac{1}{\epsilon^2 b^2_{[\alpha_n]}} \sum_{k=1}^{[\lambda\alpha_n]} a_k^2 E X^2_{nk} I(|X_{nk}| \leq c_{[\alpha_n]}) + o(1) \\
\leq\ & \frac{1}{\epsilon^2 a^2_{[\lambda\alpha_n]}} \sum_{k=1}^{[\lambda\alpha_n]} a_k^2 P(|X| \geq c_{[\lambda\alpha_n]}) \\
& + \frac{1}{\epsilon^2 b^2_{[\lambda\alpha_n]}} \sum_{k=1}^{[\lambda\alpha_n]} a_k^2 E X^2 I(|X| \leq c_{[\lambda\alpha_n]}) + o(1) \\
=\ & o(1), \text{ by (2.6) and (2.5).}
\end{aligned}$$

This completes the proof. □

3. Main results

Applying Lemma 2.1 and Lemma 2.2, we establish some limit theorems as follows.

THEOREM 3.1. *Let* $\{X_{nk}|1 \leq k \leq n, n \geq 1\}$ *be an array of row* NA *random variables satisfying* $P(|X_{nk}| > x) \leq P(|X| > x)$. *Let*

$\{a_n|n \geq 1\}$, $\{b_n|n \geq 1\}$ and $\{T_n|n \geq 1\}$ satisfy the hypothesis of Lemma 2.1. If (2.4) hold then the WLLN

$$\frac{\sum_{k=1}^{T_n} a_k(X_{nk} - E(X'_{nk}|\mathcal{F}_{n,k-1}))}{b_{[\alpha_n]}} \to 0 \text{ in probability as } n \to \infty,$$

where $X'_{nk} = -c_{[\alpha_n]}I(X_{nk} < c_{[\alpha_n]}) + X_{nk}I(|X_{nk}| \leq c_{[\alpha_n]}) + c_{[\alpha_n]}I(X_{nk} > c_{[\alpha_n]})$.

Proof. Let $X'_{nk} = -c_{[\alpha_n]}I(X_{nk} < c_{[\alpha_n]}) + X_{nk}I(|X_{nk}| \leq c_{[\alpha_n]}) + c_{[\alpha_n]}I(X_{nk} > c_{[\alpha_n]})$. Then $\{X'_{nk}|1 \leq k \leq n, n \geq 1\}$ is still array of row NA random variables and also $(X'_{nk} - E(X'_{nk}|\mathcal{F}_{n,k-1}), 1 \leq k \leq T_n)$ is a martingale difference sequence. From the result of Lemma (2.1), it suffices to show that $\sum_{k=1}^{T_n} a_k(X'_{nk} - E(X'_{nk}|\mathcal{F}_{n,k-1}))/b_{[\alpha_n]} \to 0$ in probability as $n \to \infty$. For an arbitrary $\epsilon > 0$,

$$\begin{aligned}
& P\left\{\frac{|\sum_{k=1}^{T_n} a_k(X'_{nk} - E(X'_{nk}|\mathcal{F}_{n,k-1}))}{b_{[\alpha_n]}} > \epsilon\right\} \\
\leq\ & P\left\{\frac{|\sum_{k=1}^{T_n} a_k(X'_{nk} - E(X'_{nk}|\mathcal{F}_{n,k-1}))|}{b_{[\alpha_n]}} > \epsilon, T_n \leq \lambda\alpha_n\right\} \\
& + P(T_n > \lambda\alpha_n) \\
\leq\ & P(\cup_{l=1}^{[\lambda\alpha_n]}\{|\sum_{k=1}^{l} a_k(X'_{nk} - E(X'_{nk}|\mathcal{F}_{n,k-1})) > \epsilon b_{[\alpha_n]}\}) + o(1) \\
=\ & P(max_{1\leq l\leq[\lambda\alpha_n]}|\sum_{k=1}^{l} a_k(X_{nk}' - E(X'_{nk}|\mathcal{F}_{n,k-1})) > \epsilon b_{[\alpha_n]} + o(1) \\
\leq\ & \frac{1}{\epsilon^2 b_{[\alpha_n]}^2} E\left(\sum_{k=1}^{[\lambda\alpha_n]} a_k(X'_{nk} - EX'_{nk})\right)^2 + o(1) \\
& \text{by } h\acute{a}jek - R\acute{e}nji \text{ inequality} \\
\leq\ & \frac{1}{\epsilon^2 b_{[\alpha_n]}^2} \sum_{k=1}^{[\lambda\alpha_n]} a_k^2 EX_{nk}^2 I(|X_{nk}| \leq c_{[\alpha_n]}) \\
& + \frac{1}{\epsilon^2 b_{[\alpha_n]}^2} \sum_{k=1}^{[\lambda\alpha_n]} a_k^2 c_{[\alpha_n]}^2 P(|X_{nk}| \geq c_{[\alpha_n]}) + o(1)
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{\epsilon^2 a^2_{[\lambda\alpha_n]}} \sum_{k=1}^{[\lambda\alpha_n]} a_k^2 EX^2 I(|X| \le c_{[\lambda\alpha_n]}) \\
&\quad + \frac{1}{\epsilon^2 b^2_{[\lambda\alpha_n]}} \sum_{k=1}^{[\lambda\alpha_n]} a_k^2 P(|X| \ge c_{[\lambda\alpha_n]}) + o(1) \\
&= o(1), \text{ by (2.6) and (2.5).}
\end{aligned}$$

This completes the proof. □

The next corollary is an immediate consequence of Theorem 3.1 by taking $a_n = 1,\ b_n = n^{1/p},\ \alpha_n = n,\ n \ge 1$ and it is a similar result to Theorem 5.2.6 of Chow and Teicher ([5]) when $p = 1$.

