Anti-Plane Shear Behavior of an Arbitrarily Oriented Crack in Bonded Materials with a Nonhomogeneous Interfacial Zone

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The anti-plane shear problem of bonded elastic materials containing a crack at an arbitrary angle to the graded interfacial zone is investigated in this paper. The interfacial zone is modeled as a nonhomogeneous interlayer of finite thickness with the continuously varying shear modulus between the two dissimilar, homogeneous half-planes. Formulation of the crack problem is based upon the use of the Fourier integral transform method and the coordinate transformations of basic field variables. The resulting Cauchy-type singular integral equation is solved numerically to provide the values of mode III stress intensity factors. A comprehensive parametric study is then presented of the influence of crack obliquity on the stress intensity factors for different crack size and locations and for different material combinations, in conjunction with the material nonhomogeneity within the graded interfacial zone.

Key Words: Bonded Materials, Oblique Crack, Functionally Graded Materials, Nonhomogeneous Interfacial Zone, Singular Integral Equation, Mode III Stress Intensity Factors

1. Introduction

With the recent advances in the field of functionally graded materials, the utilization of such materials in the form of an interlayer for joining the dissimilar constituents has been suggested as one of the highly viable and effective applications in engineering practice. As a result, several shortcomings arising from the stepwise property mismatch inherent in the conventional bonded or layered media could be alleviated (Lee and Erdogan, 1995). This is because the graded materials exhibit the relatively smooth spatial variations of thermomechanical properties, which can be treated as nonhomogeneous solids on a continuum mechanics basis (Suresh and Mortensen, 1997).

In the context of fracture mechanics, considerable attention has also been directed toward the investigation of effects of the material nonhomogeneity on the near-tip field. For the material nonhomogeneity specified in terms of the exponential variation of the elastic modulus, the inverse square-root character of the crack-tip field was proposed by Delale and Erdogan (1983). It has thereafter been confirmed that the above conjecture holds true for the general nonhomogeneous materials (Eischen, 1987 ; Schovanec and Walton, 1988 ; Martin, 1992 ; Jin and Noda, 1994), provided the elastic properties are simply continuous and piecewise differentiable near and at the crack tip. In particular, using the eigenfunction expansion, Eischen (1987) derived the near-tip field in a nonhomogeneous cracked body, demonstrating that the corresponding stress field possesses not only the square-root singularity, but also the same angular distributions around the crack tip as those in the homogeneous material. The evolution of the functionally graded materials thus appears to have resolved the issues of long-standing interest in the analytical studies of fracture mechanics, i.e., the oscillatory or the nonsquare-root

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crack-tip singularities in piecewise homogeneous bonded media with the ideal interface of zero thickness (Rice, 1988 ; Romeo and Ballarini, 1995). Subsequently, a series of benchmark solutions to some crack problems that entail graded nonhomogeneous properties has been obtained by Erdogan and his coworkers, which is well documented in a review paper (Erdogan, 1995). One of the most recent contributions, among others, is attributable to Choi (2001) where the plane problem of an arbitrarily oriented crack in bonded half-planes with a graded interfacial zone was considered.

The objective of this paper is to investigate the anti-plane shear counterpart of the problem previously considered by one of the authors (Choi, 2001). It should now be mentioned that the anti-plane shear problems may have practical significance and applications in their own right in such situations as torsion or three-dimensional problems in which the third fracture mode is separable. In the latter case, superimposed on the solution of the in-plane deformation problem, the solution supplemented in this paper would aid further in studying the general fracture problem in bonded materials with a graded interfacial zone. In addition, the problem under consideration serves the purpose of generalizing those considered by Erdogan et al. (1991) and Ozturk and Erdogan (1993) where only the case of a crack perpendicular to or lying along the nominal interface with the interfacial zone was examined, respectively, under anti-plane shear loading. With the aforementioned in mind, it is worthwhile to report some new results regarding the anti-plane shear crack in bonded materials.

To solve the proposed crack problem, the Fourier integral transform method is employed together with the coordinate transformations of pertinent field variables. As a result, formulation of the crack problem is reduced to an integral equation with a Cauchy-type singular kernel. The mode III stress intensity factors are defined and evaluated in terms of the solution to the integral equation. In the numerical results, the values of stress intensity factors are obtained as functions of various geometric and materials parameters of the bonded media with a graded interfacial zone. To be noted is that a number of past studies for the anti-plane shear behavior of an inclined crack in the piecewise homogeneous bonded media can be found in literature (Bassani and Erdogan, 1979 ; Hwang et al., 1992 ; Kondo, 1992 ; Wang and Meguid, 1996 ; and the other references quoted therein).

2. Problem Statement and Formulation

Consider the anti-plane shear problem as shown in Fig. 1, where the two dissimilar elastic half-planes are bonded through a graded interfacial zone. The global geometric coordinates (x, y) and the local crack coordinates (x_1, y_1) are used. The oblique crack of length $2c$ is directed along the line $a<x_1<b$ and $y_1=0$, with its inclination angle $0° \leq \theta \leq 90°$ measured counterclockwise from the x-axis and its distance d designated from the interfacial zone. Let the shear moduli of the homogeneous half-planes be given by μ_j, $j=1, 3$, and the interfacial zone be treated as a nonhomogeneous interlayer of thickness h. The shear modulus of the interlayer $\mu_2(x)$ is assumed to follow an exponential variation as (Erdogan et al., 1991)

$$\mu_2(x) = \mu_1 e^{\beta x}, \ \beta = -\frac{1}{h} \ln\left(\frac{\mu_3}{\mu_1}\right) \tag{1}$$

where the nonhomogeneity parameter β specified above fulfils the continuous transition of the

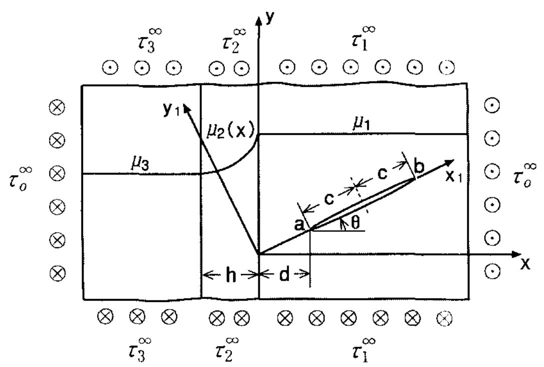

Fig. 1 Bonded half-planes with a crack at an arbitrary angle to the graded interfacial zone subjected to anti-plane shear loading

elastic properties from one half-plane to the other.

