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REPRESENTATION OF L^1-VALUED CONTROLLER ON BESOV SPACES

JIN-MUN JEONG AND DONG-HWA KIM

ABSTRACT. This paper will show that the relation

$$L^1(\Omega) \subset C_0(\overline{\Omega}) \subset H_{p,q} \tag{1.1}$$

if $1/p' - 1/n(1-2/q') < 0$ where $p' = p/(p-1)$ and $q' = q/(q-1)$ where $H_{p,q} = (W_0^{1,p}, W^{-1,p})_{1/q,q}$ We also intend to investigate the control problems for the retarded systems with $L^1(\Omega)$-valued controller in $H_{p,q}$

1. Introduction

Let Ω be a bounded domain in $\mathcal{R}^n$ with smooth boundary $\partial\Omega$. Let $\mathcal{A}(x, D_x)$ be an elliptic differential operator of second order as follows.

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

where $\{a_{i,j}(x)\}$ is a positive definite symmetric matrix for each $x \in \Omega$, $b_i \in C^1(\overline{\Omega})$ and $c \in L^\infty(\Omega)$

If we put that $A_0u = -\mathcal{A}(x, D_x)u$ then it is known that A_0 generates an analytic semigroup in $W^{-1,p}(\Omega)$ where $W^{-1,p}(\Omega)$ is the dual

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space of $W_0^{1,p'}(\Omega), p' = p/(p-1)$ as is seen in [5]. Therefore, from the interpolation theory it is easily seen that the operator A_0 generates an analytic semigroup in $H_{p,q} = (W_0^{1,p}, W^{-1,p})_{1/q,q}$. In the section 4, we will show that the relation

$$L^1(\Omega) \subset C_0(\overline{\Omega}) \subset H_{p,q} \tag{1.1}$$

if $1/p' - 1/n(1-2/q') < 0$ where $p' = p/(p-1)$ and $q' = q/(q-1)$. Hence we intend to investigate the control problem for the following retarded system with $L^1(\Omega)$-valued controller:

$$\begin{aligned} \text{(1.2)} \quad \frac{d}{dt}u(t) =& A_0u(t) + A_1u(t-h) \\ &+ \int_{-h}^0 a(s)A_2u(t+s)ds + \Phi_0 w(t), \quad t \in (0,T] \\ \text{(1.3)} \qquad u(0) =& g^0, \quad u(s) = g^1(s) \qquad\qquad s \in [-h, 0), \end{aligned}$$

in the space $H_{p,q}$. Here, $A_\iota u = -\mathcal{A}_\iota(x, D_x)u$, $\iota = 1, 2$, where $\mathcal{A}_\iota(x, D_x)$ are second order linear differential operators with real coefficients. The kernel $a(\cdot)$ belongs to $L^{q'}(-h, 0)$ where h is a fixed positive number and the controller Φ_0 is a bounded linear operator from some Banach space U to $L^1(\Omega)$. The initial data g^0, g^1 are given functions so that needed for the construction of solution semigroup for (1.2) and (1.3) and of $L^1(\Omega)$- valued controller. From the relation (1.1) we shall deal with approximate controllability and observability for the system (1.2) and (1.3) in the space $H_{p,q}$ choosing p and q such that $1/p' - 1/n(1-2/q') < 0$.

In view of Sobolev's embedding theorem we may also consider $L^1(\Omega) \subset W^{-1,p}(\Omega)$ if $1 < p < n/(n-1)$ as is seen in [5]. Hence, we can investigate the system (1.2) and (1.3) in the space $W^{-1,p}(\Omega)$ considering Φ_0 as an operator into $W^{-1,p}(\Omega)$. Furthermore, it is known that $W^{-1,p}(\Omega)$ is ζ-convex and the initial value problem

$$\begin{aligned} \text{(1.4)} \qquad \frac{d}{dt} &= A_0u(t) + f(t), \quad t \in (0,T], \\ u(0) &= u_0 \end{aligned}$$

has a unique solution $u \in L^q(0,T;W_0^{1,p}(\Omega) \cap W^{1,q}(0,T;W^{-1,p}(\Omega))$ for any $u_0 \in H_{p,q}$ and $f \in L^q(0,T;W^{-1,p}(\Omega))$ (see Theorem 3.1 in [5]). Thereafter, we can apply the method of G. D. Blasio, K. kunisch and E. Sinestrari [3] to the system (1.2) and (1.3) to show the existence and uniqueness of the solution

$$u \in L^q(0,T;W_0^{1,p}(\Omega) \cap W^{1,q}(0,T;W^{-1,p}(\Omega)) \subset C([0,T];H_{p,q})$$

