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Retrieving the Time History of Displacement from Measured Acceleration Signal

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It is intended to retrieve the time history of displacement from measured acceleration signal. In this study, the word retrieving means reconstructing the time history of original displacement signal from already measured acceleration signal not just extracting various information using relevant signal processing techniques. Unlike extracting required information from the signal, there are not many options to apply to retrieve the time history of displacement signal, once the acceleration signal is measured and recorded with given sampling rate. There are two methods, in general, to convert measured acceleration signal into displacement signal. One is directly integrating the acceleration signal in time domain. The other is dividing the Fourier transformed acceleration signal by the scale factor of $-\omega^2$ and taking the inverse Fourier transform of it. It turned out both the methods produced a significant amount of errors depending on the sampling resolution in time and frequency domain when digitizing the acceleration signals. A simple and effective way to convert the time history of acceleration signal into the time history of displacement signal without significant errors is studied here with the analysis on the errors involved in the conversion process.

Key Words : Time History, Acceleration Signal, Displacement Signal, Nyquist Frequency Leakage, Zero Padding, Scale Factor

Nomenclature

A_k : Discrete Fourier coefficient of acceleration signal
D_k : Discrete Fourier coefficient of displacement signal
T : Total sampling time
V_k : Discrete Fourier coefficient of velocity signal
$X(f)$: Fourier transform of displacement signal
$a(t)$: Time history of acceleration signal
a_r : Time array digitized from a continuous acceleration signal
ε_V : Relative error in evaluating the velocity from an acceleration signal
f_0 : Frequency of the signal
f_{Ny} : Nyquist frequency of the measurement

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1. Introduction

There are few displacement type transducers to measure the structural vibration signals or the machine monitoring signals of relatively high frequency components. Accelerometers are the most widely used transducers to measure the vibration responses of structures with the assumption that the information on the displacement can be extracted later from measured acceleration data. Recently, expensive laser equipments are used to measure the displacement response of the structures, but they actually measure the velocity of the structure and convert it into displacement with appropriate signal processing algorithm rather than directly measure the displacement signal itself. Whether using laser equipments or accelerometers, the information on the amplitudes, frequencies, decay rates, and phase differences of the measured accelerations or the velocities are usually the main objectives of the signal analysis involved in the structural vibration test.

Many signal processing techniques are available in the literature to extract these parameters of the signal ranging from classical Fourier transform methods (Bendat and Piersol, 1991 ; Newland, 1993 ; Papoulis, 1991) to relatively sophisticated ones (Chen and Mechefske, 2001 ; Lee, 1999 ; Luo and et. al, 2000 ; Tang, 2000). If the acceleration signal is impulse or random type, there may be no practical reason to retrieve the time history of velocity or displacement besides extracting the maximum amplitude or the frequency contents from the signal. But, in the structural vibration responses, sometimes it is necessary to retrieve measured acceleration signals in the form of velocities and displacements in cases such as active control of the structure or implementing the proper orthogonal decomposition (Feeny and Kappagantu, 1998). While it is quite easy to extract the information on the frequency components and the root mean square values of amplitudes of the corresponding velocities and displacements from the measured acceleration signals, it is not an easy task to retrieve the time histories of the structural responses in the form of velocities and displacements.

Figure 1 shows both the displacement and acceleration responses of a free-free beam impacted with hammer. Both the signals are generated using the finite element model of the beam with the Nyquist frequency that contains up to the 7th mode of the beam. If one intends to directly measure the displacement response of the beam, he cannot find adequate transducer to achieve the goal. The most convenient choice is using the accelerometer and getting the acceleration response of the beam. The objective of this study is to reconstruct the time history of the displacement response of the structure as shown in Fig. 1(a) using the already measured acceleration response as shown in Fig. 1(b).

