

CONVERGENCE IN METRIC DIFFERENTIAL GEOMETRY

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ABSTRACT. We use geometric properties of Gromov-Hausdorff-convergence to present a way to construct rough but natural invariants of metric geometry.

1. Introduction

Recent researches in riemannian geometry focus on the structure of families of objects rather than that of the objects themselves. And, in the last two decades of the 20th century, the so called Alexandrov spaces came to the center of riemannian geometry. Introduction of non-differentiable spaces into riemannian geometry made us understand the differential geometric invariants such as curvatures as simple metric concepts not necessarily involving the infinitesimality. This was the beginning of the metric differential geometry.

One of the central concept in the development of this field is that of Hausdorff distance revived by Gromov. Originally the Hausdorff distance had been considered as too rough a concept to deal with differentiable objects. But Gromov changed this concept into an intrinsic invariant with a clever use of iso-distance embeddings [3]. This concept focused mathematician's attention and there has been many researches relating the concept and geometric invariants.

In this paper we first survey on various geometric properties which is preserved under the Gromov-Hausdorff(GH)-convergence and also some unusual examples. This leads us to define a rough but natural geometric invariants which fits nicely with GH-convergence. Then we present a simple result explaining the naturality of such a viewpoint. Throughout

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this paper all the spaces are assumed to be a compact metric spaces unless specified otherwise.

2. Invariants of metric spaces

We introduce an example among the many results concerning families of geometric objects [2, 5, 6, 8].

THEOREM. *Let M be a compact differentiable manifold and G a compact Lie subgroup of $\text{Diff}(M)$. Then there exists a C^3 -neighborhood of G such that, for each compact Lie subgroup $F \subset U$ one can find a C^1 -diffeomorphism $\eta : M \rightarrow M$ with $\eta F \eta^{-1} \subset G$.*

This result explains the natural phenomenon that small changes possibly break symmetries but never create one and the result obtained is an optimal one from the given assumptions. But this theorem deals with riemannian metrics on a fixed manifold and therefore it lacks the consideration of the possible topological changes in the base manifold. Especially the collapsing and or the loss of topology in GH-convergences are natural phenomena which could not be covered under the methods of the theorem above.

Many results in the 1980's avoid such restrictions. Among which those of Gromov, Grove, etc. got peoples' attentions. To mention a few, there are the finiteness theorem of betti numbers and fundamental groups of Gromov, and the finiteness theorem of homotopy types by Grove and Petersen, the finiteness of diffeomorphism types by Cheeger and Anderson, etc. For details refer to Petersen's book [10].

The results mentioned above assume the bounds on sectional curvatures, diameters, volumes, etc. and sometimes on the Ricci curvatures, norms of the curvature operator, or their integrals. These are most important results in the modern riemannian geometry and give useful informations of the riemannian spaces involved. And what is interesting is that many of the assumptions in these theorems involve geometric invariants which are invariant under the GH-convergence. Conforming to this, many of these results has been generalized to the Alexandrov spaces.

In the study of metric geometry we find a surprising fact that the GH-limit of riemannian spaces with common lower curvature bound has the same curvature bound. This gives a first and rather unusual example of infinitesimal invariant preserved under the GH-convergence. Spaces with common lower curvature bound are known to show nice behaviors with

respect to the GH-convergence. A convergent sequence of minimizing geodesics converge to a minimizing geodesic when the spaces converge in GH-distance. When the spaces have the same curvature bounds the converse holds [7].

But this does not hold without the curvature assumptions. In fact, we can find a GH-convergent sequence of riemannian spaces $M_n \rightarrow X$ such that X has a minimizing geodesic which is not realized as a limit of minimizing geodesics of M_n [9]. The problems involving convergence of spaces turns out to be crucial in studying spaces with singularities, and one finds it important to distinguish between the properties which are preserved and not preserved under the GH-convergences.

There is a well known method of testing GH-convergence [4]. In simple words, it says that a GH-convergent sequence of spaces looks approximately same as a Lipschitz convergent nets. A convergence like this resembles the uniform convergence of mappings. Therefore it is hard to expect differential geometric properties to be preserved under such a convergence. This makes the above properties of preserving the curvature bounds and of being a geodesic are special properties.