Since Φ_0 is a mapping into $W^{-1,p}(\Omega)$ not into $H_{p,q}$, we cannot express the solution $u(t;0,\Phi_0 w)$ of system of (1.2) and (1.3) with $g=0$ using the solution semigroup $S(t)$. Here, the solution semigroup for the system (1.2) and (1.3) is defined by

$$S(t)g = (u(t;g,0), u_t(\cdot;g,0))$$

where $g = (g^0, g^1) \in Z_{p,q} = H_{p,q} \times L^q(-h,0;W_0^{1,p}(\Omega))$, $u(t;g,0)$ is the solution of (1.2) and (1.3) with $\Phi_0 = 0$ and $u_t(\cdot,g,0)$ is the function $u_t(s;g,0) = u(t+s;g,0)$ defined in $[-h,0]$. Therefore, we have to define the approximate controllability and observability in $W^{-1,p}(\Omega)$ using the fundamental solution as is seen in definition 5.1 in [5]. Here, we note that in order to existing of fundamental solution of system (1.2) and (1.3), we must need the assumption that $a(\cdot)$ is Hölder continuous.

In this paper, assuming that $a(\cdot)$ has only to belong to $L^{q'}(-h,0)$, with the aid of the relation (1.1) we can define the approximate controllability and observability in $H_{p,q}$ without using the fundamental solution. We define the set of attainability by

$$R = \{\int_0^t S(t-\tau)\Phi w(t)d\tau : w \in L^q(0,t;U), \quad t > 0\}$$

where $\Phi w = (\Phi_0 w, 0)$. We say that the system (1.2) and (1.3) is approximately controllable if R is dense in $Z_{p,q}$ and the adjoint system is observability if $\phi \in Z_{p',q'}$, $\Phi_0^* v(t;\phi) \equiv 0$ implies $\phi = 0$. where $v(t;\phi)$ is a solution the following adjoint system of (1.2) and (1.3).

(1.4)

$$\frac{d}{dt}v(t) = A_0^* v(t) + A_1^* v(t-h) + \int_{-h}^0 a(s)A_2 u(t+s)ds,$$

$$(1.5) \qquad v(0) = \phi^0, \quad v(s) = \phi^1(s), \qquad s \in [-h,0),$$

where $\phi = (\phi^0, \phi^1) \in Z_{p',q'}$. The structural operator $F : Z_{p,q} \longrightarrow Z^*_{p',q'}$ is defined by

$$Fg = (g^0, A_1 g^1(-h-s) + \int_{-h}^{0} a(\tau) A_2 g^1(\tau - s) d\tau).$$

We will show that if F is an isomorphism, then the approximate controllability of (1.2) and (1.3) is equivalent to the observability of (1.4) and (1.5). Finally, we remark that in the space $W^{-1,p}(\Omega)$ we can not define the attainability set using solution semigroup $S(t)$ and it is said that the system (1.4) and (1.5) is observable if $\phi \in Z_{p',q'}$, $\Phi_0^* v(t;\phi) \equiv 0$ almost everywhere implies $\phi = 0$.

2. Notations

Let Ω be a region in an n-dimensional Euclidean space $\mathcal{R}^n$ and closure $\overline{\Omega}$. $C^m(\Omega)$ is the set of all m-times continuously differential functions on Ω. $C_0^m(\Omega)$ will denote the subspace of $C^m(\Omega)$ consisting of these functions which have compact support in Ω. $W^{m,p}(\Omega)$ is the set of all functions $f = f(x)$ whose derivative $D^\alpha f$ up to degree m in distribution sense belong to $L^p(\Omega)$. As usual, the norm is then given by

$$||f||_{m,p,\Omega} = (\sum_{\alpha \le m} ||D^\alpha f||^p_{p,\Omega})^{\frac{1}{p}}, \quad 1 \le p < \infty$$
$$||f||_{m,\infty,\Omega} = \max_{\alpha \le m} ||D^\alpha u||_{\infty,\Omega},$$

where $D^0 f = f$. In particular, $W^{0,p}(\Omega) = L^p(\Omega)$ with the norm $||\cdot||_{p,\Omega}$. Let $p' = p/(p-1)$, $1 < p < \infty$. $W^{-1,p}(\Omega)$ stands for the dual space $W_0^{1,p'}(\Omega)^*$ of $W_0^{1,p'}(\Omega)$ whose norm is denoted by $||\cdot||_{-1,p,\infty}$.