There are two methods, in general, to convert an already measured acceleration signal into displacement signal (McConnell, 1995). One is directly integrating the acceleration signal in time domain. The other is dividing the Fourier transformed acceleration signal by the scale factor of $-\omega^2$ and taking the inverse Fourier transform of it. It turned out both the method produced a significant amount of errors depending on the sampling resolution to digitize the response signals (Han, 2001). It is well known that to have better resolution in time domain, one has to compromise with the coarse resolution in frequency domain and visa verse with given number of sampling points. Therefore,. with a fixed resolution in time and frequency domain, converting high frequency signals in time domain and converting low frequency signals in frequency domain will produce biased errors. The errors involved in the converting process are stated, and an effective way to convert acceleration signal into displacement signal without significant errors are studied here. Since the procedure to convert an acceleration signal into displacement signal requires the same procedure of converting the acceleration signal into velocity signal successively, this paper will be focused on the final objective of retrieving the displacement signal from the measured acceleration signal.

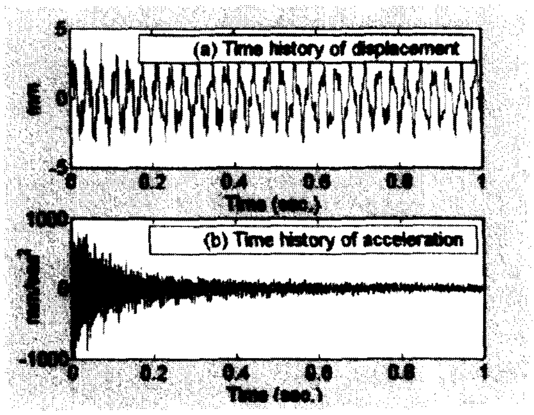

Fig. 1 Displacement and acceleration responses of an impacted free-free beam

2. Converting Acceleration Signal into Displacement Signal in Frequency Domain

Suppose that the continuous time history of acceleration response of the structure $a(t)$ is not known and only equally spaced samples are available. This acceleration signal is represented by the discrete series $\{a_r\}$, $r=0, 1, 2, \cdots, (N-1)$, where $t=r\cdot\Delta t$. The discrete Fourier transform of the series $\{a_r\}$ is given by

$$A_k=\frac{1}{N}\sum_{k=0}^{N-1} a_r e^{-j(2\pi kr/N)} \qquad k=0, 1, 2, \cdots, (N-1) \tag{1}$$

and the inverse discrete Fourier transform is given by

$$a_r=\sum_{k=0}^{N-1} A_k e^{j(2\pi kr/N)} \qquad r=0, 1, 2, \cdots, (N-1) \tag{2}$$

It is important to note that although the discrete Fourier transform given in Eq. (1) does not provide enough information to allow the continuous time series $a(t)$ to be obtained, it does allow all the discrete values of the series $\{a_r\}$ to be retrieved exactly (Newland, 1993).

From the properties of the Fourier transform of the integrals, the discrete Fourier transform of the velocity and the displacement signals are given as

$$V_k=\frac{1}{j2\pi k}A_k \qquad k=0, 1, 2, \cdots, (N-1) \tag{3}$$

$$D_k=-\frac{1}{(2\pi k)^2}A_k \qquad k=0, 1, 2, \cdots, (N-1) \tag{4}$$

Time histories of the velocities and the displacements are obtained by taking the inverse Fourier transform of the coefficients in Eqs. (3) and (4) as follows.

$$v_k=\sum_{k=0}^{N-1} V_k e^{-j(2\pi kr/N)} \qquad r=0, 1, 2, \cdots, (N-1) \tag{5}$$

$$d_r=\sum_{k=0}^{N-1} D_k e^{j(2\pi kr/N)} \qquad r=0, 1, 2, \cdots, (N-1) \tag{6}$$

The error involved in the transformation of acceleration into velocities and displacements are from the scale factor of $1/j2\pi k$ and $-1/(2\pi k)^2$ in Eqs. (3) and (4). This means that the velocity and displacement levels drop off at a rate of 3dB and 6dB per doubling the frequency and, therefore, the Fourier coefficients of high frequency components of the velocity or the displacement can be less than the Fourier coefficients of low frequency noise components after the conversion.

3. Converting Acceleration Signal into Displacement Signal in Time Domain

The velocity and displacement of a signal can be obtained by directly integrating the acceleration signal in time domain using the following definition.

$$v(t)=\int_0^t a(t)\,dt+v_0 \tag{7}$$

$$d(t)=\int_0^t v(t)\,dt+d_0 \tag{8}$$

Here v_0 and d_0 are the initial velocity and displacement, respectively.

