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CHAOTIC HOMEOMORPHISMS OF C INDUCED BY HYPERBOLIC TORAL AUTOMORPHISMS AND BRANCHED COVERINGS OF $\bar{C}$

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Abstract. It is well known that there exists a regular branched covering map from T^2 onto $\bar{C}$ iff the ramification indices are $(2,2,2,2)$, $(2,4,4)$, $(2,3,6)$ and $(3,3,3)$. In this paper we construct (countably many) chaotic homeomorphisms induced by hyperbolic toral automorphism and regular branched covering map corresponding to the ramification indices $(2,2,2,2)$. And we also gave an example which shows that the above construction of a chaotic map is not true in general if the ramification indices is $(2,4,4)$ and also show that there are no chaotic homeomorphisms induced by hyperbolic toral automorphism and regular branched covering map corresponding to the ramification indices $(2,3,6)$ and $(3,3,3)$.

1. Introduction

Let $p : T^2 \to \bar{C}$ be a regular branched covering map from a torus onto the Riemann sphere. Then the ramification indices corresponding to $p : T^2 \to \bar{C}$ are $(2,2,2,2)$, $(2,4,4)$, $(2,3,6)$ or $(3,3,3)$. Conversely, if the ramification indices are $(2,2,2,2)$, $(2,4,4)$, $(2,3,6)$ or $(3,3,3)$ then there exists a regular branched covering map $p : T^2 \to \bar{C}$ whose ramification indices are $(2,2,2,2)$, $(2,4,4)$, $(2,3,6)$ or $(3,3,3)$ by Riemann-Hurwitz formula. See [7, p.232], [8, p.29].

Let $(2,2,2,2)$ be the ramification indices for the Riemann sphere. It is well known that the regular branched covering map corresponding to this, is the Weierstrass $\mathcal{P}$ function. Lattès [6] (See also 3, p.291) gives a chaotic rational function $R(z) = \frac{z^4+\frac{1}{2}g_2z^2+\frac{1}{16}g_2^2}{4z^3-g_2z}$ on $\overline{C}$ which is induced

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by the Weierstrass $\mathcal{P}$ function and the linear map $L(z) = 2z$ on the complex plane C.

Let $L : C \to C$ be a hyperbolic automorphism, i.e., $L(x) = Ax$, where A is a 2×2 integer matrix, $|det(A)| = 1$ and hyperbolic. Then a hyperbolic toral automorphism $L_A : T^2 \to T^2$, which is induced by L, is a chaotic map [3, p.192].

Now let $L_A : T^2 \to T^2$ be a hyperbolic toral automorphism and let $\mathcal{P} : T^2 \to \bar{C}$ be the Weierstrass $\mathcal{P}$ function. Then we have a commutative diagram which induces a homeomorphism of C onto itself (See Figure in Lemma 3.1). In this paper we construct (countably many) chaotic homeomorphisms, which is not holomorphic, induced by hyperbolic toral automorphism and regular branched covering map corresponding to the ramification indices $(2,2,2,2)$, which is Weierstrass $\mathcal{P}$ function (Theorem 3.1). And we also show that the above construction of a chaotic map is not true in general if the ramification indices is $(2,4,4)$. That is, there are no chaotic homeomorphisms induced by hyperbolic toral automorphism $L_A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and regular branched covering map corresponding to the ramification indices $(2,4,4)$ (Theorem 3.2). We also remark that the above construction of chaotic map is impossible when the ramification indices are $(2,3,6)$ and $(3,3,3)$ (Remark 3.1).

2. Hyperbolic toral automorphisms and branched coverings

In this section we briefly review chaotic map, hyperbolic toral automorphism and branched coverings of the Riemann sphere $\bar{C}$, based on [3], [4] and [7].

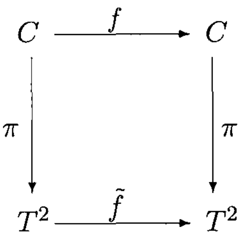
Let $f : M \to M$ be a map of metric space M. A map $f : M \to M$ is *chaotic* iff f has sensitive dependence on initial conditions, f is topologically transitive and periodic points are dense in M. We refer to the reader [3] for detailed definition and examples of chaotic map.