If X is a Banach space and $1 < p < \infty$, $L^p(0,T;X)$ is the collection of all strongly measurable functions from $(0,T)$ into X the p-th powers of norms are integrable. $C^m([0,T];X)$ will denote the set of all m-times continuously differentiable functions from $[0,T]$ into X.

If X and Y are two Banach spaces, $B(X,Y)$ is the collection of all bounded linear operators from X into Y, and $B(X,X)$ is simply written as $B(X)$. For an interpolation couple of Banach spaces X_0 and X_1, $(X_0,X_1)_{\theta,p}$ and $[X_0,X_1]_\theta$ denote the real and complex interpolation spaces between X_0 and X_1, respectively. Let $B^s_{p,q}(\Omega)$ denote the Besov space and $B^s_{p,q}(\Omega)$ will be denote the subspace of $B^s_{p,q}(\Omega)$ consisting of those functions which have compact support in Ω.

3. Preliminaries

Let Ω be a bounded domain in $\mathcal{R}^n$ with smooth boundary $\partial\Omega$. Consider an elliptic differential operator of second order

$$\mathcal{A}(x,D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

where $\{a_{i,j}(x)\}$ is a positive definite symmetric matrix for each $x \in \overline{\Omega}$. The operator

$$\mathcal{A}'(x,D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)\cdot) + c(x)$$

is the formal adjoint of $\mathcal{A}$.

For $1 < p < \infty$ we denote the realization of $\mathcal{A}$ in $L^p(\Omega)$ under the Dirichlet boundary condition by A_p

$$D(A_p) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega),$$
$$A_p u = \mathcal{A}u \quad \text{for} \quad u \in D(A_p).$$

For $p' = p/(p-1)$, we can also define the realization $\mathcal{A}'$ in $L^{p'}(\Omega)$ under Dirichlet boundary condition by $A'_{p'}$:

$$D(A'_{p'}) = W^{2,p'}(\Omega) \cap W^{1,p'}_0(\Omega),$$
$$A'_{p'} u = \mathcal{A}' u \quad \text{for} \quad u \in D(A'_{p'})$$

It is known that $-A_p$ and $-A'_{p'}$ generate analytic semigroups in $L^p(\Omega)$ and $L^{p'}(\Omega)$, respectively, and $A_p^* = A'_{p'}$. From the result of R. Seeley [10] (see also H. Triebel [14;p. 321]) we obtain that

$$[D(A_p), L^p(\Omega)]_{\frac{1}{2}} = W_0^{1,p}(\Omega)$$

and hence, may consider that

$$D(A_p) \subset W_0^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{-1,p}(\Omega) \subset D(A'_{p'})^*.$$

Let $(A'_{p'})'$ be the adjoint of $A'_{p'}$ considered as a bounded linear operator from $D(A'_{p'})$ to $L^{p'}(\Omega)$. Let $\widetilde{A}$ be the restriction of $(A'_{p'})'$ to $W_0^{1,p}(\Omega)$. Then by the interpolation theory, the operator $\widetilde{A}$ is an isomorphism from $W_0^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$. Similarly, we consider that the restriction $\widetilde{A'}$ of $(A_p)' \in B(L^{p'}(\Omega), D(A_p)^*)$ to $W_0^{1,p'}(\Omega)$ is an isomorphism from $W_0^{1,p'}(\Omega)$ to $W^{-1,p'}(\Omega)$. Furthermore, as is seen in proposition 3.1 in J. M. Jeong [5], it is known that $\widetilde{A}$ and $\widetilde{A'}$ generate analytic semigroups in $W^{-1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$, respectively, and the inequality

$$||(\widetilde{A})^{is}||_{B(W^{-1,p}(\Omega))} \leq Ce^{\gamma|s|}, \quad -\infty < s < \infty,$$

holds for some constants $C > 0$ and $\gamma \in (0, \pi/2)$. We set

$$H_{p,q} = (W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1}{q},q}, \tag{3.1}$$

for $q \in (1, \infty)$ Since $\widetilde{A}$ is an isomorphism from $W_0^{1,p}(\Omega)$ onto $W^{-1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ and $W^{-1,p}(\Omega)$ are ζ-convex spaces, it is easily seen that $H_{p,q}$ is also ζ-convex. From the interpolation theory and definitions of the operator $\widetilde{A}$ and the space $H_{p,q}$ we can see the following results.