The first source of error occurred in the conversion process is due to the time resolution of the digitized acceleration signal. Bias error of the numerical quadrature using the trapezoidal rule to convert the acceleration into velocity is given as (Burden, et al, 1979)

$$E_2=\frac{(\Delta t)^3}{12}\ddot{a}(t) \quad 0<t<\Delta t \tag{9}$$

For a pure sinusoidal signal, the relationship between the acceleration and the velocity is given as

$$\ddot{a}(t)=-(2\pi f_0)^3 v(t) \tag{10}$$

where f_0 is the frequency of the signal. And the Nyquist frequency of the measurement is determined by the sampling resolution Δt as

$$f_{Ny}=\frac{1}{2\Delta t} \tag{11}$$

Therefore, the relative error in evaluating the velocity from the acceleration of a pure sinusoidal signal is given as

$$\varepsilon_v=\frac{\pi^3}{12}\left(\frac{f_0}{f_{Ny}}\right)^3 \tag{12}$$

and the relationship between the frequency of the signal to be integrated within a certain amount of error and the Nyquist frequency of the measurement is determined as follows.

$$f_0=\sqrt[3]{\frac{12\varepsilon_v}{\pi^3}}f_{Ny}=0.7287\sqrt[3]{\varepsilon_v}\,f_{Ny} \tag{13}$$

For example, if we want to reconstruct the time history of the velocity by directly integrating a sinusoidal acceleration signal within 5% of error, the signal frequency should be less than $0.2685f_{Ny}$, and within 1% of error, the signal frequency should be less than $0.1570f_{Ny}$.

The second source of the error comes from the fact that there is no information available on the initial conditions involved with each integration scheme. Uncertain value of initial velocity will produce a dc component during the successive integration of the conversion process. One of the methods to eliminate the error due to the uncertain initial value of the signal is filtering out the dc component in every integration scheme or extrapolating the acceleration signal to find out the corresponding initial velocity.

4. Statement of the Error in the Conversion Process

Let's examine the conversion error by defining an acceleration signal measured with a digital signal analyzer that has a fixed number of sampling points. Therefore, the sampling resolution of the measured signal in both time and frequency domain depends on the record time T. To check the errors involved in the conversion process, consider the following three different pure sinusoidal acceleration signals. The frequencies of the signals are 20 Hz, 20.3 Hz, 800 Hz, respectively, and the amplitudes are $(2\pi f_0)^2$ where f_0 is the frequency of each signal. The reason for defining different amplitudes of the accelerations is to make the amplitudes of all the corresponding displacement signals be 1 unit for comparison. It is assumed that all of the signals were measured with a digital signal analyzer that has 2048 sampling points. Since the record time is fixed to be 1 sec., the time and frequency resolution of the digitized signal are fixed, which are 1/2047 second and 1 Hz, respectively. The objective is to convert these acceleration signals into displacement signals with the assumption that these a cceleration signals are the best representations of the structural responses obtained by the digital-analog converter of the analyzer.

Displacement signals retrieved from the acceleration signal of 20 Hz using both the frequency domain method and the time domain method are compared with the corresponding theoretical displacement signal in Fig. 2. The retrieved displacement signals using both the methods are identical, and we can see the retrieved signals are identical with the theoretical one without any distortion for this 20 Hz signal. The result of this 20 Hz pure sinusoidal signal confirms that the numerical algorithms used in both the conversion methods are correct for this special case. The author believes that most people think that any other types of structural signals can also be retrieved as nice as this special case of the signal, and that this kind of simple job is not worth of paying special research attention. But that is not true, and unfortunately, the author cannot find any other literatures reported on this issue.

