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A NEW PROOF OF MACK'S CHARACTERIZATION OF PCS-ALGEBRAS

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Abstract. Let A be a C^*-algebra and K_A its Pedersen's ideal. A is called a PCS-algebra if the multiplier $\Gamma(K_A)$ of K_A is the multiplier $M(A)$ of A. J. Mack [5] characterized PCS-algebras by weak compactness on the spectrum of A. We give a new simple proof of this Mack's result using the concept of semicontinuity and N. C. Phillips' description of $\Gamma(K_A)$.

1. Introduction and preliminaries

Let A denote a C^*-algebra and $M(A)$ the set of all multipliers of A in the bidual A^{**}, the enveloping von Neumann algebra of A. In their memoir [4], Lazar and Taylor made an extensive study on the multipliers (double centralizers) $\Gamma(K_A)$ of Pedersen's ideal K_A of A and defined a special class of C^*-algebras. They called A a PCS-algebra if $\Gamma(K_A) = M(A)$ holds and showed that for C^*-algebras with Hausdorff spectrum, an algebra is PCS if and only if the spectrum is pseudocompact. Then John Mack [5] gave a topological characterization for the spectrum of an arbitrary PCS-algebra. He showed that a C^*-algebra is PCS if and only if the spectrum is weakly compact.

Later, N. C. Phillips [8] got a new description of $\Gamma(K_A)$ as an inverse limit of C^*-algebras (pro C^*-algebra). In this paper, we give a new simple proof of J.Mack's theorem characterizing PCS-algebras using the Phillips' result and the concept of semicontinuity in [3] and [6]. Let $\widehat{A}$ be the spectrum of A with Jacobson topology(hull-kernel topology). For an open subset C of $\widehat{A}$, note that there corresponds a closed two sided

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ideal denoted by $I(C)$. For a closed two sided ideal I, the spectrum of I is homeomorphic to the open subset $\widehat{A} \setminus hull(I)$ of $\widehat{A}$ (See [7]).

Now we will briefly review the theory of weakly compact sets.

DEFINITION 1.1. A topological space is called *weakly compact* (or *feebly compact, lightly compact*) if each infinite pair-wise disjoint family of open subsets has an accumulation point (i.e., a point for which each neighborhood meets infinitely many members of the family). Note that this is equivalent to the requirement that each locally finite pairwise disjoint collection of open subsets be finite.

THEOREM 1.2. ([5, Theorem 2.4], [2, Theorem 1]) *For any topological space X, the following are equivalent.*

(a) *X is weakly compact.*
(b) *For each decreasing sequence $\{V_n\}$ of nonempty open subsets of X, the intersection $\bigcap \overline{V}_n$ is nonempty.*
(c) *Each nonnegative, locally bounded, lower semicontinuous function on X is bounded.*
(d) *Each locally finite collection of nonempty open subsets of X is finite.*

PROOF. (a)$\Rightarrow$(b): Let $\{V_n\}$ be a decreasing sequence of nonempty open subsets of X such that $\bigcap \overline{V_n}$ is empty. We will construct an infinite locally finite class of pairwise disjoint open subsets of X. Put $U_1 = X \setminus \overline{V}_{m_1}$, where V_{m_1} is the first element in $\{V_n\}_{n\in\mathbf{N}}$ whose closure is not X, and let $U_k = V_{m_{(k-1)}} \setminus \overline{V}_{m_k}$, where V_{m_k} is the first element in $\{V_n\}_{n>m_{(k-1)}}$ whose closure does not contain $V_{m_{(k-1)}}$. Then the class $\{U_k\}$ is clearly an infinite pairwise disjoint family of non-empty open subsets of X. Since every member of an open cover $\{X \setminus \overline{V_n}\}_{n\in\mathbf{N}}$ of X can intersect with only finitely many members of $\{U_k\}$, the class $\{U_k\}$ is locally finite.

(b)$\Rightarrow$(c): Let $f : X \to \mathbf{R}^+$ be a locally bounded and lower semicontinuous function. Since f is locally bounded, there exists a real valued, upper semicontinuous function g such that $f \leq g$. Define $V_n = \{x : f(x) > n\}$. Then $\overline{V_n} \subset \{x : g(x) \geq n\}$. Since g is strictly real-valued, $\bigcap \overline{V_n}$ is empty. By (c) it follows that V_n is empty for some n. Hence $f(x) \leq n$ for all x in X.

(c)$\Rightarrow$(d): Suppose $\mathcal{W}$ is an infinite, locally finite family of nonempty open subsets of X. Pick a sequence $\{W_n\}$ of distinct elements of $\mathcal{W}$. Define $f(x) = sup\{n : x \in W_n\}$ for $x \in \bigcup W_n$ and $f(x) = 0$ otherwise. Then for any real number $r \geq 0$, it is easy to show that $\{x : f(x) >$

$r\} = \bigcup_{n>r} W_n$. Thus f is lower semicontinuous. For any $x \in X$, there is a neighborhood G which meets at most finitely many members of $\mathcal{W}$, hence f is bonded on G. However f is unbounded since the sets $\{W_n\}$ are distinct nonempty and $f(x_n) \geq n$ for $x_n \in W_n$. Thus we have constructed an unbounded, locally bounded, lower semicontinuous function on X.