With $w_j(x, y)$, $j=1, 2, 3$, referring to the z-component of the displacement vector under anti-plane shear loading, the non-vanishing stress components and the governing equations are expressed as

$$\tau_{jxz}=\mu_j \frac{\partial w_j}{\partial x},\ \tau_{jyz}=\mu_j \frac{\partial \omega_j}{\partial y};\ j=1, 2, 3 \tag{2}$$

$$\nabla^2 w_j+\beta \frac{\partial w_j}{\partial x}=0;\ j=1, 2, 3 \tag{3}$$

in which the numeric subscripts denote the constituent materials and $\beta \neq 0.0$ for the graded interlayer ($j=2$) and $\beta=0.0$ for the homogeneous half-planes ($j=1, 3$).

The interface and the regularity conditions are imposed in the (x, y) coordinate system as

$$w_1(0, y)=w_2(0, y),\quad w_2(-h, y)=w_3(-h, y);\ |y|<\infty \tag{4a}$$

$$\tau_{1xz}(0, y)=\tau_{2xz}(0, y),\quad \tau_{2xz}(-h, y)=\tau_{3xz}(-h, y);\ |y|<\infty \tag{4b}$$

$$w_1(+\infty, y)=0,\quad w_3(-\infty, y)=0;\ |y|<\infty \tag{4c}$$

and the mixed conditions on the plane of the crack are written in the (x_1, y_1) coordinate system

$$\tau_{1y_1z_1}(x_1, +0)=\tau_{1y_1z_1}(x_1, -0);\ 0<x_1<\infty \tag{5a}$$

$$w_1(x_1, +0)=w_1(x_1, -0);\quad 0<x_1<a,\ b<x_1<\infty \tag{5b}$$

$$\tau_{1y_1z_1}(x_1, +0)=f(x_1);\ a<x_1<b \tag{5c}$$

where $f(x_1)$ describes the arbitrary crack surface traction and the relations between the two coordinates, (x, y) and (x_1, y_1), are given as

$$x_1=mx+ny,\ y_1=-nx+my \tag{6a}$$

$$m=\cos\theta,\ n=\sin\theta \tag{6b}$$

Note that for the bonded media subjected to anti-plane shear stresses, τ_o^∞ and τ_j^∞, $j=1, 2, 3$, applied sufficiently far away from the crack region (see Fig. 1), it is required that these stresses be prescribed in such a manner as to produce one of the constant strains at points remote from the crack as

$$\tau_2^\infty(x)=\tau_1^\infty e^{\beta x},\ \tau_3^\infty=\tau_1^\infty \frac{\mu_3}{\mu_1} \tag{7}$$

and the superposition principle may stipulate the statically self-equilibrating traction on the crack surface, which is necessary for investigating the local crack-tip behavior. As a result, the equivalent crack surface traction in Eq. (5c) for the crack angle θ is obtained as

$$f(x_1)=-m\tau_1^\infty+n\tau_0^\infty;\ a<x_1<b \tag{8}$$

For the homogeneous half-plane with an arbitrarily oriented crack ($x>0$ and $\beta=0.0$), the state of displacement and stresses can be expressed as the sum of two parts in the (x, y) coordinates:

$$w_1(x, y)=w_1^{(1)}(x, y)+w_1^{(2)}(x, y) \tag{9a}$$

$$\tau_{1ij}(x, y)=\tau_{1ij}^{(1)}(x, y)+\tau_{1ij}^{(2)}(x, y);\quad (i, j)=(x, z),\ (y, z) \tag{9b}$$

or in the (x_1, y_1) coordinates:

$$w_1(x_1, y_1)=w_1^{(1)}(x_1, y_1)+w_1^{(2)}(x_1, y_1) \tag{10a}$$

$$\tau_{1ij}(x_1, y_1)=\tau_{1ij}^{(1)}(x_1, y_1)+\tau_{1ij}^{(2)}(x_1, y_1);\quad (i, j)=(x_1, z_1),\ (y_1, z_1) \tag{10b}$$

where the superscript (1) denotes the infinite plane with a crack and the superscript (2) is for the half-plane without the crack.

In order to find the expressions for the field components in the homogeneous full-plane containing the crack along $a<x_1<b$ and $y_1=0$, the general solutions for the displacements in the upper ($y_1>0$) and lower ($y_1<0$) regions are first obtained by solving the governing equation in Eq. (3) based on the Fourier integral transform method and those for the stresses are obtainable from the constitutive relations in Eq. (2). Upon imposing the traction equilibrium as in Eq. (5a) and introducing an unknown function as the derivative of the displacement jump across the crack surfaces, i.e.,

$$\phi(x_1)=\frac{\partial}{\partial x_1}[w_1^{(1)}(x_1, +0)-w_1^{(1)}(x_1, -0)];\quad 0<x_1<\infty \tag{11}$$

the displacement and stresses for the cracked full-plane in the (x_1, y_1) coordinates can be then given as

$$w_1^{(1)}(x_1, y_1) = -\frac{1}{2\pi}\int_a^b \phi(t)\tan^{-1}\left(\frac{t-x_1}{y_1}\right)dt \quad (12a)$$

$$\tau_{1x_1z_1}^{(1)}(x_1, y_1) = \frac{\mu_1}{2\pi}\int_a^b \frac{y_1\phi(t)}{y_1^2+(t-x_1)^2}dt \quad (12b)$$