The retrieved displacement signal of 20.3 Hz using the time domain method is identical with

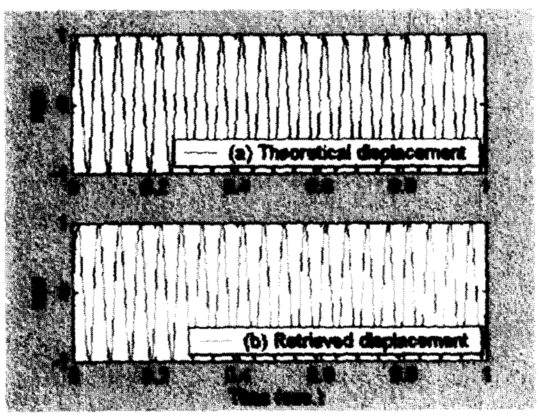

Fig. 2 Time histories of theoretical and retrieved displacement signal from an acceleration signal of 20 Hz using both the frequency domain method and the time domain method

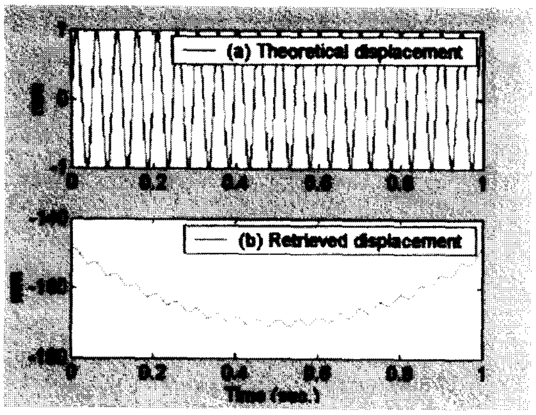

Fig. 3 Time histories of theoretical and retrieved displacement signal from an acceleration signal of 20.3 Hz using the frequency domain method

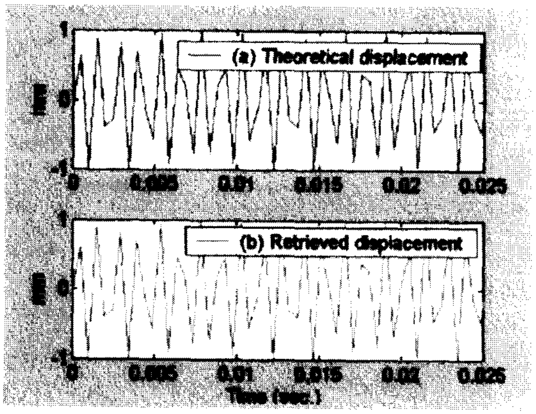

Fig. 4 Time histories of theoretical and retrieved displacement signal from an acceleration signal of 800 Hz using the frequency domain method

the theoretical one, which is not necessary to be repeated again, but the retrieved signal using the frequency domain method is totally different from the theoretical signal as shown in Fig. 3.

For the signal of 800 Hz, the retrieved displacement signals using the frequency domain method and time domain method are shown in Figs. 4 and 5, respectively. Figure 4 shows the retrieve displacement signal using the frequency domain method, and it is identical with the theoretical signal even though both the signals do not look like pure sinusoidal signal. On the other hand, the displacement signal retrieved from the

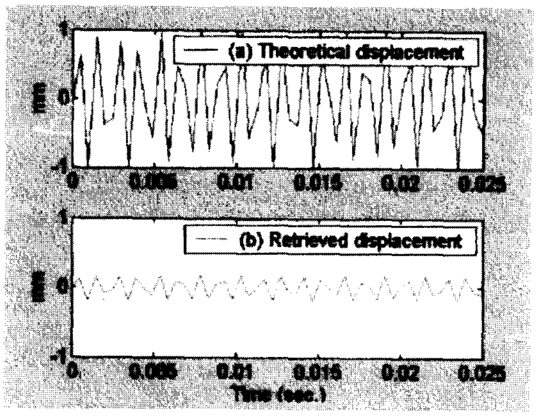

Fig. 5 Time histories of theoretical and retrieved displacement signal from an acceleration signal of 800 Hz using the time domain method

same acceleration signal using the time domain method fails to express rapidly changing peak values of the signal as shown in Fig. 5.

Let's explain the error involved in the conversion process with the results of the above three cases of the signals.