First of all, topology is not, in general, preserved over to the GH-limits. This is obvious if we imagine a surface with a shrinking handle which vanishes in the limit. Topological invariants are not preserved under GH-convergence of spaces either. In the above example of the surface with shrinking handle, neither the fundamental group nor the homology groups are preserved.

Also, topological dimensions changes in GH-limits. In fact a one dimensional net-space consisting of horizontal and vertical lines with shrinking meshes converges to a two dimensional metric space. The situation is the same for the Hausdorff dimension because one can construct the von Koch curve as a limit of one dimensional inner metric spaces.

GH-limit does not preserve the curvatures in general. This is seen from the fact that some of the singularities are formed under curvature bound as in the forming of a cone point [11]. The only known exception in this case is the lower bound of the sectional curvature as was mentioned earlier. Not always fitted into the convergent case but it is worth mentioning that the quotient space by isometric group action preserves the curvature lower bound.

But the volume of a space does not behave well. As is seen in the net-space above the change in dimension induces change in volume. Even if we restrict ourselves to the case when the limit space has the same

dimension as that of the converging spaces, it is not the case as is seen in the following example. Volume is a continuous function of the space only with the lower bound of Ricci curvature [1].

EXAMPLE. Let $X, Y = [0, 1]$ and let the inner metric space M_n be the quotient space of $X \cup Y$ identifying the points $0, \frac{1}{n}, \frac{2}{n}, \dots, 1$ of both spaces in pairs. As n increases the space GH-converges to $[0, 1]$, and each M_n has its one dimensional volume 2, but the limit space has its volume 1. The volume is not continuous.

On the contrary, invariants defined combinatorially from the distance function behaves well under GH-convergence. For example, the diameter and the radius of compact spaces are continuous functions with respect to the GH-convergence. A bit more involved is the q -extent or excess which is defined as the maximum of the average distance between q points in a space.

Related to the above, the problem asking what the GH-limits of compact riemannian spaces look like is also important. For the riemannian spaces with common curvature lower bound, the GH-limit space is always an Alexandrov space with the same lower curvature bound. It is an interesting question if the family of Alexandrov spaces is the completion of the space of riemannian manifolds under the GH-distance.

3. A new concept

What we do in studying geometry is modelling the relationship of the positions of what we see in the reality. Currently we explain this using differentiable models and this is the central theme in differential geometry. But if one asks what do we see really, even if we do not answer by saying that we see the atoms, it is obvious that we do not analyze infinity of information that we receive through our eyes. Our sense of smoothness from many but finitely many set of information tells us that it is possible to talk about rough continuity and differentiability of such finite information. What we see in this section is that an old pseudo-invariant viewed in a new perspective gives a new invariant.

DEFINITION 1. X is a compact metric space and \mathcal{A} is an open cover of X . The homology (cohomology) of the simplicial complex defined as the nerve $N(\mathcal{A})$ of this cover is called the *covering homology (cohomology)* of X determined by \mathcal{A} and is denoted by $H_*(X; N(\mathcal{A}))$ ($H^*(X; N(\mathcal{A}))$).

We have the continuity property of this invariant under the GH-convergence.

THEOREM 1. *Let X_i be a sequence of compact metric spaces converging to X in GH-distance. If we take a finite open cover \mathcal{A} on X , then for sufficiently large i , there are exists open covers \mathcal{A}_i in X_i whose nerves $N(\mathcal{A}_i)$ are simplicially isomorphic to $N(\mathcal{A})$. Hence the covering homology and cohomology groups are isomorphic to each other for sufficiently large i 's (The number i are depending on the open cover \mathcal{A}).*

PROOF. Let $\mathcal{A} = \{U_\alpha | \alpha = 1, \dots, n\}$ be a finite open cover for X . Then all nonempty intersections of elements in \mathcal{A} are open and there are open balls contained in each intersection. Since the number of intersections are finite, there is a minimum of maximum radius's of the balls contained in each intersection. We denote this minimum by ℓ . Since X is a normal space and \mathcal{A} has a finite number of elements, we can find an open cover $\{V_\alpha | \alpha = 1, \dots, n\}$ in X such that $d_{GH}(V_\alpha, U_\alpha) < \ell$ and $\bar{V}_\alpha \subset U_\alpha$ for each $U_\alpha \in \mathcal{A}$. We set $\mathcal{B} = \{\bar{V}_\alpha\}$ which is a closed cover of X . Now we will show that $N(\mathcal{A})$ is simplicially isomorphic to $N(\mathcal{B})$.