$$\tau_{1y_1z_1}^{(1)}(x_1, y_1) = \frac{\mu_1}{2\pi}\int_a^b \frac{(t-x_1)\phi(t)}{y_1^2+(t-x_1)^2}dt \quad (12c)$$

where the function ϕ is subjected to the following continuity and single-valuedness conditions outside the crack line:

$$\phi(x_1) = 0 \,;\ 0 < x_1 < a,\ b < x_1 < \infty \quad (13a)$$

$$\int_a^b \phi(x_1)\,dx_1 = 0 \quad (13b)$$

For the second part of the solution, the displacement and stresses in the (x, y) coordinates with the regularity condition in Eq. (4c) are readily obtained in terms of the Fourier integrals

$$w_1^{(2)}(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(s)\,e^{-|s|x-isy}ds \quad (14a)$$

$$\tau_{1xz}^{(2)}(x, y) = -\frac{\mu_1}{2\pi}\int_{-\infty}^{\infty} |s|\,A(s)\,e^{-|s|x-isy}ds \quad (14b)$$

$$\tau_{1yz}^{(2)}(x, y) = -\frac{\mu_1 i}{2\pi}\int_{-\infty}^{\infty} s\,A(s)\,e^{-|s|x-isy}ds \quad (14c)$$

where $A(s)$ is an arbitrary unknown function, s is the transform variable, and $i = (-1)^{1/2}$.

The general solutions for the displacement and stress components in the nonhomogeneous interlayer ($-h < x < 0$ and $\beta \neq 0.0$) can be expressed as

$$w_2(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{2} B_j(s)\,e^{r_jx-isy}ds \quad (15a)$$

$$\tau_{2xz}(x, y) = \frac{\mu_1 e^{\beta x}}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{2} B_j(s)\,r_j e^{r_jx-isy}ds \quad (15b)$$

$$\tau_{2yz}(x, y) = -\frac{\mu_1 e^{\beta x} i}{2\pi}\int_{-\infty}^{\infty} s\sum_{j=1}^{2} B_j(s)\,e^{r_jx-isy}ds \quad (15c)$$

where $B_j(s)$, $j = 1, 2$, are arbitrary unknown functions and $r_j(s)$, $j = 1, 2$, are given as

$$r_1 = -\frac{\beta}{2} - \sqrt{\frac{\beta^2}{4}+s^2},\quad r_2 = -\frac{\beta}{2} - \sqrt{\frac{\beta^2}{4}+s^2} \quad (16)$$

For the half-plane on the left-hand side ($x < -h$ and $\beta = 0.0$), the general solutions for the displacement and stress components that satisfy the regularity condition in Eq. (4c) are also obtained as

$$w_3(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(s)\,e^{|s|x-isy}ds \quad (17a)$$

$$\tau_{3xz}(x, y) = \frac{\mu_3}{2\pi}\int_{-\infty}^{\infty} |s|\,C(s)\,e^{|s|x-isy}ds \quad (17b)$$

$$\tau_{3yz}(x, y) = -\frac{\mu_3 i}{2\pi}\int_{-\infty}^{\infty} s\,C(s)\,e^{|s|x-isy}ds \quad (17c)$$

where $C(s)$ is an arbitrary unknown function.

It is seen that the general solutions in the basic elasticity formulation involve four unknowns $A(s)$, $B_j(s)$, $j = 1, 2$, and $C(s)$, in addition to the auxiliary function ϕ. The interface conditions in Eqs. (4a) and (4b) can be applied to express these unknowns in terms of ϕ, which then becomes the only unknown to be determined by using the crack surface condition in Eq. (5c).

3. Derivation of the Integral Equation

In order to apply the interface conditions, the field components in Eqs. (9a) and (9b) defined in the (x, y) coordinates for the cracked half-plane are employed. The full-plane solutions in Eq. (12) obtained in the (x_1, y_1) coordinates should thus be transformed such that

$$w_1(x, y) = w_1^{(1)}(x_1, y_1) + w_1^{(2)}(x, y) \quad (18a)$$

$$\tau_{1xz}(x, y) = m\tau_{1x_1z_1}^{(1)}(x_1, y_1) - n\tau_{1y_1z_1}^{(1)}(x_1, y_1) + \tau_{1xz}^{(2)}(x, y) \quad (18b)$$

and from Eqs. (12), (14a), and (14b), together with the aid of Eq. (6a), it can be shown that

$$w_1(x, y) = -\frac{1}{2\pi}\int_a^b \phi(t)\tan^{-1}\left(\frac{t-mx-ny}{my-nx}\right)dt + \frac{1}{2\pi}\int_{-\infty}^{\infty} A(s)\,e^{-|s|x-isy}ds \quad (19a)$$

$$\tau_{1xz}(x, y) = \frac{\mu_1}{2\pi}\int_a^b \phi(t)\left[\frac{y-nt}{x^2+y^2+t^2-2t(mx+ny)}\right]dt - \frac{\mu_1}{2\pi}\int_{-\infty}^{\infty} |s|\,A(s)\,e^{-|s|x-isy}ds \quad (19b)$$