The Fourier transform of a single frequency signal is the delta function of $\delta(f-f_0)$ along the frequency axis, where f_0 is the frequency of the signal. Therefore, theoretically, all the other Fourier coefficients are zero except at the corresponding frequency value. But due to the digitization error, each Fourier coefficients actually have some small values, and when the Fourier coefficient of the frequency component of the signal is divided by the correction factor of $-(2\pi k)^2$, there appears distortion in the Fourier coefficients along the frequency axis as shown in Fig. 6, especially around the low frequency region. The amount of distortion is much severer when there is leakage in the measured signal as in Fig. 7, in which case the difference between the maximum value of the Fourier coefficient and the minimum value of the coefficients are relatively small, therefore, the scale factor plays significant role in converting the acceleration into the displacement. Suppose that the measured signal has signal to noise ratio of less than 60dB, or the signal is a damped one, which is common in practice, then the discrete Fourier coefficients of the frequency

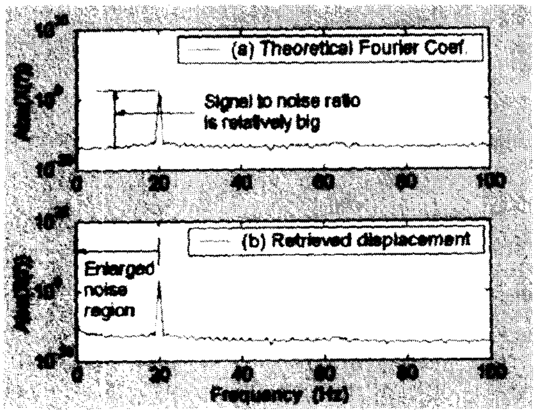

Fig. 6 Absolute values of Fourier coefficients of displacement retrieved from 20 Hz acceleration signal

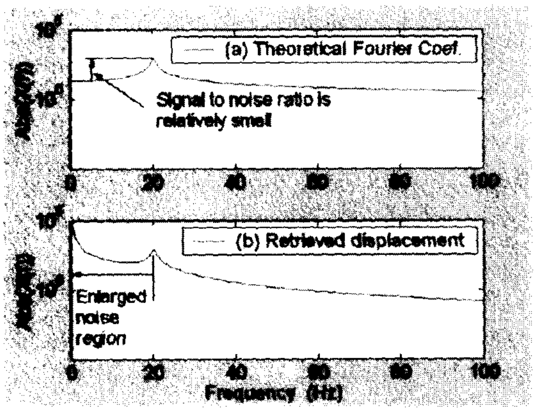

Fig. 7 Absolute values of Fourier coefficients of displacement retrieved from 20.3 Hz acceleration signal

component at high value of k can be decreased by more than 60dB after divided by the factor of $-(2\pi k)^2$. This will cause the actual value of high frequency component appears less than the values of low frequency noise components and the displacement signal appears to have very big low frequency components, which will distort the converted displacement as shown in Fig. 3.

Let's illustrate the situation with the acceleration of 20.3 Hz pure sinusoidal signal. In this case, the actual Fourier coefficients lies between 20 Hz and 21 Hz due to the leakage. Since the dominant frequency components are split, their magnitudes appear not so great as those of non-significant frequency components, for example, the Fourier coefficient at $f=0$. To reconstruct the time history of the corresponding displacement signal, these relatively small values of the dominant components are divided by $-(2\pi f)^2$, and the resulted Fourier coefficients become less than those of non-significant frequency components as illustrated in Table 1. This is the reason why the low frequency noise components of the signal are magnified for the case of the signal with leakage.

Table 1 Fourier coefficients of the acceleration of 20.3 Hz pure sinusoidal signal and the Fourier coefficients of displacement that are divided by the scale factor of $-(2\pi f)^2$

Frequency	F.C. of acceleration		F.C. divided by $-(2\pi f)^2$	
	Real	Imaginary	Real	Imaginary
0	−3.38e5	0	−3.38e5	0
20	−1.16e7	8.33e6	7.37e2	−5.28e2
21	4.88e6	−3.33e6	−2.80e2	2.10e2