Proposition 3.1. *The operator $\widetilde{A}$ and $\widetilde{A'}$ generate analytic semigroups in $H_{p,q}$ and $H_{p',q'}$, respectively.*

Proof. Since $-A_p$ and $-\widetilde{A}$ generate analytic semigroup in $L^p(\Omega)$ and $W^{-1,p}(\Omega)$, respectively (see Proposition 3.1 in J. M. Jeong [5]), there exists an angle $\gamma \in (0, \frac{\pi}{2})$ such that

$$\Sigma = \{\lambda : \gamma \leq \arg \lambda \leq 2\pi - \gamma\} \subset \rho(A_p) \cap \rho(\widetilde{A}), \tag{3.2}$$

$$||(\lambda - A_p)^{-1}||_{B(L^p(\Omega))} \leq C/|\lambda|, \quad \lambda \in \Sigma, \tag{3.3}$$

$$||(\lambda - \widetilde{A})^{-1}||_{B(W^{-1,p}(\Omega))} \leq C/|\lambda|, \quad \lambda \in \Sigma, \tag{3.4}$$

where $\rho(A_p)$ is the resolvent set of A_p. In view of (3.3)

$$||A_p(\lambda - A_p)^{-1}u||_{p,\Omega} = ||(\lambda - A_p)^{-1}A_p u||_{p,\Omega} \leq \frac{C}{|\lambda|}||A_p u||_{p,\Omega},$$

for any $u \in D(A_p)$, we have

$$||(\lambda - A_p)^{-1}||_{B(D(A_p))} \leq \frac{C}{|\lambda|}. \tag{3.5}$$

From (3.3) and (3.5) it follows that

$$||(\lambda - \widetilde{A})^{-1}||_{B(W_0^{1,p}(\Omega))} \leq \frac{C}{|\lambda|} \tag{3.6}$$

and, hence from (3.4), (3.6) and the definition of the space $H_{p,q}$ we have that

$$||(\lambda - \widetilde{A})^{-1}||_{B(H_{p,q})} \leq \frac{C}{|\lambda|}.$$

Therefore we have shown that $-\widetilde{A}$ generates an analytic semigroup in $H_{p,q}$.

PROPOSITION 3.2. *There exists a constant $C > 0$ such that*

$$||\widetilde{A}^{is}||_{B(H_{p,q})} \le Ce^{\gamma|\lambda|}, s \in \mathcal{R},$$

where γ is the constant in (3.2).

Proof. From Theorem 1 of Seeley [9] and Proposition 3.2 of J. M. Jeong [5] there exists a constant $C > 0$ such that

$$(3.7) \qquad ||(A_p)^{\epsilon+is}||_{B(L^p(\Omega))} \le Ce^{\gamma|s|},$$

$$(3.8) \qquad ||\widetilde{A}^{\epsilon+is}||_{B(W^{-1,p}(\Omega))} \le Ce^{\gamma|s|},$$

for any $s \in \mathcal{R}$ and $\epsilon > 0$. From (3.7) it follows

$$(3.9) \qquad ||(A_p)^{\epsilon+is}||_{B(D(A_p))} \le Ce^{\gamma|s|},$$

and hence, from (3.7) and (3.9) we obtain

$$(3.10) \qquad ||\widetilde{A}^{\epsilon+is}||_{B(W_0^{1,p}(\Omega))} \le Ce^{\gamma|s|}.$$

Hence from (3.1), (3.8) and (3.10) we have shown that

$$||\widetilde{A}^{\epsilon+is}||_{B(H_{p,q})} \le Ce^{\gamma|s|}.$$

So the proof is complete.