(d)$\Rightarrow$(a): This is obvious by definition. □

2. Main result

A topological space X is called *pseudocompact* if every continuous complex valued function on X is bounded. When A is commutative and has pseudocompact spectrum the equality $\Gamma(K_A) = M(A)$ holds, that is, every multiplier of Pedersen's ideal K_A is bounded. So Lazar and Taylor made the following definition:

DEFINITION 2.1. Let A be a C^*-algebra. A is called a *PCS-algebra* if each element of $\Gamma(K_A)$ is bounded.

THEOREM 2.2. (Phillips [8, Theorem 4]) *Let A be a C^*-algebra. Then, for any approximate identity (e_λ) for A contained in K_A, we have*

$$\Gamma(K_A) \cong \varprojlim_{a \in (K_A)_+} M(I_a) \cong \varprojlim_{\lambda} M(I_{e_\lambda}),$$

where I_b is the closed two sided ideal generated by b.

REMARK. This theorem enables us to consider the self-adjoint elements of $\Gamma(K_A)$ as (unbounded) self-adjoint operators on the universal Hilbert space H_u of A: If h is in $\Gamma(K_A)_{sa}$, the set of self-adjoint elements of $\Gamma(K_A)$, then there exists h_λ in $\mathrm{M}(\mathrm{I}_{e_\lambda})_{sa}$ for all λ such that $h_\mu p_\lambda = h_\lambda$ for $\mu \geq \lambda$ where p_λ is the open central projection corresponding to I_{e_λ}; and h can be identified with the net (h_λ). For each λ, h_λ gives a projection valued measure $E_S^\lambda(h_\lambda)$ on $p_\lambda H_u$. Note that $(E_S^\lambda(h_\lambda))_\lambda$ is an increasing net of projections in A^{**} for every Borel set $S \subset \mathbf{R}$. Now we let $E(S)$ be the limit projection of $(E_S^\lambda(h_\lambda))_\lambda$ in A^{**}. Then $(E(S))$ forms a projection valued measure on H_u. Hence the operator that corresponds to $(E(S))$ is a densely defined self-adjoint operator on H_u, and will be denoted again by h. Then $hp_\lambda = h_\lambda$ for all λ and ah and hb belong to K_A for all a and b in K_A.

Note that for a self-adjoint element x in the bidual A^{**} of A, Pedersen [6] defined a bounded real valued function $\check{x}$ on the spectrum $\widehat{A}$ by the formula $\widetilde{\pi}(c(x)) = \check{x}(t)1$ whenever $(\pi, H) \in t$. Here, $c(x)$ denotes the central cover of x and $\widetilde{\pi}$ the canonical extension of π to A^{**}. If $x \geq 0$ then $\check{x}(t) = \|\widetilde{\pi}(x)\|$. Note that we may apply the same formula to define $\check{h}$ for self-adjoint h in $\Gamma(K_A)$. In this case $\check{h}$ may not be bounded but it is bounded on each open subset $\widehat{A} \setminus hull(I_{e_\lambda})$ $(hp_\lambda \in M(I_{e_\lambda}))$.

Now we give a new simple proof of J. Mack's Theorem on PCS-algebras.

THEOREM 2.3. ([5, Theorem 3.5]) *A C^*-algebra A is a PCS-algebra if and only if $\widehat{A}$ is weakly compact.*

PROOF. Assume that $\widehat{A}$ is weakly compact and m is a selfadjoint element in $\Gamma(K_A)$. Let $f_n = (0 \vee id) \wedge n$. Then $f_n(m_+)$ is bounded and hence it belongs to $M(A)_{sa}$ by the spectral mapping theorem [4, Theorem 5.19]. By Pedersen [6, Theorem 2.5 & Theorem 4.6], $f_n(m_+)\check{}$ is a nonnegative lower semicontinuous function on $\widehat{A}$ for all $n \in \mathbf{N}$. Since m_+ is the limit of the increasing sequence $\{f_n(m_+)\}$, $(m_+)\check{}$ is also the limit of the increasing sequence $\{f_n(m_+)\check{}\}$. Therefore it must be a nonnegative lower semicontinuous function on $\widehat{A}$. For every π in $\widehat{A}$, $\pi(e_\lambda) \neq 0$ for some e_λ. This implies that π belongs to an open subset $\widehat{A} \setminus hull(I_{e_\lambda})$. Now Theorem 2.2 above shows the boundedness of m on $p_\lambda H_u$ and hence $(m_+)\check{}$ is bounded on the open subset, that is, $(m_+)\check{}$ is locally bounded . Similarly $((-m)_+)\check{}$ is also a non-negative locally bounded lower semicontinuous function on $\widehat{A}$. By the given assumption and Theorem 1.2 above, they are bounded. This shows that m is bounded and so are arbitrary elements in $\Gamma(K_A)$.

If we assume that $\widehat{A}$ is not weakly compact, then there exists an infinite locally finite sequence $\{W_n\}$ of open subsets of $\widehat{A}$. Pick a nonzero positive element $a_n \in I(W_n)$ for each $n \in N$, and let $m = \sum_{n=1}^{\infty} na_n/||a_n||$. Then the local finiteness of $\{W_n\}$ implies the local boundedness of m, that is, mp_λ is bounded for all λ. Since every finite sum $\sum_{n=1}^{N} na_n/||a_n||$ is in $M(A)_{sa}$, mp_λ belongs to $M(I_{e_\lambda})_{sa}$ by [3, 2.18] and [6, Theorem 2.5]. Again by Theorem 2.2 above, m is a multiplier of K_A, and it is clearly unbounded. Hence we are done by the contrapositive statement. □

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