Upon substituting Eqs. (15), (17), and (19) into Eqs. (4a) and (4b), the expressions for the unknowns, $A(s)$, $B_j(s)$, $j = 1, 2$, and $C(s)$, can be obtained as

$$A(s) = \frac{(r_1-|s|)(r_2-|s|)(e^{-r_1h}-e^{-r_2h})}{2s\Delta(s)}I(s) \quad (20a)$$

$$B_1(s)=\frac{|s|}{s}\frac{(r_2-|s|)e^{-r_2h}}{\Delta(s)}I(s) \qquad (20b)$$

$$B_2(s)=-\frac{|s|}{s}\frac{(r_1-|s|)e^{-r_1h}}{\Delta(s)}I(s) \qquad (20c)$$

$$C(s)=\frac{|s|}{s}\frac{(r_2-r_1)e^{-(r_1+r_2-|s|)h}}{\Delta(s)}I(s) \qquad (20d)$$

where $\Delta(s)$ and $I(s)$ are given by

$$\Delta(s)=(r_1+|s|)(r_2-|s|)e^{-r_2h} -(r_1-|s|)(r_2+|s|)e^{-r_1h} \qquad (21)$$

$$I(s)=i\int_a^b\phi(t)e^{-|s|tm+istn}dt \qquad (22)$$

Subsequently, the traction component, $\tau_{y_1z_1}$, in the (x_1, y_1) coordinates for the cracked half-plane is written from Eq. (10b), with its second part transformed as

$$\tau_{y_1z_1}(x_1, y_1)=\tau^{(1)}_{1y_1z_1}(x_1, y_1)-n\tau^{(2)}_{1xz}(x, y) +m\tau^{(2)}_{1yz}(x, y) \qquad (23)$$

$$x=mx_1-ny_1,\ y=nx_1+my_1 \qquad (24)$$

so that the traction along the crack plane can be expressed from Eqs. (12c), (14b), and (14c) as

$$\frac{2\pi}{\mu_1}\lim_{y_1\to+0}\tau_{1y_1z_1}(x_1, y_1) =\int_a^b\frac{\phi(t)}{t-x_1}dt +\int_{-\infty}^{\infty}(|s|n-ism)A(s)e^{-(|s|m+isn)x_1}ds;\quad 0<x_1<\infty \qquad (25)$$

where the first term on the right-hand side is the integral with a Cauchy singular kernel $1/(t-x_1)$.

After making use of the expression for $A(s)$ in Eq. (20a), the crack surface condition in Eq. (5c) can be derived as a singular integral equation

$$\int_a^b\frac{\phi(t)}{t-x_1}dt+\int_a^b k(x_1, t)\phi(t)dt =\frac{2\pi}{\mu_1}f(x_1);\ a<x_1<b \qquad (26)$$

where the kernel $k(x_1, t)$ is obtained as

$$k(x_1, t)=\int_0^{\infty}[m\cos ns(t-x_1) -n\sin ns(t-x_1)]\Lambda(s)e^{-s(t+x_1)m}ds \qquad (27a)$$

$$\Lambda(s)=\frac{\beta(e^{-h\sqrt{\beta^2+4s^2}}-1)}{2s(e^{-h\sqrt{\beta^2+4s^2}}-1)-\sqrt{\beta^2+4s^2}(e^{-h\sqrt{\beta^2+4s^2}}+1)} \qquad (27b)$$

Now, before proceeding with the solution of the integral equation, it may be worthwhile to consider the three particular cases. The first is the case of βc=constant and $h\to\infty$, for which the problem would be that of a cracked homogeneous half-plane bonded to a nonhomogeneous half-plane. It can then be shown that the function Λ (s) is expressed as

$$\Lambda(s)=\frac{2s-\sqrt{\beta^2+4s^2}+\beta}{2s+\sqrt{\beta^2+4s^2}-\beta} \qquad (28)$$

and in the second particular case of μ_3/μ_1= constant and $h\to 0.0$ or $|\beta|\to\infty$, the problem would reduce to that of two bonded half-planes that are piecewise homogenous with the shear moduli μ_1 and μ_3. Upon noting the following limiting behavior

$$\lim_{|\beta|\to\infty}h\sqrt{\beta^2+4s^2}=-\ln\left(\frac{\mu_3}{\mu_1}\right)\lim_{|\beta|\to\infty}\sqrt{1+4\left(\frac{s}{\beta}\right)^2} =-\ln\left(\frac{\mu_3}{\mu_1}\right) \qquad (29)$$

the function $\Lambda(s)$ in Eq. (27b) is simplified as

$$\Lambda(s)=\Lambda_0=\frac{\mu_1-\mu_3}{\mu_1+\mu_3} \qquad (30)$$

and the corresponding kernel in Eq. (27a) can be evaluated in closed form:

$$k(x_1, t)=\Lambda_0\left[\frac{m^2(t+x_1)-n^2(t-x_1)}{m^2(t+x_1)^2+n^2(t-x_1)^2}\right] \qquad (31)$$

which is identical to the one previously obtained by Bassani and Erdogan (1979).

In the other limiting case of μ_3/μ_1=constant and $h\to\infty$ (or βc=constant and $h\to 0.0$), it is obvious from Eqs. (27a) and (27b) that the kernel $k(x_1, t)$ would vanish and the integral equation in Eq. (26) degenerates to that for an infinite homogenous plane with a line crack, where the closed form solution is obtainable. To be further pointed out is that the functions $\Lambda(s)$ in Eqs. (27b), (28), and (30) are the ones that can be obtained when the problems are formulated for the special case of a crack perpendicular to the bonded interface. Hence, the effect of the crack obliquity other than $\theta=0°$ is reflected by virtue of the trigonometric functions involved in Eq. (27a).

4. The Solution and Stress Intensity Factors

The integrand of the kernel in Eq. (27a) retains the exponentially decaying behavior as the variable s becomes large, although when $\theta \to 90°$ or $(t+x_1) \to 0.0$, the convergence rate of the related improper integral is rendered relatively shower than otherwise. When the elastic properties are continuous and not necessarily differentiable near and at the crack tip, the Cauchy singular kernel in Eq. (26) solely contributes to the dominant part of the integral equation for $d=0.0$ as well as $d>0.0$. Consequently, the near-tip stress field would be characterized by the inverse square-root singularity, independent of the crack orientation as described by Erdogan et al. (1991) and Ozturk and Erdogan (1993) for the limiting crack angles $\theta=0°$ and $\theta=90°$, respectively. These notable features are in contrast to the dependence of the order of crack-tip stress singularity on both the elastic constants of the constituents and the angle at which the crack tip intersects the interface in piecewise homogeneous bonded media (Bassani and Erdogan, 1979; Kondo, 1992).

