# Dynamic Analysis of Bending-Torsion Coupled Beam Structures Using Exact Dynamic Elements

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## ABSTRACT

Beams are often subject to bending-torsion coupled vibration due to mass coupling and/or stiffness coupling. This paper proposes a dynamic analysis method using the exact dynamic element for bending-torsion coupled vibration of general plane beam structures with joints. The exact dynamic element matrix for a bending-torsion coupled beam is derived, and the detailed procedure of using the exact dynamic element matrix is also presented. Three examples are provided for validating and illustrating the proposed method. The numerical study proves the proposed method to be useful for dynamic analysis of bending-torsion coupled beam structures with joints.

**Keywords** : Bending-torsion coupled vibration, Composite beam, Non-symmetric beam, Stiffness coupling, Mass coupling, Exact dynamic elements

## 1. Introduction

Beams are often subject to bending-torsion coupled vibrations due to either stiffness coupling or mass coupling. It is well known from the literature that stiffness coupling is present in composite beams [1-4], and mass coupling is involved in non-symmetric beams due to the difference between the mass axis and the elastic axis [5-9]. In recent years, bending-torsion coupled vibration of beams has attracted much attention from many researchers with the increasing use of composite beams and non-symmetric beams. In particular, the dynamic stiffness matrix method has been often adopted [5, 7-9] for bending-torsion coupled vibration problems in order to reduce the system matrix size or to obtain exact solutions. However, exact solutions of bending-torsion coupled vibrations for general beam structures supported and/or connected by joints with damping have been rarely discussed. It is believed to be desirable to develop a systematic method for attaining exact solutions of bending-torsion coupled vibration of distributed-parameter beam structures with joints.

This paper presents a dynamic analysis method using exact dynamic elements for general beam structures with joints, which are subject to stiffness coupling and/or mass coupling in bending and torsion. To this end, the exact dynamic element method (EDEM), which was proposed and proved useful in [10-12], is applied to a general, unified beam equation, which accounts for both the stiffness and mass coupling effects. The derivation procedure to obtain the exact dynamic element matrix for a uniform bending-torsion coupled beam is presented in detail. However, unlike other applications of the EDEM published in the past few years, the exact dynamic element matrix for the bending-torsion coupled beam necessitates a numerical procedure. Once the exact dynamic element matrix is derived, a beam structure can be modeled by assembling discretized elements in the same manner as in the finite element method (FEM). The advantages of using exact dynamic elements are evident as already discussed in the literature: e.g., it can deliver exact solutions for distributed-parameter systems, a great reduction for the system matrix size is also expected since a uniform beam segment, regardless of the length, can be modeled by a single element, and changing parameters for any uniform beam section can be easily accomplished [10-12]. In addition, since the proposed formulation not only provides us with the exact dynamic element matrix, which is equivalent to element matrices in the finite element method but also makes use of the same nodal coordinate system as the finite element method for beam elements, the proposed modeling method can be incorporated with the finite element method for modeling and analysis of a complicated system.

In order to validate the proposed method, three numerical examples are presented. A simple, open box cantilever beam is considered as the first numerical example, wherein eigenvalues by the proposed method are compared with those from a reference. Another example of application for a U-shaped beam hinged at two positions is presented, which is also quoted from a reference. In the final example, a general beam structure with joints is analysed as a rigorous application of the proposed method. The numerical study shows that the proposed method is very useful for the dynamic analysis of general distributed-parameter beam structures subject to bending-torsion coupling.