Suppose that $U_{\alpha_1} \cap \dots \cap U_{\alpha_k} \neq \emptyset$ and $U_\beta \cap (U_{\alpha_1} \cap \dots \cap U_{\alpha_k}) \neq \emptyset$. We choose the corresponding elements \bar{V}_{α_i} and \bar{V}_β in \mathcal{B} . There is a point z such that $B(z; \ell) \subset (U_{\alpha_1} \cap \dots \cap U_{\alpha_k}) \cap U_\beta$. It is easily seen that $z \in (\bar{V}_{\alpha_1} \cap \dots \cap \bar{V}_{\alpha_k}) \cap \bar{V}_\beta$ as $d_{GH}(\bar{V}_{\alpha_i}, U_{\alpha_i}) < \ell$ and $d_{GH}(\bar{V}_\beta, U_\beta) < \ell$.

Let \mathcal{C} be a collection of all nonempty intersections of elements in \mathcal{B} . Then, all elements in \mathcal{C} are closed and the number of elements in \mathcal{C} is finite.

If $\bigcap_{\alpha=1}^{\alpha=n} \bar{V}_\alpha \neq \emptyset$, there is only one n -simplex. On the other hand, if $\bigcap_{\alpha=1}^{\alpha=n} \bar{V}_\alpha = \emptyset$, then there are distinct elements in \mathcal{C} (i.e., there are $A, B \in \mathcal{C}$ with $A \cap B = \emptyset$).

In the later case, there is a positive number η such that $\text{dist}(A, B) = \inf\{d(a, b) | a \in A, b \in B\} > \eta$ for all distinct elements $A, B \in \mathcal{C}$.

For $\frac{\eta}{2}$, we have that $d_{GH}(X_i, X) < \frac{\eta}{2}$ for sufficiently large i . Let d_i be a metric on $X_i \amalg X$ extending the metrics on X_i and X such that $X_i \subset B_{d_i}(X; \frac{\eta}{2})$ and $X \subset B_{d_i}(X_i; \frac{\eta}{2})$.

The proof is complete if we can find a nerve in X_i which is simplicially isometric to $N(\mathcal{B})$.

We set

$$\begin{aligned} F_\alpha^i &:= \{x_i \in X_i \mid d(x_i, x) < \frac{\eta}{2}, \text{ for each } x \in \bar{V}_\alpha\} \\ &= \bigcup_{x \in \bar{V}_\alpha} (B_{d_i}(x; \frac{\eta}{2}) \cap X_i) \\ &= B_{d_i}(\bar{V}_\alpha; \frac{\eta}{2}) \cap X_i. \end{aligned}$$

Then, $\mathcal{A}_i = \{F_\alpha^i\}$ is an open cover of X_i . Now, we show that $N(\mathcal{A}_i)$ is simplicially isometric to $N(\mathcal{B})$.

It is clear that $F_\alpha^i \cap F_\beta^i \neq \emptyset$ if $\bar{V}_\alpha \cap \bar{V}_\beta \neq \emptyset$ from the construction of F_α^i and F_β^i .

Suppose that $F_\alpha^i \cap F_\beta^i \neq \emptyset$, where $\bar{V}_\alpha \cap \bar{V}_\beta = \emptyset$.

If we choose a point $x_i \in F_\alpha^i \cap F_\beta^i$, then we can find points $x_\alpha \in \bar{V}_\alpha$ and $x_\beta \in \bar{V}_\beta$ such that

$$d_i(x_\alpha, x_\beta) \leq d_i(x_\alpha, x_i) + d_i(x_i, x_\beta) < \eta.$$

But, $\text{dist}(\bar{V}_\alpha, \bar{V}_\beta) > \eta$. This implies that $F_\alpha^i \cap F_\beta^i = \emptyset$.

In the former case, $\bigcap_{\alpha=1}^n \bar{V}_\alpha \neq \emptyset$, we can easily find an open cover in X_i such that the nerve of this cover is one n -simplex if $d_{GH}(X, X_i) < D < \infty$.