We can quantitatively determine whether the low frequency noise components become significant or not after converted in the frequency domain. Since each discrete Fourier coefficient is divided by the scale factor of $-(2\pi k)^2$, its value will be decreased by 6 in dB scale per doubling the k value. Therefore, by plotting the absolute values of the Fourier coefficients in dB scale along the frequency domain, we can find out whether the dominant frequency component will be greater than the noise components or not after conversion. Any dominant acceleration signal components whose magnitudes of the Fourier coefficients are above this 6dB line starting from the zero frequency will be greater than the noise components after conversion. The magnitude of the Fourier coefficients corresponding to the 20 Hz and 800 Hz sinusoidal signal in Fig. 8(a) are above the 6dB line when plotted in dB scale, therefore, the retrieved displacement signals have little distortion as shown in the results of Figs. 2 and 4. These cases of the exact conversion are possible because the signal components are the integer multiples of the frequency resolution, which means there is no leakage during the mea-

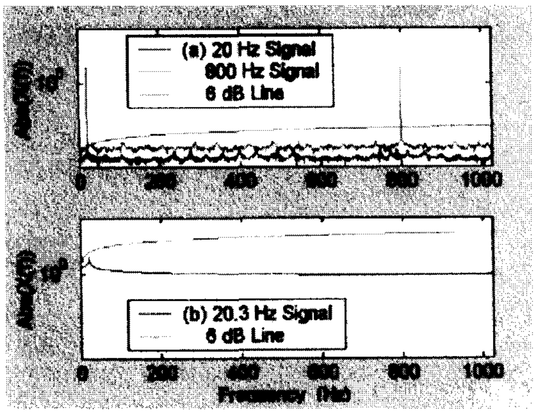

Fig. 8 Absolute values of Fourier coefficients of the original acceleration signal
(a) 20 Hz and 800 Hz
(b) 20.3 Hz

surement of the signal. On the other hand, the magnitude of Fourier coefficient corresponding to 20.3 Hz appears below the 6dB line as shown in Fig. 8(b), therefore, the converted displacement signal is contaminated by the significant low frequency noise components as shown in Fig. 3.

The reason for both the 20 Hz and 20.3 Hz acceleration signals being nicely converted into displacement signals with the time domain method is that the frequencies of the signals are much less than the Nyquist frequency. But the time domain method is not good for 800 Hz signal, in which case the frequency component is relatively high compared to the Nyquist frequency as stated in Eq. (13). There is one more thing to mention here. The digitized theoretical displacement signal itself is not a good representative of the original pure sinusoidal signal because of the poor sampling resolution. In this example, the total sampling points are 2048, and there are only 3 (2048/800) points available to digitize one period of the signal. As stated earlier, even though digitized signal does not provide enough information of the original continuous signal, its Fourier conversion is exact as is shown in the result of Fig. 4.

From the results of the retrieved time histories of the displacements from the single frequency acceleration signals discussed above we can draw following conclusions. When there is no leakage in the signal, even though the condition can seldom be satisfied in real situations, time history of displacement signal of a given measured acceleration signal can be retrieved nicely by using frequency domain method. In this case, the frequency domain method can be applied for both high and low frequency signals. On the other hand, the frequency domain method is not good for the signals measured with leakage or the signals decaying out. The direct integration method can retrieve low frequency signals whether they have leakage or not, but the method does not work well for the signals with relatively high frequency components. Therefore, if one intends to retrieve the displacement signal later with the measured acceleration signal, he has to measure the signal with sufficiently fine sampling rate to satisfy the condition given in Eq. (13).

5. Conversion of Multi-Frequency Signal

As stated above, both the frequency domain method and the time domain method produce a certain amount of errors depending on the frequency component of the signal during the conversion process of retrieving the displacement from the measured acceleration. The frequency domain method works well with the acceleration signal measured without leakage. On the other hand, the time domain method works well with the acceleration signal whose frequency is well below the Nyquist frequency. But these conditions are seldom satisfied with real acceleration signals encountered in the structural vibration test. In practice, structural response consists of both high and low frequency components and leakage always happens in the measurement. Furthermore, once a structural response is measured with a signal analyzer, there is little choice to control the frequency components of the signal as well as the time resolution. Considering the errors involved in the conversion process, it is better to apply the frequency domain method to retrieve the displacement from the measured acceleration, once the signal is measured and recorded and contains relatively high frequency components.