## 2. Modeling of bending-torsion coupled beams

### 2.1 Equations of motion of a beam element

Figure 1 shows a typical, non-symmetric beam and the associated coordinate system. The governing equations for such a general beam subject to both mass coupling and stiffness coupling in bending and torsion can be represented, neglecting warping effect, [5,8] as

$$EI\frac{\partial^2\theta}{\partial x^2}+kAG\left(\frac{\partial u}{\partial x}-\theta\right)+K\frac{\partial^2\varphi}{\partial x^2}=\rho I\frac{\partial^2\theta}{\partial t^2} \quad (1\text{-}1)$$

$$kAG\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial\theta}{\partial x}\right)=m\left(\frac{\partial^2 u}{\partial t^2}-y_\alpha\frac{\partial^2\varphi}{\partial t^2}\right) \quad (1\text{-}2)$$

$$GJ\frac{\partial^2\varphi}{\partial x^2}+K\frac{\partial^2\theta}{\partial x^2}=I_\alpha\frac{\partial^2\varphi}{\partial t^2}-my_\alpha\frac{\partial^2 u}{\partial t^2} \quad (1\text{-}3)$$

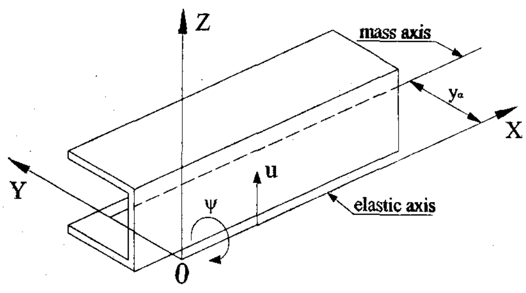


Fig. 1 Typical non-symmetric beam and the coordinate system

where $u$, $\theta$, and $\varphi$ are the transverse, angular and torsional displacements of the beam, respectively. $\rho$, $G$ and $E$ are the density, shear modulus and Young's modulus, respectively. $A$, $I$ and $J$ are the area, the diametral moment of inertia and the polar moment of inertia, respectively, $k$ being the shape factor that is dependent on the cross sectional shape. $K$ is the coupling stiffness between bending and torsion. $m, I_\alpha$ and $y_\alpha$ are the mass per unit length, the polar mass moment of inertia per unit length of the beam and the distance between the mass and elastic axes. Equations (1-1) to (1-3) can be rewritten in a spatial state equation form as

$$\frac{\partial u}{\partial x}=\theta-\frac{F}{kAG} \quad (2\text{-}1)$$

$$\frac{\partial\theta}{\partial x}=\frac{GJ}{GJ\cdot EI-K^2}M-\frac{K}{GJ\cdot EI-K^2}T \quad (2\text{-}2)$$

$$\frac{\partial\varphi}{\partial x}=\frac{EI}{GJ\cdot EI-K^2}T-\frac{K}{GJ\cdot EI-K^2}M \quad (2\text{-}3)$$

$$\frac{\partial F}{\partial x}=-m\left(\frac{\partial^2 u}{\partial t^2}-y_\alpha\frac{\partial^2\varphi}{\partial t^2}\right) \quad (2\text{-}4)$$

$$\frac{\partial M}{\partial x}=F+\rho I\frac{\partial^2\theta}{\partial t^2} \quad (2\text{-}5)$$

$$\frac{\partial T}{\partial x}=-my_\alpha\frac{\partial^2 h}{\partial t^2}+I_\alpha\frac{\partial^2\varphi}{\partial t^2} \quad (2\text{-}6)$$

where $F, M$ and $T$ are the corresponding force, moment and torque, respectively.

## 2.2 Derivation of exact dynamic element matrix

The Laplace transformation of equations (2-1) to (2-6) with respect to time, with zero initial conditions, leads to

$$\frac{\partial \boldsymbol{\Psi}(s,x)}{\partial x} = \boldsymbol{B}(s)\boldsymbol{\Psi}(s,x) \tag{3}$$

where,

$$\boldsymbol{\Psi}(s,x) = \left\{u^* \theta^* \varphi^* F^* M^* T^*\right\}^T,$$

$$\mathbf{B}(s) = \begin{bmatrix} 0 & 1 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & -c \\ 0 & 0 & 0 & 0 & -c & d \\ -e & 0 & f & 0 & 0 & 0 \\ 0 & g & 0 & 1 & 0 & 0 \\ -f & 0 & h & 0 & 0 & 0 \end{bmatrix}$$