The auxiliary function $\phi(t)$ can therefore be expressed as (Muskhelishvili, 1953)

$$\phi(t)=\frac{g(t)}{\sqrt{(t-a)(b-t)}}\,;\ a<t<b \tag{32}$$

where $g(t)$ is an unknown function bounded and nonzero at $t=a$ and $t=b$. In the normalized interval

$$\begin{Bmatrix} x_1 \\ t \end{Bmatrix}=\frac{b-a}{2}\begin{Bmatrix} \xi \\ \eta \end{Bmatrix}+\frac{b+a}{2}\,;\ -1<(\xi,\ \eta)<1 \tag{33}$$

the solution to the integral equation can be expanded into the series of the Chebyshev polynomial of the first kind T_n as

$$\phi(t)=\phi(\eta)=\frac{1}{\sqrt{1-\eta^2}}\sum_{n=0}^{\infty} c_n T_n(\eta)\,;\ |\eta|<1 \tag{34}$$

where c_n, $n\geq 0$, are the unknown coefficients. It is noted that $c_0=0$ and the orthogonality for T_n can be used to satisfy the single-valuedness condition in Eq. (13b).

Upon substituting Eqs. (32)-(34) into Eq. (26), truncating the series at $n=N$, and using the integral formula of the Chebyshev polynomial (Abramowitz and Stegun, 1972), the integral equation is regularized as

$$\sum_{n=1}^{N} c_n\left[\pi U_{n-1}(\xi)+\frac{b-a}{2}\int_{-1}^{1}\frac{k(\xi,\ \eta)\,T_n(\eta)}{\sqrt{1-\eta^2}}d\eta\right]=\frac{2\pi}{\mu_1}f(\xi)\,;\ |\xi|<1 \tag{35}$$

where U_n is the Chebyshev polynomial of the second kind. To solve the above functional equations, the zeros of $T_N(\xi)$ are chosen as a set of collocation points which are concentrated near the ends $\xi=\pm 1$:

$$T_N(\xi_j)=0,\quad \cos^{-1}\xi_j=\frac{\pi}{2}\frac{2j-1}{N}\,;\ j=1,\ 2,\ \cdots,\ N \tag{36}$$

and the integral equation can be recast into a system of linear algebraic equations for c_n, $1\leq n\leq N$, by evaluating the equations in Eq. (35) at N station points ξ_j, $1\leq j\leq N$.

After the values of c_n, $1\leq n\leq N$, are determined, the integral equation in Eq. (26) provides the traction component, $\tau_{1y_1z_1}(x_1,\ 0)$, outside as well as inside the crack region. As a result, the mode III stress intensity factors at the crack tips, a and b, can be defined and evaluated in terms of the solution to the integral equation as

$$K_{\rm III}(a)=\lim_{x_1\to a^-}\sqrt{2(a-x_1)}\,\tau_{1y_1z_1}(x_1,\ 0)=\frac{\mu_1}{2}\sqrt{\frac{b-a}{2}}\sum_{n=1}^{N}(-1)^n c_n\,;\ x_1<a \tag{37a}$$

$$K_{\rm III}(b)=\lim_{x_1\to b^+}\sqrt{2(x_1-b)}\,\tau_{1y_1z_1}(x_1,\ 0)=-\frac{\mu_1}{2}\sqrt{\frac{b-a}{2}}\sum_{n=1}^{N} c_n\,;\ x_1>b \tag{37b}$$

where due to the continuity of shear moduli through the graded nonhomogeneous interlayer, the defined stress intensity factors are equally valid even when the crack terminates at the nominal interface with the interlayer.

5. Numerical Results and Discussion

The integral equation in Eq. (26) is solved to provide the values of the mode III stress intensity factors for various combinations of geometric (θ, $h/2c$, d/c) and material (μ_3/μ_1) parameters of the bonded media with a graded nonhomogeneous interlayer. Both the uniform crack surface traction and the remote stresses of anti-plane shear type are considered as the external loadings. The kernel $k(x_1, t)$ in Eq. (27a) with a semi-infinite integration interval is evaluated by using the Gauss-Legendre quadrature formula, while the integral in Eq. (35) is evaluated based on the Gauss-Chebyshev quadrature formula (Davis and Rabinowitz, 1984). No more than thirty-term expansion in Eqs. (35) and (36) suffices in obtaining the solution with a desired degree of accuracy for the geometric and material configurations considered in this study.

To confirm the validity of the numerical results to be examined henceforth, the stress intensity factors for two special crack orientations in (Erdogan et al., 1991 ; Ozturk and Erdogan, 1993) are reproduced subjected to the uniform traction on the crack surfaces as $f(x_1) = -\tau_o$ in Eq. (5c). One is for a crack at a right angle to the interlayer as $\theta=0°$ and $d/c=0.0$, together with the results plotted in Fig. 2 as a function of interlayer thickness $h/2c$ for the two different material combinations. The other one is for an interface crack as $\theta=90°$ and $d/c=0.0$, the results of which are shown in Fig. 3 as a function of shear moduli ratios μ_3/μ_1 for some values of $h/2c$. It is seen from these figures that the current values of the stress intensity factors are in excellent agreement with those reported in literature that are now added by solid circles.

The stress intensity factors for the arbitrarily oriented crack loaded by the uniform traction on its surfaces are next presented in Figs. 4-6. With the location and size of the crack specified as $d/c=0.0$ and $h/2c=0.5$, Figure 4 illustrates the results as a function of the crack orientation angle θ for different shear moduli ratios μ_3/μ_1. As expected, the values of the stress intensity factors increase with decreasing μ_3/μ_1. It is then observed that the crack tip a closer to the interlayer is more sensitive to the variation of μ_3/μ_1, while the crack tip b is affected by the angle θ to a larger extent. In addition to such a generic trend, the stress intensity factors are being enlarged with increasing θ when the crack is in the stiffer constituent ($\mu_3/\mu_1 < 1.0$), with the implication of higher likelihood of brittle fracture for the greater crack obliquity. The opposite response may be prevailing when $\mu_3/\mu_1 > 1.0$ such that the fracture resistance is enhanced by the nearby stiffer material for the greater θ.