The following simple characterization of a chaotic map, which is proved by Touhey [9], is very useful to prove whether a map is chaotic or not. For example, we can easily check that the inverse of chaotic homeomorphism is also chaotic by this characterization. For the proof of the following proposition, he applied [1] which showed that sensitive dependence on initial conditions is implied by the remaining two conditions.

PROPOSITION 2.1. [9] *A map $f : M \to M$ is chaotic iff for every pair of non-empty open sets U and V of M, f has a periodic orbit Γ such that $\Gamma \cap U \neq \emptyset$ and $\Gamma \cap V \neq \emptyset$.*

We remark that holomorphic function $f : M \to M$ is chaotic iff $J(f) = M$, where $J(f)$ is the Julia set of f.

Let Λ be the lattice induced by $w_1, w_2 \in C$ with $w_1/w_2 \notin R$. Let π be the identification map of C such that $\pi(z) = \pi(z + nw_1 + mw_2)$ for $n, m \in Z$. Then we have the torus T^2 induced by π. In particular if $w_1 = 1$ and $w_2 = i$ then we call Λ the square lattice and if $w_1 = e^{\frac{\pi}{3}i}$ and $w_2 = e^{-\frac{\pi}{3}i}$ then we call Λ the triangular lattice, which we will use later. Now let $f : C \to C$ be a function such that $f(z) - f(z + nw_1 + mw_2)$ belongs to the lattice points for all points $z \in C$. It follows that $\pi \circ f(z) = \pi \circ f(z + nw_1 + mw_2)$ and therefore f induces a well defined map $\tilde{f} : T^2 \to T^2$ with $\tilde{f}(\pi(z)) = \pi \circ f(z)$ such that the following diagram commutes.



Let $L : C \to C$ be a linear map whose matrix representation is an integer matrix A. Then $\tilde{L}$ is clearly well-defined on T^2 which is induced by the square lattice. We call $\tilde{L}$ a *toral automorphism*, denoted by L_A.

DEFINITION 2.1. Let $L(x) = Ax$, where A is a 2×2 matrix satisfying
1. All entries of A are integers.
2. $det(A) = \pm 1$.
3. A is hyperbolic, i.e., A has no eigenvalues on unit circle. Then we call $L_A : T^2 \to T^2$ a *hyperbolic toral automorphism.*

PROPOSITION 2.2. [3, p.192] *Let L_A be a hyperbolic toral automorphism of T^2. Then L_A is chaotic map.*

We remark that the inverse of hyperbolic toral automorphism is also hyperbolic by the definition above. Therefore the inverse is also chaotic by the proposition above. We also can see that the inverse is chaotic since the inverse of chaotic map is also chaotic as we mentioned before Proposition 2.1.

Let $f: X \to X$ and $g: Y \to Y$ be two maps. f and g are said to be *topologically conjugate* iff there exists a homeomorphism $h: X \to Y$ such that $h \circ f = g \circ h$. The homeomorphism h is called a *topological conjugacy.* If the map h is finite to one and onto map in the above then f and g are said to be *topologically semi-conjugate.* Now we can easily prove the following lemma using Proposition 2.1.

LEMMA 2.1. *Let $f: X \to X$ and $g: Y \to Y$ be two maps. If f and g are topologically semi-conjugate and if $f: X \to X$ is chaotic then $g: Y \to Y$ is also chaotic.*

PROOF. Let U, V be open sets of Y. It suffice to find periodic orbit which intersects U and V by Proposition 2.1. Let p be a topological semi-conjugacy from X onto Y and consider $p^{-1}(U)$ and $p^{-1}(V)$. Since f is chaotic there exists a periodic orbit Γ such that $\Gamma \cap p^{-1}(U) \neq \emptyset$ and $\Gamma \cap p^{-1}(V) \neq \emptyset$. Note that $pf^n(x) = g^n p(x)$ for $x \in X$. Hence if Γ has period n with periodic point x_0 then $p\Gamma = \{p(x_0), pf(x_0), \cdots, pf^n(x_0)\} = \{p(x_0), gp(x_0), \cdots, g^n p(x_0)\}$ is also periodic orbit with period n. Since $p\Gamma \cap U \neq \emptyset$ and $p\Gamma \cap V \neq \emptyset$, this proves the lemma. □