In fact, we set

$$\begin{aligned} E_\alpha^i &:= \{x_i \in X_i \mid d(x_i, x) < \frac{D}{2}, \text{ for each } x \in \bar{V}_\alpha\} \\ &= B_{d_i}(\bar{V}_\alpha; \frac{D}{2}) \cap X_i. \end{aligned}$$

Obviously, the collection $\{E_\alpha^i\}$ is an open cover for X_i and $\bigcap_\alpha E_\alpha^i \neq \emptyset$. \square

It is well-known that the Čech homology(cohomology) is the direct limit of the covering homologies. But, the homology(cohomology) is not preserved by the Gromov-Hausdorff convergence. Moreover, in Theorem 1, if we choose a slightly different open cover from \mathcal{A} , then we may have the different covering homology(cohomology) for the same space X_i .

Here is an example explaining this.

EXAMPLE. In Figure 1, the arcs l_i such that $l_i \cap S^1 = \{a, b\}$ converge to the small subarc ab in S^1 and $l_i \cup \{\text{the small subarc } ab\}$ are simple closed curves. If we set $X_i = S^1 \cup l_i$, then X_i converge to S^1 in GH-sense.

We know that $H_1(X_i) = \mathbb{Z} \oplus \mathbb{Z}$. But, the homology of the limit space is $H_1(S^1) = \mathbb{Z}$.

In Figure 2, l_i is an arc from a to b whose length is equal to $\frac{1}{i} +$ the length of the small subarc in S^1 from a to b and $\frac{1}{i+1} < \text{dist}(l_i, S^1) \leq \frac{1}{i}$. Let $X_i = S^1 \cup l_i$. Then, $\lim_{i \rightarrow \infty} d_{GH}(X_i, X) = 0$.

Let $\mathcal{A} = \{B(p, \pi + 2\epsilon), B(q, \pi + 2\epsilon)\}$ be an open cover in S^1 for small positive number ϵ , where $d(p, q) = \pi$ (d is an inner metric in S^1). We choose i such that $\frac{1}{2\epsilon} > i > \frac{1}{4\epsilon}$. We consider a slightly different open cover $\mathcal{A}' = \{B(p', \pi + 2\epsilon), B(q', \pi + 2\epsilon)\}$ where p' and q' are ϵ -shifting points of p and q , respectively, toward the south pole S (i.e., $d(p, p') = d(q, q') = \epsilon$).

Then, \mathcal{A} is also an open cover for X_i . But, \mathcal{A}' is not a cover for X_i .

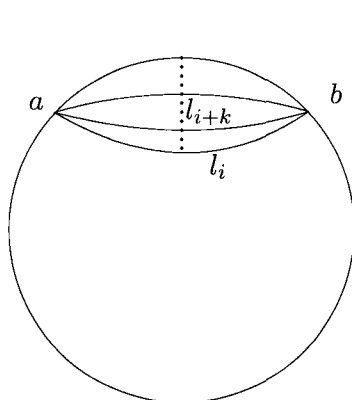


Figure 1

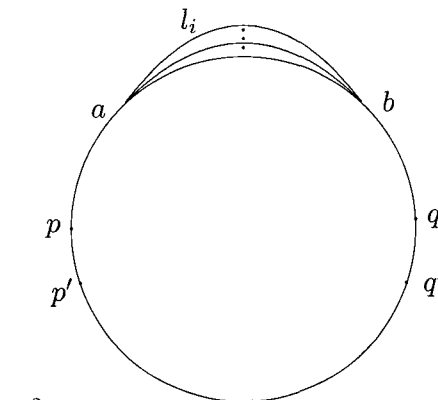


Figure 2

4. Concluding remarks

The preceding theorem says an obvious fact that a combinatorial concept from a finite number of information behaves continuously under the GH-convergence which is natural for such information. But this result suggests that it is possible to develop the theory of pseudo-continuum consisting of finite information in parallel with the theory of the continuum with infinite information. And as is obvious, all the results we have about smooth spaces pose a question to be answered in this category. These ideas center around semi-infinitesimal to global concepts rather

than infinitesimal ones and it is possible to give another viewpoint in metric differential geometry.

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