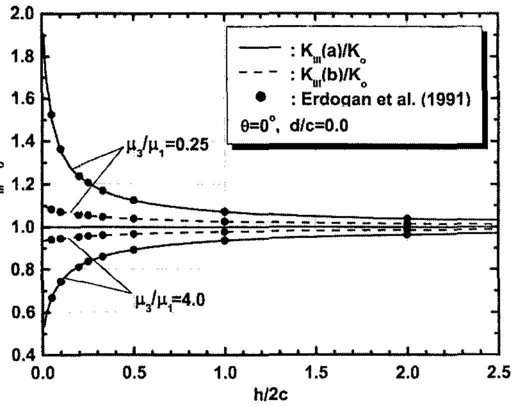

Fig. 2 Variations of stress intensity factors K_{III}/K_o for a crack perpendicular to the graded interlayer as a function of $h/2c$ under uniform crack surface traction $f(x_1) = -\tau_o$ for different values of μ_3/μ_1 ($\theta=0°$, $d/c=0.0$, and $K_o=\tau_o c^{1/2}$)

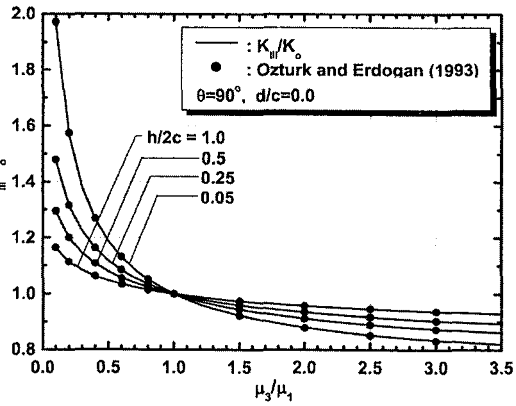

Fig. 3 Variations of stress intensity factors K_{III}/K_o for a crack along the interface with the graded interlayer as a function of μ_3/μ_1 under uniform crack surface traction $f(x_1) = -\tau_o$ for different values of $h/2c$ ($\theta=90°$, $d/c=0.0$, and $K_o=\tau_o c^{1/2}$)

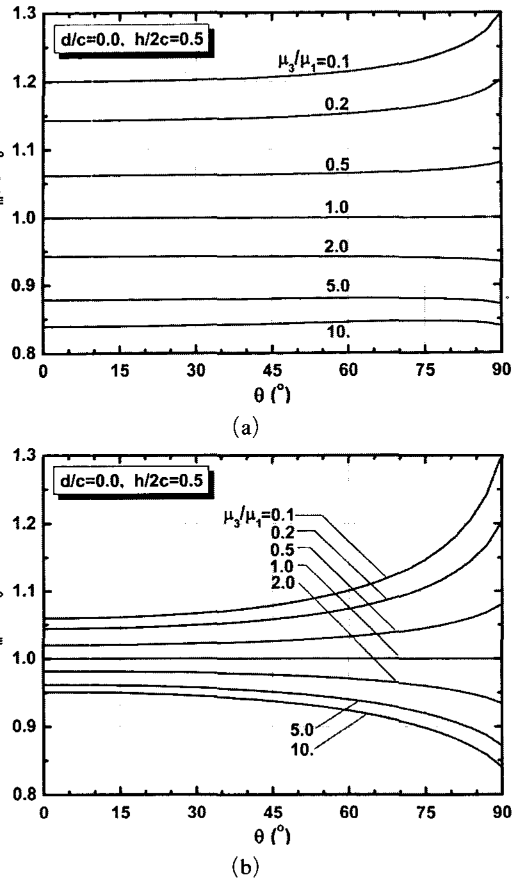

Fig. 4 Variations of stress intensity factors (a) $K_{III}(a)/K_o$ and (b) $K_{III}(b)/K_o$ versus crack orientation angle θ under uniform crack surface traction $f(x_1)=-\tau_o$ for different values of μ_3/μ_1 (d/c=0.0, $h/2c$=0.5, and $K_o=\tau_o c^{1/2}$)

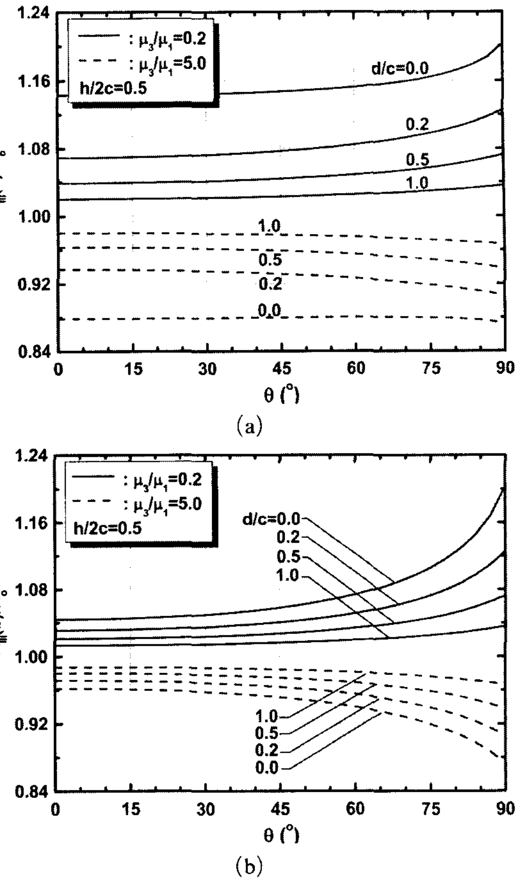

Fig. 5 Variations of stress intensity factors (a) $K_{III}(a)/K_o$ and (b) $K_{III}(b)/K_o$ versus crack orientation angle θ under uniform crack surface traction $f(x_1)=-\tau_o$ for different values of d/c and μ_3/μ_1 ($h/2c$=0.5 and $K_o=\tau_o c^{1/2}$)

The effect of the crack location d/c in conjunction with that of the crack orientation is provided in Fig. 5 for the fixed interlayer thickness as $h/2c$=0.5. In this case, the stress intensity factors for μ_3/μ_1=5.0 decrease as the crack is located closer to the interface with the interlayer, due to the more pronounced constraints by the nearby stiffer constituent. The reverse behavior may persist for μ_3/μ_1=0.2. Also observed is that the greater is the crack distance d/c from the interlayer, the less sensitive are the values of the stress intensity factors to the variations of crack angle θ and material parameter μ_3/μ_1 as well.