Let$f: C \to S$ be a function from the complex plane to a Riemann surface. We call f *doubly periodic* iff $f(z + nw_1 + mw_2) = f(z)$ for $m, n \in Z$ and some $w_1, w_2 \in C$ such that $w_1/w_2 \notin R$. A holomorphic function f is called *meromorphic* iff it has no singular points other than poles. And a meromorphic doubly periodic function is called an *elliptic function.*

We remark that we may consider an elliptic function $f: C \to S$ as a meromorphic function from T^2 to S by the double periodicity of an elliptic function. In particular we may consider an elliptic function $f: C \to C$ as $f: T^2 \to \bar{C}$ by defining $f(z) = \infty$, where z is a pole.

Let Λ be the lattice induced by $w_1, w_2 \in C$ with $w_1/w_2 \notin R$. Define

$$\mathcal{P}(z) = \frac{1}{z^2} + {\sum}' \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right),$$

where the sum runs over all nonzero elements $w \in \Lambda$. Then $\mathcal{P}(z)$ is a meromorphic doubly periodic function with periods w_1 and w_2, i.e., an elliptic function. We call $\mathcal{P}(z)$ the Weierstrass $\mathcal{P}$ function.

The following basic important properties of Weierstrass $\mathcal{P}$ function can be found in several places in the literature on elliptic function theory, for example [10], [5].

Properties of the Weierstrass $\mathcal{P}$ function

(1) $\mathcal{P}(z)$ is an even function.

(2) $\mathcal{P}(z)$ has only one pole of order 2 at the point corresponding to the lattice points.

(3) $\mathcal{P}'(z) = -2\sum \frac{1}{(z-w)^3}$ is an odd function, where the sum runs over all $w \in \Lambda$.

(4) $\frac{w_1}{2}$, $\frac{w_2}{2}$ and $\frac{w_1+w_2}{2}$ are the only zeros of $\mathcal{P}'(z)$ and therefore $\mathcal{P}(z)$ has branch points of index 2 at $\frac{w_1}{2}$, $\frac{w_2}{2}$, and $\frac{w_1+w_2}{2}$. In fact, $\mathcal{P}'(\frac{w_1}{2}) = -\mathcal{P}'(-\frac{w_1}{2}) = -\mathcal{P}'(-\frac{w_1}{2} + w_1) = -\mathcal{P}'(\frac{w_1}{2})$. So $\mathcal{P}'(\frac{w_1}{2}) = 0$. Similarly, $\mathcal{P}'(\frac{w_2}{2}) = \mathcal{P}'(\frac{w_1+w_2}{2}) = 0$.

(5) Let $\mathcal{P}(\frac{w_1}{2}) = e_1$, $\mathcal{P}(\frac{w_2}{2}) = e_2$ and $\mathcal{P}(\frac{w_1+w_2}{2}) = e_3$. Then,
  (i) $e_1 + e_2 + e_3 = 0$,
  (ii) $e_1e_2 + e_2e_3 + e_3e_1 = -\frac{g_2}{4}$ and
  (iii) $e_1e_2e_3 = \frac{g_3}{4}$, where $g_2 = 60\sum_{w\neq 0} w^{-4}$ and $g_3 = 140\sum_{w\neq 0} w^{-6}$ for $w \in \Lambda$.

  Now let $w_1 = 1$ and $w_2 = i$, i.e., Λ is the square lattice. Then $\mathcal{P}(iz) = -\mathcal{P}(z)$ and $\mathcal{P}(\bar{z}) = \overline{\mathcal{P}(z)}$ since $i\Lambda = \Lambda$ and $\bar{\Lambda} = \Lambda$ respectively. Hence we have $\mathcal{P}(\frac{1}{2}i) = -\mathcal{P}(\frac{1}{2})$. Therefore $e_1 = -e_2$. Moreover $\mathcal{P}(\frac{1}{2}i) = \mathcal{P}(-\frac{1}{2}i) = \mathcal{P}(\overline{\frac{1}{2}i}) = \overline{\mathcal{P}(\frac{1}{2}i)}$. Hence e_1 and e_2 are real numbers and $e_3 = 0$ by Formula (i).