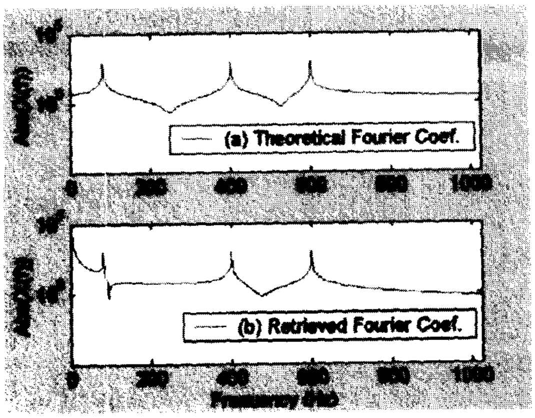

Fig. 9 Absolute values of Fourier coefficients of theoretical and retrieved displacement from an acceleration signal with 3 sinusoidal components

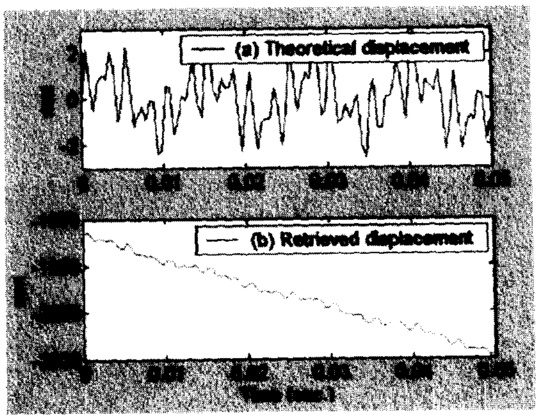

Fig. 10 Time histories of theoretical and retrieved displacement from an acceleration signal with 3 sinusoidal components

The main source of the error involved in the conversion process using the frequency domain method is that low frequency components become significant after divided by the scale factor of $-\omega^2$. To convert a more general signal with minimum error using the frequency domain method, there must be some measures to prevent the low frequency components from being significant. Any frequency components of the measured acceleration signal whose absolute values of the Fourier coefficients appear below the 6dB line starting from the dc component along the frequency axis as shown in Fig. 8(b) can be less than the noise components during the conversion process. Assuming that every other signal component is noise except the significant ones appeared in the Fourier transform of the acceleration, zero padding the noise components could reduce the effect of the conversion factor. Of course, we can use digital band pass filter on the measured and recorded signal to eliminate non-significant components of the Fourier transform of the acceleration signal. But, it is much more difficult to adjust the band width of the filter than suggested zero padding method. Zero padding is simply neglecting all of the unwanted frequency components of the signal. Fourier coefficients of the displacement signal retrieved from an acceleration signal with 80.3 Hz, 400.3 Hz, 600.3 Hz sinusoidal components and those of theoretical displacement

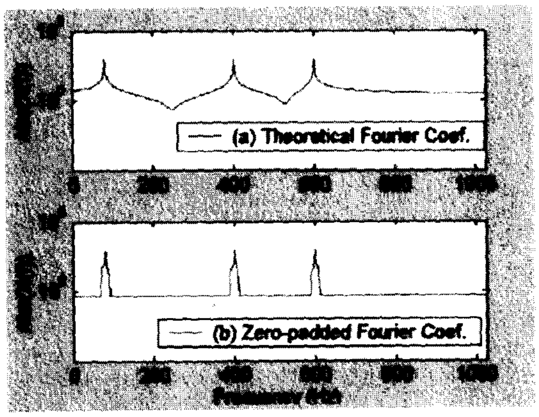

Fig. 11 Absolute values of Fourier coefficients of theoretical displacement compared with the converted and zero padded displacement signal

signal are shown in Fig. 9. 80.3 Hz, 400.3 Hz, 600.3 Hz are chosen as an example of low, medium and high frequency components mixed in the signal. As expected, the low frequency noise components become significant, and inverse Fourier transform of these coefficients produce a totally different displacement signal as in Fig. 10. Zero padding the frequency components except the significant sinusoidal components would modify the Fourier coefficients of the signal to be retrieved as shown in Fig. 11. Retrieved displacement signal from these modified Fourier coefficients nicely match with the theoretical one ex-