With the crack located at d/c=0.5 from the interlayer, the effect of the interlayer thickness $h/2c$ on the behavior of the arbitrarily oriented crack is presented in Fig. 6. To be noted from this figure is that an increase in $h/2c$ results in some reduction in the magnitude of the stress intensity factors for μ_3/μ_1=0.2, whereas the reverse response is observed for μ_3/μ_1=5.0. Aforementioned feature with the variation of the interlayer thickness, including the limiting case of $h/2c$=0.0, indicates that the presence of the graded interlayer of greater thickness would play the more effective role of shielding the crack that exists in the stiffer side of the bonded materials. If the thickness ratio $h/2c$ were further increased,

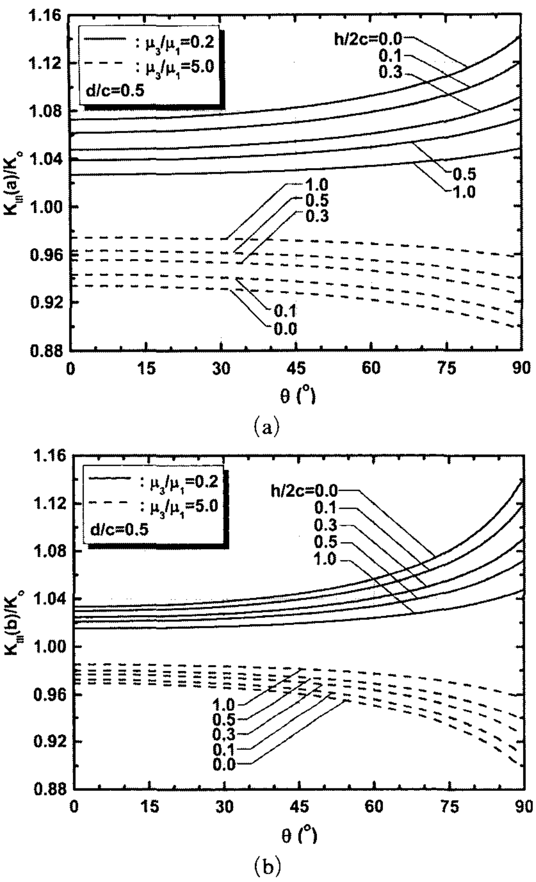

Fig. 6 Variations of stress intensity factors (a) $K_{III}(a)/K_o$ and (b) $K_{III}(b)/K_o$ versus crack orientation angle θ under uniform crack surface traction $f(x_1)=-\tau_o$ for different values of $h/2c$ and μ_3/μ_1 ($d/c=0.5$ and $K_o=\tau_o c^{1/2}$)

the influence of θ would be insignificant such that the solutions would tend to those for a crack in the infinite homogeneous plane.

In the sequel, the oblique crack in the bonded materials subjected to far-field anti-plane shear loading (see Fig. 1) is considered. With the equivalent crack surface traction given by Eq. (8), the stress intensity factors owing to such remote loading can readily be evaluated by using those obtained under the condition of uniform traction on the crack surfaces as

$$K_{III}=\sqrt{c}\,(m\tau_1^{\infty}-n\tau_o^{\infty})K_r \tag{38}$$

where K_r refers to the normalized stress intensity

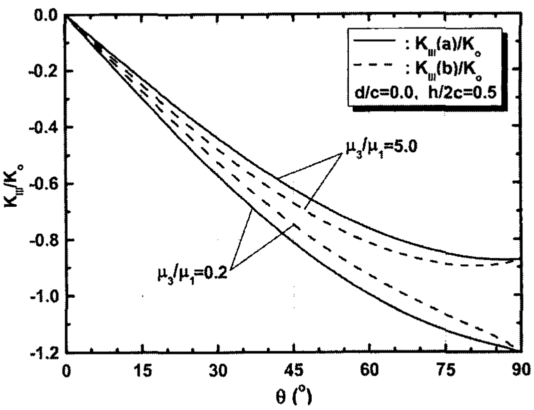

Fig. 7 Variations of stress intensity factors K_{III}/K_o versus crack orientation angle θ under remote loading of $\tau_o^{\infty}\neq 0$ and $\tau_j^{\infty}=0$, $j=1, 2, 3$, for different values of μ_3/μ_1 ($d/c=0.0$, $h/2c=0.5$, and $K_o=\tau_o^{\infty}c^{1/2}$)

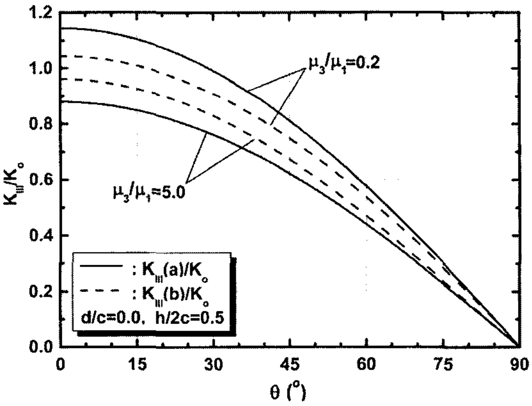

Fig. 8 Variations of stress intensity factors K_{III}/K_o versus crack orientation angle θ under remote loading of $\tau_o^{\infty}=0$ and $\tau_j^{\infty}\neq 0$. $j=1, 2, 3$, for different values of μ_3/μ_1 ($d/c=0.0$, $h/2c=0.5$, and $K_o=\tau_1^{\infty}c^{1/2}$)

factor for the uniform crack surface traction.