(6) The Weierstrass function not only gives us an example of an elliptic function but enables us to describe the structure of all elliptic functions. In fact,let $f(z)$ be an arbitrary elliptic function with periods w_1, w_2 and $\mathcal{P}(z)$ be the Weierstrass $\mathcal{P}$ function with the same periods. Then there exist rational functions $R(z)$ and $R_1(z)$ such that $f(z) = R(\mathcal{P}(z)) + R_1(\mathcal{P}(z))\mathcal{P}'(z)$. In particular, if f is an even function then $f(z) = cR(\mathcal{P}(z))$, where $R(\mathcal{P}(z)) = \frac{\prod(\mathcal{P}(z)-\mathcal{P}(a_i))}{\prod(\mathcal{P}(z)-\mathcal{P}(b_i))}$ where the $a_i's$ and $b_i's$ are those zeros and poles of f respectively which are not lattice points and a_i and b_i are the representatives of the pairs $(a_i, -a_i)$ and $(b_i, -b_i)$.

In particular, if all the poles of an elliptic functions lie at lattice points, then $f(z) = R(\mathcal{P}(z)) + R_1(\mathcal{P}(z))\mathcal{P}'(z)$ where R and R_1 are polynomial functions (See for details [10], [5]).

REMARK 2.1. By the property (6), if f is an even elliptic function which has the pole only on the lattice points, then $f(z) = R(\mathcal{P}(z))$ where R is a polynomial function, g_2 and g_3 are the numbers in Property (5).

DEFINITION 2.2. Let $f : M \to S$ be a holomorphic map between complex manifolds. If $f(z_0) = w_0$ with $f'(z_0) = 0$ then we call z_0 a branch point and w_0 a ramified point. Moreover if

$$f(z) = w_0 + c_n(z - z_0)^n + c_{n+1}(z - z_0)^{n+1} + \cdots$$

with $n \geq 1$ and $c_n \neq 0$, then we call the integer $n = n(z_0)$ the branch index or the local degree of f at z_0. Therefore $n(z) \geq 2$ if and only if z is branch point.

DEFINITION 2.3. Let $p : M \to S$ be a holomorphic surjective map between complex manifolds. Let B_p be the set of ramified points. Then we call the map $p : M \to S$ a branched covering map if and only if

(1) $p : M - p^{-1}(B_p) \to S - B_p$ is a topological covering map,
(2) for every $x \in S$ there exists a connected open set W of x such that for every component U of $p^{-1}(W)$,
  (i) $p^{-1}(x) \cap U = \{q\}$ is one point and
  (ii) $p|_U : U \to W$ is surjective and proper (hence finite).

A branched covering of S is a complex manifold M together with branched covering map $p : M \to S$.

A branched covering $p : M \to S$ is called *regular* if the covering transformation group G_p acts transitively, i.e., for $x, y \in p^{-1}(s)$, there exists $\sigma \in G_p$ such that $\sigma(x) = y$.

A regular branched covering map $p : M \to S$ has the following special properties:

(1) We can identify S with the quotient manifold M/G_p, and therefore the map p can be identified with the quotient map.
(2) The branch index $n(z)$ depends only on the target point $p(z)$, i.e., $n(z_1) = n(z_2)$ if $p(z_1) = p(z_2)$. So we can define the ramification function $\nu : S \to \{1, 2, 3, \cdots\}$ for p by setting $\nu(w)$ equal to the common value of $n(z)$ for all points z in $p^{-1}(w)$. We will call $\nu(w)$ the *ramification index* of w or will also call the *branch* index of w.