If $1 < p < n/(n-1)$, then we see that $L^1(\Omega) \subset W^{-1,p}(\Omega)$ from Sobolev's embedding theorem. Furthermore, we will show that $L^1(\Omega) \subset H_{p,q}$ in the section 4. Therefore, from Propositions 3.1 and 3.2 we can apply Theorem 3.1 and Proposition 4.1 of J. M. Jeong [5] to the abstract Cauchy problem in the space $H_{p,q}$. Hence, the following equation may be considered as an equation in both $H_{p,q}$ and $W^{-1,p}(\Omega)$.

$$(3.11) \quad \frac{d}{dt}u(t) = A_0u(t) + A_1u(t-h) + \int_{-h}^{0} a(s)A_2u(t+s)ds + \Phi_0 w(t), \quad t \in (0,T]$$

$$(3.12) \quad u(0) = g^0, \quad u(s) = g^1(s) \qquad s \in [-h, 0),$$

where $A_0 = -\widetilde{A}$, $A_\iota u$ ($\iota = 1,\ 2$) are the restriction $W_0^{1,p}(\Omega)$ of the following linear differential operators $-\mathcal{A}_\iota(\iota = 1,\ 2)$ with real coefficient:

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

where $a^\iota_{i,j} = a^\iota_{j,i} \in C^1(\overline{\Omega})$, $b^\iota_i \in C^1(\overline{\Omega})$, $c^\iota \in L^\infty(\Omega)$. The kernel $a(\cdot)$ belongs to $L^{q'}(-h,0)$ and the controller Φ_0 is a bounded linear operator from some Banach space U to $L^1(\Omega)$. For $q \in (1,\infty)$ we set

$$Z_{p,q} = H_{p,q} \times L^q(-h, 0; W_0^{1,p}(\Omega)). \tag{3.13}$$

Let $g = (g^0, g^1) \in Z_{p,q}$ and $w \in L^q(0,T;U)$. Then as is seen in Proposition 4.1 of J.M. Jeong [5] a solution of the equation (3.11) and (3.12) exists and is unique for each $T > 0$, moreover, we have

$$\begin{aligned}||u||_{L^q(0,T,W_0^{1,p}(\Omega))\cap W^{1,q}(0,T,W^{-1,p}(\Omega))} &\leq c(||g^0||_{H_{p,q}} \\ &+ ||g^1||_{L^q(-h,0,W_0^{1,p}(\Omega))} + ||w||_{L^q(0,T,U)}),\end{aligned}$$

where c is a constant. Thus, we can define the solution semigroup for the system (3.11) and (3.12) as follows [3, Theorem 4.1]:

$$S(t) = (u(t;g,0), u_t(\cdot;g,0))$$

where $g = (g^0, g^1) \in Z_{p,q}$, $u(t;g,0)$ is a solution of (3.11), (3.12) with $\Phi_0 = 0$ and $u_t(\cdot, g, 0)$ is the function $u_t(s;g,0) = u(t+s;g,0)$ defined in $[-h, 0]$. It is also known that $S(t)$ is a C_0-semigroup in $Z_{p,q}$.

We introduce the adjoint problem of (3.11) and (3.12):

$$\begin{aligned}\frac{d}{dt}v(t) =& A_0^*v(t) + A_1^*v(t-h) \\ &+ \int_{-h}^{0} a(s)A_2^*v(ts)ds, \quad t \in (0,T],\end{aligned} \tag{3.14}$$

$$v(0) = \phi^0, \quad v(s) = \phi^1(s), \quad s \in [-h, 0), \tag{3.15}$$

where $A_0^* = -\widetilde{A'}$ and $A_\iota^* (\iota = 1,\ 2)$ are the adjoint operators of A_ι. It is easily seen that $A_\iota^* \in B(W_0^{1,p'}(\Omega), W^{-1,p'}(\Omega))$ for $\iota = 0,\ 1,\ 2$ and hence (3.14) and (3.15) is an equation in both $H_{p',q'}$ and $W^{-1,p'}$. We can also define the solution semigroup $S_T(t)$ of (3.14) and (3.15) by

$$S_T(t)\phi = (v(t;\phi), v_t(\cdot,\phi))$$

for $\phi = (\phi^0, \phi^1) \in Z_{p',q'}$, where $v(t;\phi)$ is the solution of (3.14) and (3.15).