The variations of corresponding stress intensity factors are plotted in Figs. 7-9 as a function of the crack orientation angle θ, for the crack location and interlayer thickness fixed as $d/c=0.0$ and $h/2c=0.5$. As shown in Fig. 7, with the loading being imposed as $\tau_o^{\infty}\neq 0$ and $\tau_j^{\infty}=0$, $j=1, 2, 3$, it is clear from the geometry of the bonded media that such an external load would result in the zero value of the stress intensity factors when $\theta=0°$ and their magnitudes increase as θ increases.

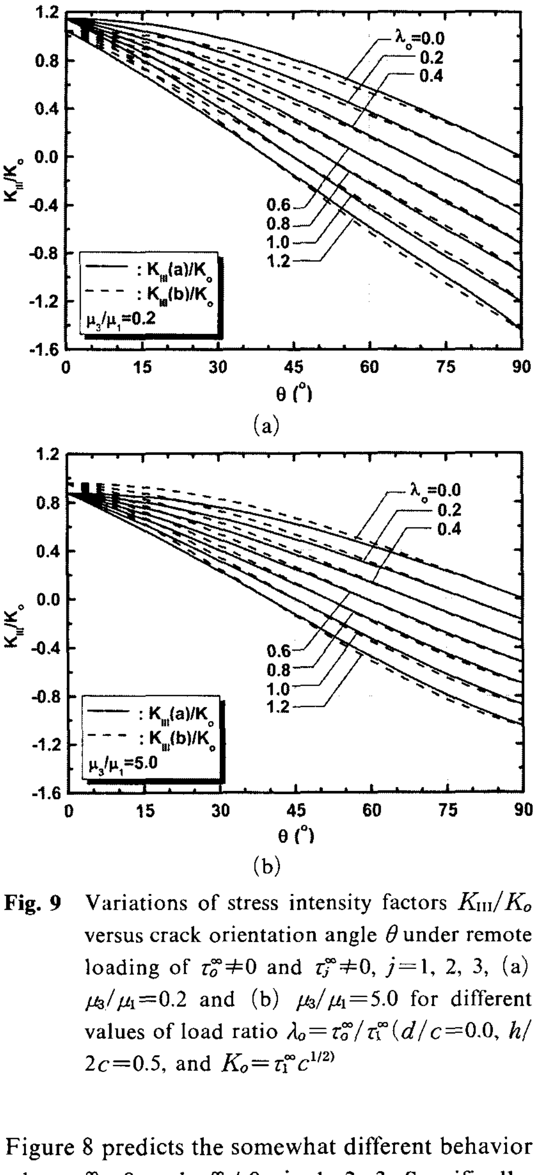

Fig. 9 Variations of stress intensity factors K_{III}/K_o versus crack orientation angle θ under remote loading of $\tau_o^\infty \neq 0$ and $\tau_j^\infty \neq 0$, $j=1, 2, 3$, (a) $\mu_3/\mu_1=0.2$ and (b) $\mu_3/\mu_1=5.0$ for different values of load ratio $\lambda_o=\tau_o^\infty/\tau_1^\infty$ ($d/c=0.0$, $h/2c=0.5$, and $K_o=\tau_1^\infty c^{1/2}$)

Figure 8 predicts the somewhat different behavior when $\tau_o^\infty=0$ and $\tau_j^\infty \neq 0$, $j=1, 2, 3$. Specifically, the values of the stress intensity factors decrease as θ increases, approaching zero when the crack is becoming an interfacial one as $\theta=90°$. For the loading condition of $\tau_o^\infty \neq 0$ and $\tau_j^\infty \neq 0$, $j=1, 2, 3$, the results are illustrated in Figs. 9(a) and 9(b) for $\mu_3/\mu_1=0.2$ and $\mu_3/\mu_1=5.0$, respectively. In order to prescribe the remote loading condition in a quantitative manner, it is assumed that $\tau_o^\infty=\lambda_o\tau_1^\infty$ where λ_o is a parameter to measure the degree of load ratio. It then appears that the effect of the load ratio λ_o becomes more pronounced for the greater θ and more noteworthy for the crack in the stiffer constituent as $\mu_3/\mu_1=0.2$. Of particular interest in these figures is that for $\lambda_o>0.0$, the values of the stress intensity factors change sign as the crack orientation angle θ increases, implying that there may exist certain crack obliquities that render the mode III crack free from the singularity, i.e. $\theta=\cot^{-1}\lambda_o$, for any given material combinations.

6. Summary and Conclusions

The anti-plane shear problem of bonded elastic half-planes with a crack at an arbitrary angle to the graded nonhomogeneous interlayer has been investigated. Formulation of the crack problem ended up with the derivation of a Cauchy-type singular integral equation, with the corresponding mode III stress intensity factors evaluated in terms of the solution to the integral equation. In the numerical results, parametric studies were conducted with the following conclusions:

(1) For an arbitrarily oriented crack under the uniform crack surface traction, the crack tip away from the interlayer is more sensitive to the variation of the crack angle, whereas the crack tip closer to the interlayer is affected by the shear moduli ratios to a greater extent than the other crack tip.

(2) The stress intensities are enlarged with increasing θ when the crack is in the stiffer constituent, with the implication that the likelihood of brittle fracture is higher for the crack located parallel to or along the interface with the interlayer.

(3) For the crack in the less stiff constituent, the severity of crack-tip stress state becomes intensified as the thickness of the interlayer or the crack-tip distance from the interlayer increases. When the crack is in the stiffer constituent such that the adjacent uncracked constituent is relatively compliant, the opposite behavior prevails.

(4) Under the far-field anti-plane shear loading condition prescribed in terms of the load ratio, the values of the mode III stress intensity factors vanish at certain crack orientation angle without regard to the material combinations.

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