DEFINITION 2.4. A pair(S, ν) consisting of a complex manifold S and a ramification function $\nu : S \to \{1, 2, 3, \cdots\}$ which takes the value $\nu(w) = 1$ except at isolated points will be called an orbifold. In case S is a Riemann surface we call (S, ν) a Riemann surface orbifold.

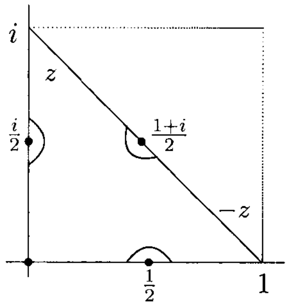
By the "ramification indices" we mean the list of values of the ramification function at the s ramified points, ordered so that $\nu(w_1) \leq \nu(w_2) \leq \cdots \leq \nu(w_s)$.

Let $p : T^2 \to \bar{C}$ be a regular branched covering map from a torus onto the Riemann sphere . Then the ramification indices corresponding to $p : T^2 \to \bar{C}$ are $(2,2,2,2)$, $(2,4,4)$, $(2,3,6)$ or $(3,3,3)$. Conversely, if $(\bar{C}, \nu)$ be an orbifold whose ramification indices are $(2,2,2,2)$, $(2,4,4)$, $(2,3,6)$ or $(3,3,3)$ then there exists a regular branched covering map $p : T^2 \to \bar{C}$ whose ramification indices are $(2,2,2,2)$, $(2,4,4)$, $(2,3,6)$

or $(3,3,3)$ by Riemann-Hurwitz formula. See [7, pp.229–233] and [8] for detailed definitions and properties of ramification indices and branched coverings.

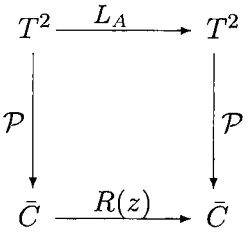
Let Λ be the square lattice. Now let $(2,2,2,2)$ be the ramification indices. Then the branched covering corresponding to the ramification indices $(2,2,2,2)$ is just the Weierstrass $\mathcal{P}$ function Moreover the geometry of Weierstrass $\mathcal{P}$ function is just quotient map from T^2 onto $\bar{C}$ such that z and $-z$ have the same image. [3, p.292]

Geometry of Weierstrass $\mathcal{P}$ function



The following is one of the famous examples of chaotic maps, so called Lattès example.

PROPOSITION 2.3. [6], (See also [3, pp.291–294]) *Let Λ be the lattice induced by 1 and i, so called the square lattice. Let $L(z) = 2z$ be the linear map of C onto itself and let $L_A : T^2 \to T^2$ be the toral automorphism induced by L. Then we have the following commutative diagram*



and therefore the map $R = \frac{z^4+\frac{1}{2}g_2z^2+\frac{1}{16}g_2^2}{4z^3-g_2z}$ *from $\bar{C}$ onto itself is chaotic.*

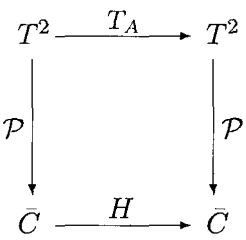
THEOREM 2.1. *Let $(2,4,4)$ be the ramification indices. Then the branched covering map from T^2 to $\bar{C}$ corresponding to this indices is $(\mathcal{P}(z))^2$ whose ramified points are 0, ∞ and e_1^2 with index 4,4, and 2 respectively.*