Here, the asterisk represents the Laplace transform of the corresponding state variable, s being the Laplace variable for time. In addition,

$$a = \frac{1}{kAG},\ b = \frac{GJ}{GJ \cdot EI - K^2},$$
$$c = \frac{K}{GJ \cdot EI - K^2},\ d = \frac{EI}{GJ \cdot EI - K^2}$$

$$e = ms^2,\ f = my_\alpha s^2,\ g = \rho I s^2,\ h = I_\alpha s^2$$

For a beam with cross section symmetric about two principal axes, $y_\alpha = 0$ . A beam without stiffness coupling effect can be dealt with by setting $K = 0$ . One can set $kAG = \infty$, $\rho I = 0$ or $a = g = 0$ in order to treat an Euler-Bernoulli beam model.

The Laplace transformation of equation (3) for the spatial coordinate $x$ , with consideration of boundary values at $x = 0$, may yield

$$\tilde{\boldsymbol{\Psi}}(s,\lambda) = [\lambda \boldsymbol{I} - \boldsymbol{B}]^{-1}\boldsymbol{\Psi}(s,0) \tag{4}$$

Here, $\lambda$ is the Laplace variable for the spatial coordinate, and $\tilde{\boldsymbol{\Psi}}(s,\lambda)$ represents the spatial Laplace transform for $\boldsymbol{\Psi}(s,x)$ . Evaluation of the exact dynamic element matrix requires an analytical expression for $[\lambda \boldsymbol{I} - \boldsymbol{B}]^{-1}$ . The following mathematical relation is used:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} Adj(\mathbf{A}) \tag{5}$$

where Adj($\boldsymbol{A}$) represents the adjoint matrix for $\boldsymbol{A}$. The determinant for $[\lambda \boldsymbol{I} - \boldsymbol{B}]$ can be written as

$$\det[\lambda \boldsymbol{I} - \boldsymbol{B}] = (\lambda^2 - \alpha^2)(\lambda^2 - \beta^2)(\lambda^2 - \gamma^2) \tag{6-1}$$

where $\alpha, \beta, \gamma$ may be determined by solving the following third order algebraic equation:

$$\det[\lambda \boldsymbol{I} - \boldsymbol{B}] = \lambda^6 + v_1\lambda^4 + v_2\lambda^2 + v_3 = 0 \tag{6-2}$$

where

$$v_1 = -(dh + bg + ea),\ v_2 = -(dh + bg + ea)$$
$$v_3 = f^2bgad + f^2db\text{-}f^2c^2\text{-}f^2gac^2 + c^2he\text{-}bedh\text{-}begadh + egac^2h$$

The adjoint matrix for $[\lambda \boldsymbol{I} - \boldsymbol{B}]$ is also expressed analytically with the help of the symbolic operation provided by Matlab [13]. Then, the inverse Laplace transformation of equation (4) for $x$ gives the following:

$$\boldsymbol{\Psi}(s,x) = \boldsymbol{C}(s,x)\boldsymbol{\Psi}(s,0) \tag{7}$$

where $\boldsymbol{C}(s,x)$ is a kind of exact transfer matrix. Substitution of the forces and displacements at $x = 0$ and $x = \xi$ into equation (7) and rearrangement of the variables in equation (7) yield

$$\begin{Bmatrix} F_1^* \\ M_1^* \\ T_1^* \\ F(\xi)^* \\ M(\xi)^* \\ T(\xi)^* \end{Bmatrix} = \mathbf{D}^e(s,\xi) \begin{Bmatrix} u_1^* \\ \theta_1^* \\ \varphi_1^* \\ h(\xi)^* \\ \theta(\xi)^* \\ \varphi(\xi)^* \end{Bmatrix} \tag{8}$$

where,

$$D^e(s,\xi) = \begin{bmatrix} D_{11}^e(s,\xi) & D_{12}^e(s,\xi) \\ D_{21}^e(s,\xi) & D_{22}^e(s,\xi) \end{bmatrix}$$