COROLLARY 3.2. *Let $\{X_{nk}|1 \le k \le n, n \ge 1\}$ be an array of row NA random variables satisfying $P(|X_{nk}| > x) \le P(|X| > x)$. If*

$$nP\{|X| > n^{1/p}\} = o(1) \tag{3.1}$$

for some $1 \le p \le 2$ and $\{T_n|n \ge 1\}$ is a sequence of positive integer valued random variables satisfying $\frac{T_n}{n^{1/p}} \to c$ in probability where $0 < c < \infty$. Then

$$\sum_{k=1}^{T_n} \frac{X_{nk}}{T_n} - EX'_{nk} \to 0 \text{ in probability } n \to 0,$$

where $X'_{nk} = -n^{\frac{1}{p}} I(X_{nk} < n^{\frac{1}{p}}) + X_{nk} I(|X_{nk}| \le n^{\frac{1}{p}}) + n^{\frac{1}{p}} I(X_{nk} > n^{\frac{1}{p}})$.

Finally, we obtain the following corollary from Corollary 3.2 by taking $T_n = n$ and $p = 1$.

COROLLARY 3.3. *Let $\{X_{nk}|1 \le k \le n, n \ge 1\}$ be an array of row NA random variables satisfying $P(|X_{nk}| > x) \le P(|X| > x)$. If (3.1) holds for $p = 1$, then*

$$\sum_{k=1}^{T_n} \frac{X_{nk}}{n} - EX'_{nk} \to 0 \text{ in probability } n \to 0.$$

References

[1] A. Adler and A. Rosalsky, *Some general strong laws for weighted sums of stochastically dominated random variables*, Stochastic Anal. Appl. **5** (1987), 1–16.

[2] ———, *On the weak law of large number for normed weighted sums of iid random variables*, Internat, J. math. Sci. **14** (1991), 191–202.

[3] A. Adler and A. Rosalsky and R. L. Taylor, *A weak law for normed weight ed sums of random elements in Rademacher Type p Banach Space*, J. Multivariate Ana. **37** (1991), 259–268.
[4] K. Alam and K. M. Lal Saxena, *Positive dependence in multivariate distributio ns*, Comm. Statist. **A 10** (1981), 1183–1196.
[5] Y. S. Chow and H. Teicher, *Probability Theory : Independence, Interchange ability, Martingales, 2nd ed.*, Springer-Verlag, 1988.
[6] W. Feller, *A limiting theorem for random variables with infinite moments*, Amer. J. Math. **68** (1946), 257–262.
[7] A. Gut, *The weak law of large numbers for arrays*, Statist. Probab. Lett. **14** (1992), 49–52.
[8] K. Joag-Dev and F. Proschan, *Negative association of random variables with applications*, Ann. Statist. **11** (1983), 286.
[9] K. Knopp, *Theory and Application of infinite series*, 2nd english ed. Blackie, London, 1951.
[10] P. Kowalski and Z. Rychlik, *On the weak law of large numbers for randomly indexed partial sums for arrays*, Ann. Univ. Mariae Curie-sklodovska Sect. A LI. **1** (1997), 109–119.
[11] E. L. Lehmann, *Some concept of dependence*, Ann, Math. Statist. **37** (1966), 1137–1153.
[12] M. Loeve, *Probability Theory I*, 4th ed. Springer-Verlag, New York, 1970.
[13] P. Matula, *A note on the almost sure convergence of sums of negatively dependent random variables*, Statist. Probab. Lett. **15** (1992), 209–213.
[14] C. M. Newman, *Asymptotic independence and limit theorems for positively and negatively dependent random variables, in Inequalities in Statistics and Probability*, Y. L. Tong, ed.,IMS Lecture Notes Monogr. Ser. 5. Inst Math. Statist., Hayward, CA, 1984, pp. 127–140.
[15] A. Rosalsky and H. Teicher, *A limit theorem for double arrays*, Ann. probab. **9** (1981), 460–467.
[16] Q. M. Shao, *A comparison theorem on maximal inequalities between negatively associated and independent random variables*, Ann. Probab., in Press., 1998.
[17] C. Su, T. Hu and H. Liang, *On the logarithm law for strictly stationary and negatively associated arrays*, Theory of Probab. & Its Appli. **46** (2002), no. 2, 369–379.

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