PROOF. Note that the discrete subgroup of covering transformations Γ of C corresponding to the ramification indices $(2,4,4)$ is generated by the maps $z \rightarrow z+a$ and $z \rightarrow iz$ for $a \in Z[i]$, where $Z[i]$ is a cyclic group generated by the imaginary number i [4]. Then C/Γ is biholomorphic to $\bar{C}$ and the quotient map $\pi : C \rightarrow \bar{C}$ is just the branched covering map corresponding to the ramified indices $(2,4,4)$. Therefore we have induced map $\pi : T^2 \rightarrow \bar{C}$ where T^2 is induced by the square lattice. Note that the map π is an even elliptic function with periods 1 and the imaginary number i, since $\pi(z) = \pi(iz) = \pi(i^2 z) = \pi(-z)$. Now we may see that 0 is a pole of index 4 and is the only pole since the sum of indices of zeros and the sum of indices of poles are equal. Since $\pi(z)$ is an even function, $\pi(z) = \prod(\mathcal{P}(z) - \mathcal{P}(a_i))$ where $a_i's$ are the zeros of π by the Property (6) of the Weierstrass $\mathcal{P}$ function. Recall that the sum of indices of zeros and the sum of indices of poles are equal. Therefore we may see that $\frac{1+i}{2}$ is the only zero of π of index 4. Recall that $\mathcal{P}(\frac{1+i}{2}) = 0$ (Property (5) of the Weierstrass $\mathcal{P}$ function). Consequently $\pi(z) = \mathcal{P}(z)^2$. Also recall that $\mathcal{P}(\frac{1}{2}) = e_1$ and $\mathcal{P}(\frac{i}{2}) = e_2$ are ramified points of index 2 for the Weierstrass $\mathcal{P}$ function and $e_1 = -e_2$ (Property (5) of the Weierstrass $\mathcal{P}$ function). So e_1^2 is the ramified point of index 2 for $\pi(z)$. □

3. Chaotic homeomorphisms of C

In this section we construct chaotic map induced by hyperbolic toral automorphisms and the Weierstrass $\mathcal{P}$ function, which is branched covering corresponding to the ramified indices $(2,2,2,2)$. And we also show that the above construction of chaotic maps is not true in general if the ramification indices is $(2,4,4)$. That is, there are no chaotic homeomorphisms induced by hyperbolic toral automorphism $L_A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and regular branched covering map corresponding to the ramification indices $(2,4,4)$. We also remark that the above construction of chaotic map is impossible when the ramification indices are $(2,3,6)$ and $(3,3,3)$.

LEMMA 3.1. *Let Λ be the square lattice. Let $L_A : T^2 \rightarrow T^2$ be a hyperbolic toral automorphism and let $\mathcal{P} : T^2 \rightarrow \bar{C}$ be the Weierstrass $\mathcal{P}$ function. Then there exist a chaotic homeomorphism $H : \bar{C} \rightarrow \bar{C}$ such that the following diagram commutes.*



PROOF. Recall that the Weierstrass $\mathcal{P}$ function is just quotient map $\pi : T^2 \to \bar{C}$ with $\pi(z) = \pi(-z)$. Note that $T_A(-z) = -T_A(z)$ and therefore we have well defined map H by defining $H(\mathcal{P}(z)) = \mathcal{P}T_A(z)$. Since ${T_A}^{-1}$ is also linear homeomorphism, $H : \bar{C} \to \bar{C}$ is homeomorphism. Moreover H is chaotic by the semi-conjugacy of the diagram by Lemma 2.1. □

PROPOSITION 3.1. *The map $H : \bar{C} \to \bar{C}$ in Lemma 3.1 is not holomorphic chaotic map.*

PROOF. Note that L maps the lattice points to the lattice points. Therefore $L_A(0) = 0$. Since $\mathcal{P}(0) = \infty$, $H(\infty) = \infty$. Therefore we may consider $H : \bar{C} \to \bar{C}$ with $H : C \to C$. Suppose $H : C \to C$ is bijective holomorphic. Then $H = az + b$ [2, p.301]. Then H is not chaotic since the Julia set of H, $J(H)$ is the closure of the set of repelling periodic point and H is chaotic iff $J(H) = C$ by definition. This proves the proposition. □

Since we have countably many hyperbolic toral automorphisms we now have one of our main theorems in this paper.