The exact dynamic element matrix is described in the appendix. Upon substituting $\ell$ for $\xi$, one can have the following equation:

$$\begin{Bmatrix} F_1^* \\ M_1^* \\ T_1^* \\ F_2^* \\ M_2^* \\ T_2^* \end{Bmatrix} = \mathbf{D}^e(s,\ell) \begin{Bmatrix} u_1^* \\ \theta_1^* \\ \varphi_1^* \\ h_2^* \\ \theta_2^* \\ \varphi_2^* \end{Bmatrix} \quad (9)$$

where $D^e(s,\ell)$ is an exact dynamic element matrix of a bending-torsion coupled beam element in s domain. Although $D^e(s,\ell)$ is exact by definition, it is not analytical because some parameters, e.g. $\alpha, \beta, \gamma$ are obtained through numerical computation.

## 3. Dynamic Analysis

### 3.1 The global system matrix and the direct inversion

The assembling procedure for the global system matrix is the same as that of the FEM. After meshing the entire structure into uniform, distributed-parameter beam elements, and lumped inertia and joint elements, one can assemble the element dynamic matrices in the same manner as the global matrices are constructed in the FEM. This assembling procedure may result in the following system matrix equation:

$$F^*(s) = D(s)U^*(s) \quad (10)$$

where $U^*$ and $F^*$ are the Laplace transforms of the global displacement and force vectors.

From equation (10), the transfer function and frequency response function matrices can be written by

$$H(s) = D^{-1}(s) \quad (11)$$

and

$$H(j\omega) = D^{-1}(s)|_{s=j\omega} \quad (12)$$

Exact transfer function and frequency response function matrices can be obtained through direct computation of equations (11) and (12).

### 3.2 The eigenvalue problem

The eigenvalue problem associated with equation (10) is written as

$$D(s)U^*(s) = \mathbf{0} \quad (13)$$

Thus, the eigenvalues associated with equation (13) can be attained from the nontrivial solution condition, i.e.,

$$\det\{D(s)\} = 0 \quad (14)$$

Equation (14) necessitates a special algorithm for solving nonlinear equations. In this paper, a modified bisection method is adopted which is suitable for general complex equations. It is obvious that the number of eigenvalues is infinite because $D(s)$ contains transcendental functions. The corresponding eigenvectors can be readily obtained by using equation (13), once equation (14) is solved.

Using the responses at nodal points, the responses at interior points between two nodal points can be obtained from the relation as

$$\begin{Bmatrix} u^*(\xi) \\ \vartheta^*(\xi) \\ \varphi^*(\xi) \end{Bmatrix} = [N(s,\xi)] \begin{Bmatrix} u_1^* \\ \vartheta_1^* \\ \varphi_1^* \\ u_2^* \\ \vartheta_2^* \\ \varphi_2^* \end{Bmatrix} \quad (15)$$

where

$$N(s,\xi) = \left[D_{12}^e(s,\xi)\right]^{-1} \left[D_{11}^e(s,\ell) - D_{11}^e(s,\xi) \quad D_{12}^e(s,\ell)\right]$$

Equation (15) is of great use to draw exact mode shapes.

## 4. Numerical Examples

Three examples are provided here to validate the proposed method. In the first example, a simple composite beam is considered to compare the proposed method with an existing method. In the second example, a cantilever U-shaped beam hinged at two positions is analyzed. The final example illustrates the proposed method applied to a general beam supported by two joints with mixed non-symmetric and symmetric cross sections.

Figure 2 shows numerical model 1, which is composed of a uniform, composite Timoshenko beam clamped at the left end. The detailed specifications of the beam are given in Table 1. The data of this example are quoted from the reference by Banerjee [5]. In this case, the system is subject to stiffness coupling. For modeling the current system, only one element is sufficient because the beam is uniform throughout the entire length. Since the left end of the beam is clamped, the order of the system matrix is just 3. In Table 2, the natural frequencies obtained by the proposed method are compared with the ones quoted from [5]. It can be clearly seen that the natural frequencies from both methods show almost identical results.