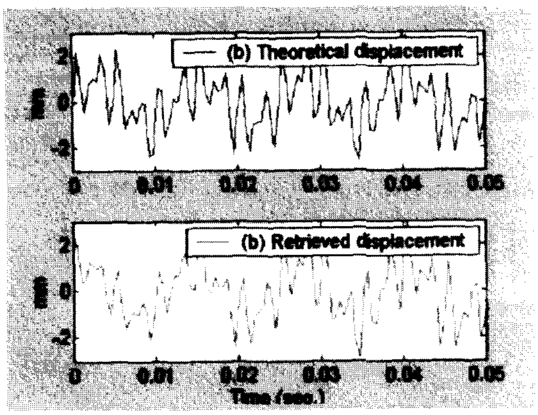

Fig. 12 Time histories of theoretical displacement and the displacement retrieved from the zero-padded Fourier coefficients

cept the starting point of the signal as shown in Fig. 12. The discrepancy at the starting point of the time history may be due to the phase information lost by simply zero padding the low frequency noise components. A better method to compensate this phase discrepancy along with the conversion process dealing with more complex signals, not just pure sinusoidal signal, will be studied further.

6. Conclusions

When the acceleration is measured with the sampling rate whose Nyquist frequency is much higher than the significant frequency components of the signal, time history of the displacement signal can be nicely retrieved by directly integrating the signal. Therefore, if one intends to retrieve the displacement signal form the acceleration signal afterward, he has to measure the signal with sufficiently fine sampling rate before the measurement. When there is no leakage in the signal, even though the condition can seldom be satisfied in real situations, time history of the displacement signal of the measured acceleration signal can also be retrieved nicely by using the frequency domain method. In this case, the frequency domain method can be applied for both high and low frequency signals. The frequency domain method is not good for the signals with leakage or the signals decayed out, because the dominant signal components become less than the low frequency noise components during the conversion process. To retrieve the time history of displacement from already measured acceleration signal with relatively high frequency components, using the frequency domain method is practically the only choice. When using the frequency domain method, zero-padding the Fourier coefficients of noise components appears to be the simplest and provides least amount of error during the retrieving process.

Acknowledgment

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References

Bendat, J. S. and Piersol, A. G., 1991, *Random data, 2nd Edition*, John Wiley & Sons Inc., New York.

Burden, R. L., Faires, J. D. and Reynolds, A. C., 1979, *Numerical Analysis*, Prindle, Weber & Schmidt, pp. 189~191.

Chen, Z. and Mechefske, C. K., 2001, "Machine Signature Identification by Analysis of Impulse Vibration Signals," *Journal of Sound and Vibrations*, Vol. 244, No. 1, pp. 155~167.

Feeny, B. F. and Kappagantu, R, 1998, "On the Physical Interpretation of proper Orthogonal Modes in Vibrations," *Journal of Sound and Vibration*, Vol. 211, No. 4, pp. 607~616.

Han, S., 2001, "Analysis of Errors in the Conversion of Acceleration into displacement," *Proceedings of 19th IMAC*, CD ROM.

Lee, S. K., 1999, "Application of the L-Wigner Distribution to the Diagnosis of Local Defects of Gear Tooth," *KSME International Journal*, Vol. 13, No. 2, pp. 144~157.

Luo, G. Y., Osypiw, D. and Irle, M., 2000, "Real-Time Condition Monitoring by Significant and Natural Frequencies Analysis of Vibration Signal with Wavelet Filter and Autocorrelation Enhancement," *Journal of Sound and Vibration*, Vol. 236, No. 3, pp. 413~430.

McConnell, K. G., 1995, *Vibration Testing, Theory and Practice*, John Wiley & Sons Inc, New York.

Newland, D. E., 1993, *An Introduction to Random Vibrations, Spectral & Wavelet Analysis*, 3rd. ed. John Wiley & Sons Inc., New York.

Papoulis, A., 1991, *Probability, Random Variables and Stochastic Processes, 3rd Edition*, McGraw-Hill International Edition.

Tang, S. K., 2000, "On the Time-Frequency Analysis of Signals That Decay Expontially with Time," *Journal of Sound and Vibration*, Vol. 242, No. 2, pp. 241～258.