THEOREM 3.1. *Let $(2, 2, 2, 2)$ be the ramification indices and $\mathcal{P}(z)$ the Weierstrass $\mathcal{P}$ function, which is branched covering of the indices $(2, 2, 2, 2)$. Let $L_A : T^2 \to T^2$ be a hyperbolic toral automorphism induced by the square lattice Λ. Then there exist (countably many) chaotic homeomorphisms of C onto itself which is not analytic such that the diagram in Lemma 3.1 commutes.*

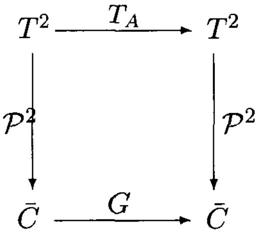
Now we consider the ramification indices $(2, 4, 4)$. Recall that the branched covering corresponding to this indices is $(\mathcal{P}(z))^2$. Also recall that $(\mathcal{P}(z))^2$ have ramified points 0, and ∞ of index 4 and ${e_1}^2$ of index 2.

We show that Theorem 3.1 is not true in general if the ramification indices is $(2, 4, 4)$ as the following theorem shows.

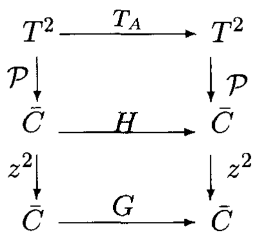
THEOREM 3.2. *Let $(2, 4, 4)$ be the ramification indices and $(\mathcal{P}(z))^2$ the branched covering corresponding to this indices. Let $L_A : T^2 \to T^2$*

be a hyperbolic toral automorphism induced by the square lattice Λ *and let* $L_A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. *Then there is no chaotic homeomorphism of* C *onto itself induced by* $(\mathcal{P}(z))^2$ *and* L_A.

PROOF. Suppose there exists chaotic homeomorphism such that the following diagram commutes.



Note that the branched covering map corresponding to this ramification indices $(2, 4, 4)$ is just composition of the Weierstrass $\mathcal{P}$ function and z^2. Hence we may consider the following diagram where H is the chaotic homeomorphism defined in Theorem 3.1.



Since the Weierstrass $\mathcal{P}$ function and z^2 are finite to one and onto map and the upper diagram commutes by Lemma 3.1, if the big diagram commutes then the lower diagram commutes. But it is impossible. In fact, let $\mathcal{P}(1/2) = e_1$, $\mathcal{P}(i/2) = e_2$ and $\mathcal{P}((1+i)/2) = e_3$. Then $e_1 = -e_2$ which are real and $e_3 = 0$ in this lattice(the square lattice). Therefore e_1 and e_2 have the same image by the map z^2. Now $T_A(1/2) = i/2$, $T_A(i/2) = (1+i)/2$ and $T_A((1+i)/2) = 1/2$. Hence $H(e_1) = e_2$, $H(e_2) = e_3 = 0$ and $H(e_3) = e_1$ by the commutativity of the upper diagram. Consider the lower diagram. Recall that e_1 and e_2 have the same image by the map z^2. But $H(e_1) = e_2$ and $H(e_2) = 0$ does not have the same image by the map z^2. So the lower diagram does not commute, and therefore the big diagram does not commute. Consequently we can not find the chaotic map such that the big diagram commutes. □

REMARK 3.1. Let $(2,3,6)$ and $(3,3,3)$ be the ramification indices. Then the transformation subgroup Γ corresponding to the ramification indices $(2,3,6)$ is generated by $z \to z+a$ and $z \to wz$ for $a \in Z[w]$ and $w = e^{\frac{\pi}{3}i}$ and the transformation subgroup Γ corresponding to the ramification indices $(3,3,3)$ is generated by $z \to z+a$ and $z \to w^2 z$ for $a \in Z[w]$ and $w = e^{\frac{\pi}{3}i}$ [4].

Then the branched covering maps corresponding to the indices $(2,3,6)$ and $(3,3,3)$ are doubly periodic maps whose periods are $e^{\frac{\pi}{3}i}$ and $e^{-\frac{\pi}{3}i}$. Consequently, the branched covering spaces are T^2 which is induced by the lattice generated by $e^{\frac{\pi}{3}i}$ and $e^{-\frac{\pi}{3}i}$, so called the triangular lattice. Now let $L : C \to C$ be a hyperbolic automorphism. Then L does not carry the lattice points to lattice points in the triangular lattice. Therefore we can not construct chaotic map of T^2 induced by these branched covering spaces and hyperbolic automorphism.

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