To demonstrate the dynamic analysis of a non-symmetric beam, a U-shaped beam, which was simulated in [9], has been considered as the next example. Figure 3 shows the numerical model, in which a uniform U-shaped beam is clamped at the left end and hinged at two positions on the elastic axis. The system is subject to mass coupling due to the non-symmetric characteristics. The detailed specifications of the model are given in Table 3. The Euler-Bernoulli beam model is used in this example. The system is divided into three elements of which nodal points are taken at hinged positions and two boundary positions. The order of the system matrix is 7. Table 4 compares the natural frequencies by the proposed method and those from the reference [9]. The table shows that the proposed method gives as accurate results as the existing method in [9].

The final example deals with a general beam structure, which has both non-symmetric and symmetric cross sections and is supported by joints. Figure 4 shows the numerical model and Table 5 describes the specifications of the model. Three elements are used to model the system. In this case, the order of the system matrix is 9. Table 6 presents eigenvalues of the model with the transverse damping coefficient of joints varied. It is clearly shown that the increase of damping changes the real parts of the eigenvalues. However, since the torsional motions are dominant for lower modes, the applied transverse damping has only a little effect on the modal damping of lower modes.

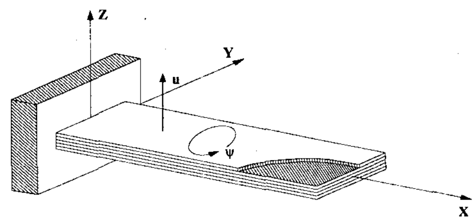


Fig. 2 Numerical model 1 : a composite beam [5]

Table 1 Specifications of numerical model 1 [5]

| *Property* | *Data* |
|---|---|
| Length, m | 0.1905 |
| Width, m | 0.0127 |
| Thickness, m | 0.00318 |
| EI, N m² | 0.2865 |
| GJ, N m² | 0.1891 |
| K, N m² | 0.1143 |
| m, kg/m | 0.0544 |
| $I_\alpha$, kg m | $0.7770\times10^{-6}$ |
| kAG, N | 6343.3 |

Table 2 Comparison of natural frequencies for numerical model 1 from the reference [5] and the proposed method

| Mode | Natural Frequencies (Hz) | |
|---|---|---|
| | Reference [5] | Proposed method |
| 1 | 30.75 | 30.75 |
| 2 | 189.8 | 189.8 |
| 3 | 518.8 | 518.8 |
| 4 | 648.3 | 648.3 |
| 5 | 986.1 | 986.2 |

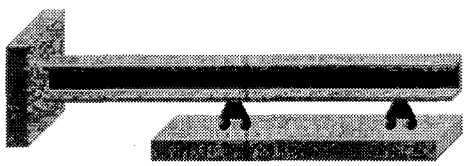

Fig. 3 Numerical model 2: a cantilever U-shaped beam hinged at two positions

Table 3 Specifications of numerical model 2

| Property | Data |
|---|---|
| Length, m | 3/3/1 |
| Width, m | 0.0889 |
| Height, m | 0.1524 |
| Thickness, m | 0.0071 |
| EI, MN m² | 1.704 |
| GJ, KN m² | 3.14 |
| m, kg/m | 17.61 |
| $I_\alpha$, kg m | 0.1342 |
| $y_\alpha$, m | 0.05626 |

Table 4 Comparison of natural frequencies for numerical model 2 from the reference [9] and the proposed method

| Mode | Natural Frequencies (Hz) | |
|---|---|---|
| | Reference [9] | Proposed method |
| 1 | 5.462 | 5.4614 |
| 2 | 16.34 | 16.3429 |
| 3 | 26.14 | 26.1382 |

The damped natural frequencies ($\omega_k$) of the first four modes slightly increase with the transverse damping while that of the fifth mode decreases with the transverse damping. Figure 5 illustrates the first three mode shapes of the system without damping, which are synthesized by equation (15).

## 5. Concluding Remarks

In this study, an exact dynamic element matrix for a bending-torsion coupled beam element was derived. The exact dynamic element matrix for the bending-torsion coupled beam element, together with the other two element matrices for lumped inertia and joint elements, was used to model the global system dynamic matrix of general beam structures. Three numerical examples were provided to show the validity and applicability of the proposed method in the dynamic analysis of bending-torsion coupled vibration of distributed parameter beam systems having joints. The proposed method provides an exact model with finite matrix size for bending-torsion coupled beam systems with joints. The proposed method allows bending-torsion couple vibration analysis in the presence of both stiffness coupling and mass coupling. The matrix size of the model is expected to be small, since most beam structures are composed of standard, uniform beam segments and any uniform beam segment can be modeled by a beam element without causing any error. In addition, the proposed method makes it easy to perform a parametric analysis on the system because any re-meshing process is not required even when beam dimensions are changed.

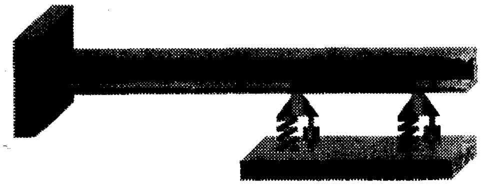

Fig. 4 Numerical model 3: a general beam structure with transverse joints

Table 5 Specifications of numerical model 3

| Element No. | #1 | #2 | #3 |
|---|---|---|---|
| Length, m | 4 | 2 | 1 |
| Width, m | 0.09 | 0.09 | 0.09 |
| Height, m | 0.16 | 0.16 | 0.16 |
| Thickness, m | 0.007 | 0.007 | 0.007 |
| EI, MN m² | 1.8021 | 2.1579 | 1.8021 |
| GJ, KN m² | 3.1726 | 4.1014 | 3.1726 |
| M, kg/m | 17.96 | 26.00 | 17.96 |
| $I_\alpha$, kg m | 0.1342 | 0.1211 | 0.0867 |
| $y_\alpha$, m | 0.05626 | 0 | 0.05632 |
| Transverse Joints (2 identical) | Damping, $c_t$, Ns/m | | Stiffness, $k_t$, MN/m |
| | 0/20/1000 | | 2 |

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Table 6 Comparison of eigenvalues for numerical model 3 with the joint damping varied

| Mode | Eigenvalue $(\sigma_k + j\omega_k)$ $-\sigma_k(rad/s)/\omega_k(Hz)$ | | |
|---|---|---|---|
| | $c_t = 0$ Ns/m | $c_t = 20$ Ns/m | $c_t = 1000$ N s/m |
| 1 | 0.000000/ 5.704251 | 0.000794/ 5.704251 | 0.039704/ 5.704303 |
| 2 | 0.000000/ 8.389381 | 0.003156/ 8.389382 | 0.157799/ 8.389635 |
| 3 | 0.000000/ 19.218267 | 0.006774/ 19.218270 | 0.338372/ 19.224855 |
| 4 | 0.000000/ 32.646829 | 0.009777/ 32.646831 | 0.486617/ 32.651903 |
| 5 | 0.000000/ 47.706620 | 0.067510/ 47.706459 | 4.591260/ 47.503232 |

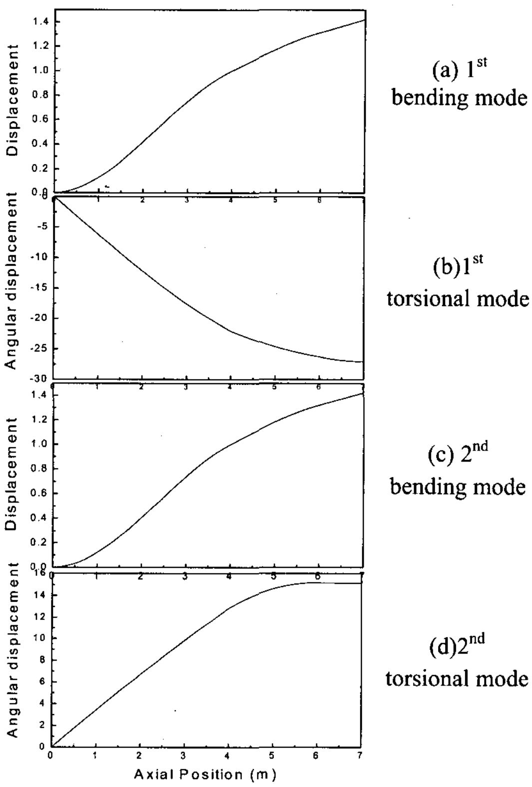


Fig. 5 First two mode shapes of numerical model 3

## Appendix: The exact dynamic element matrix of a bending-torsion coupled beam element

$$\mathbf{D}^e(s,\xi)=\begin{bmatrix} \boldsymbol{D}_{11}(s,\xi) & \boldsymbol{D}_{12}(s,\xi) \\ \boldsymbol{D}_{21}(s,\xi) & \boldsymbol{D}_{22}(s,\xi)\end{bmatrix} = \begin{bmatrix} -\mathbf{C}_{12}{}^{-1}\mathbf{C}_{11} & \mathbf{C}_{12}{}^{-1} \\ \mathbf{C}_{21}-\mathbf{C}_{22}\mathbf{C}_{12}{}^{-1}\mathbf{C}_{11} & \mathbf{C}_{22}\mathbf{C}_{12}{}^{-1}\end{bmatrix}$$

$$\mathbf{C}_{11}=\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33}\end{bmatrix} \quad \mathbf{C}_{12}=\begin{bmatrix} c_{14} & c_{15} & c_{16} \\ c_{24} & c_{25} & c_{26} \\ c_{34} & c_{35} & c_{36}\end{bmatrix}$$

$$\mathbf{C}_{21}=\begin{bmatrix} c_{41} & c_{42} & c_{43} \\ c_{51} & c_{52} & c_{53} \\ c_{61} & c_{62} & c_{63}\end{bmatrix} \quad \mathbf{C}_{22}=\begin{bmatrix} c_{44} & c_{45} & c_{46} \\ c_{54} & c_{55} & c_{56} \\ c_{64} & c_{65} & c_{66}\end{bmatrix}$$

$$c_{11}=q_5-(bg+dh)q_3+fcq_2+(-hgc^2+hbgd)q_1$$
$$c_{12}=-q_4+dhq_2-(gcfa+fc)q_1$$
$$c_{13}=-faq_3-chq_2+(fbga+bf)q_1$$
$$c_{14}=-aq_4+(b+bga+adh)q_2+(c^2h-bdh-ahbgd+ahgc^2)q_0$$
$$c_{15}=bq_3+acfq_2+(-bdh+c^2h)q_1$$
$$c_{16}=cq_3+afdq_2-(-afc^2g+afbgd-fc^2+fdb)q_0$$
$$c_{21}=-cfq_3+ebq_2-(f^2c^2-f^2db-c^2he+bedh)q_0$$
$$c_{22}=q_5-(ea+dh)q_3+fcq_2-(f^2ad-adhe)q_1$$
$$c_{23}=chq_3-bfq_2+(acf^2-ache)q_1$$
$$c_{24}=-bq_3+acfq_2+(bdh-c^2h)q_1$$
$$c_{25}=-bq_4+(bdh-c^2h+bea)q_2+(-f^2ac^2+f^2bad+ac^2he-badhe)q_0$$
$$c_{26}=-cq_4+ceaq_2+(-fc^2+fdb)q_1$$
$$c_{31}=-fdq_3+ecq_2+(-fc^2g+fbgd)q_1$$
$$c_{32}=gcq_3+fdq_2-(egca+ec)q_1$$
$$c_{33}=q_5-(bg+ea)q_3-fcq_2+(bgea+be)q_1$$
$$c_{34}=-cq_3+afdq_2-(-afc^2g+afbgd-fc^2+fdb)q_0$$
$$c_{35}=-cq_4+ceaq_2+(fc^2-fdb)q_1$$
$$c_{36}=-dq_4-(-ade-bgd+c^2g)q_2-(-ac^2eg+abgde-c^2e+dbe)q_0$$
$$c_{41}=eq_4-(bge+edh-f^2d)q_2-(-f^2c^2g+f^2bgd+eghc^2-begdh)q_0$$
$$c_{42}=-eq_3-gcfq_2-(df^2-edh)q_1$$
$$c_{43}=-fq_4+fbgq_2+(cf^2-che)q_1$$
$$c_{44}=-q_5+(bg+dh)q_3+fcq_2+(hgc^2-hbgd)q_1$$
$$c_{45}=cfq_3+ebq_2+(-f^2c^2+f^2db+c^2he-bedh)q_0$$
$$c_{46}=fdq_3+ecq_2-(-fc^2g+fbgd)q_1$$
$$c_{51}=eq_3-gcfq_2+(df^2-edh)q_1$$
$$c_{52}=gq_4-(e+gea+gdh)q_2-(-edh-egadh+df^2+gadf^2)q_0$$
$$c_{53}=-fq_3+gchq_2+(gacf^2-egach+cf^2-che)q_0$$
$$c_{54}=-q_4+dhq_2+(gcfa+fc)q_1$$
$$c_{55}=-q_5+(dh+ea)q_3+fcq_2+(f^2ad-adhe)q_1$$
$$c_{56}=-gcq_3+fdq_2+(egca+ec)q_1$$
$$c_{61}=-fq_4+fbgq_2+(-cf^2+che)q_1$$
$$c_{62}=fq_3+gchq_2+(cf^2+gacf^2-egach-che)q_0$$
$$c_{63}=hq_4-(bgh+ahe-f^2a)q_2-(bf^2+f^2bga-begah-ebh)q_0$$
$$c_{64}=faq_3-chq_2-(bf+fbga)q_1$$
$$c_{65}=-chq_3-bfq_2-(-ache+acf^2)q_1$$
$$c_{66}=-q_5+(bg+ea)q_3-fcq_2-(be+bgea)q_1$$

$$\Delta=-\{(\alpha^2-\beta^2)(\beta^2-\gamma^2)(\gamma^2-\alpha^2)\}^{-1}$$
$$A=(\beta^2-\gamma^2),\ B=(\gamma^2-\alpha^2),\ C=(\alpha^2-\beta^2)$$
$$q_0=\Delta\{\alpha^{-1}A\sinh\alpha\xi+\beta^{-1}B\sinh\beta\xi+\gamma^{-1}C\sinh\gamma\xi\}$$
$$q_1=\Delta\{A\cosh\alpha\xi+B\cosh\beta\xi+C\cosh\gamma\xi\}$$
$$q_2=\Delta\{\alpha A\sinh\alpha\xi+\beta B\sinh\beta\xi+\gamma C\sinh\gamma\xi\}$$
$$q_3=\Delta\{\alpha^2A\cosh\alpha\xi+\beta^2B\cosh\beta\xi+\gamma^2C\cosh\gamma\xi\}$$
$$q_4=\Delta\{\alpha^3A\sinh\alpha\xi+\beta^3B\sinh\beta\xi+\gamma^3C\sinh\gamma\xi\}$$
$$q_5=\Delta\{\alpha^4A\cosh\alpha\xi+\beta^4B\cosh\beta\xi+\gamma^4C\cosh\gamma\xi\}$$