The structural operator F is defined by

$$Fg = ([Fg]^0, [Fg]^1),$$
$$[Fg]^0 = g^0,$$
$$[Fg]^1(s) = A_1 g^1(-h-s) + \int_{-h}^{0} a(\tau) A_2 g^1(\tau - s) d\tau$$

for $g = (g^0, g^1) \in Z_{p,q}$. It is easy to see that $F \in B(Z_{p,q}, Z_{p',q'})$, $F^* \in B(Z_{p',q'}, Z_{p,q}^*)$. As in [5,7] we have that

$$FS(t) = S_T^*(t)F^*, \quad F^* S_T(t) = S^*(t)F^*. \tag{3.16}$$

4. Representation of $H_{p,q}$ into Besov spaces

Let Y_0 and Y_1 be two Banach spaces contained in a Banach space $\mathcal{Y}$ such that the identity mapping of $Y_\imath$ $(\imath = 0,\ 1)$ in $\mathcal{Y}$ is continuous and norms will be denoted by $||\cdot||_\imath$. The algebraic sum $Y_0 + Y_1$ of Y_0 and Y_1 is the space of all elements $a \in \mathcal{Y}$ of the form $a = a_0 + a_1$, $a_0 \in Y_0$ and $a_1 \in Y_1$. The intersection $Y_0 \cap Y_1$ and the sum $Y_0 + Y_1$ are Banach spaces with the norms

$$||a||_{Y_0 \cap Y_1} = \max\ \{||a||_0, ||a||_1\}$$

and

$$||a||_{Y_0+Y_1} = \inf_a\ \{||a_0||_0 + ||a_1||_1\}, \quad a = a_0 + a_1,\ a_\imath \in Y_\imath,$$

respectively.

DEFINITION (LIONS-PEETRE) 4.1. *We say an intermediate space Y of Y_0 and Y_1 belongs to*

(i) the class $\underline{K}_\theta(Y_0, Y_1)$, $0 < \theta < 1$, if for any $a \in Y_0 \cap Y_1$,

$$||a||_Y \leq c||a||_0^{1-\theta}||a||_1^\theta$$

where c is a constant;

(ii) the class $\overline{K}_\theta(Y_0, Y_1)$, $0 < \theta < 1$, if for any $a \in Y$ and $t > 0$ there exist $a_i \in Y_i$ $(i = 1,\ 2)$ such that $a = a_0 + a_1$ and

$$||a_0||_0 \leq ct^{-\theta}||a||_Y, \quad ||a_1||_1 \leq ct^{1-\theta}||a||_Y$$

where c is a constant;

(iii) the class $K_\theta(A_0, A_1)$, $0 < \theta < 1$, if the space Y belongs to both $\underline{K}_\theta(Y_0, Y_1)$ and $\overline{K}_\theta(Y_0, Y_1)$.

Here, we note that by replacing t with t^{-1} the condition in (ii) rewrite as follows:

$$||a_0||_0 \leq ct^{\theta}||a||_Y, \quad ||a_1||_1 \leq ct^{\theta-1}||a||_Y$$

The following result is obtained from Lions-Peetre theorem 2.3 in [6].

PROPOSITION 4.1. *For $0 < \theta_0 < \theta < \theta_1 < 1$, if the spaces X_0 and X_1 belong to the space $K_{\theta_0}(Y_0, Y_1)$ and the space $K_{\theta_1}(Y_0, Y_1)$, respectively, then*

$$(X_0, X_1)_{\frac{\theta-\theta_0}{\theta_1-\theta_0},p} = (Y_0, Y_1)_{\theta,p}.$$

In particular, if the space X_1 belongs to $K_{\theta_1}(Y_0, Y_1)$ then for $0 < \theta < \theta_1 < 1$

$$(Y_0, X_1)_{\frac{\theta}{\theta_1},p} = (Y_0, Y_1)_{\theta,p}.$$

If the space X_0 belongs to $K_{\theta_0}(Y_0, Y_1)$, then $0 < \theta_0 < \theta < 1$

$$(X_0, Y_1)_{\frac{\theta-\theta_0}{1-\theta_0},p} = (Y_0, Y_1)_{\theta,p}.$$

Let $A = -\mathcal{A}(x, D_x)$ as in section 3. Then the operator A is an isomorphism from $W_0^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$.

LEMMA 4.1. *For any $t > 0$, there exists a constant C such that*

$$||(t+A)^{-1}||_{B(W^{-1,p}(\Omega),L^p(\Omega))} \leq Ct^{-\frac{1}{2}}, \tag{4.1}$$

and

$$||(t+A)^{-1}||_{B(L^p(\Omega),W_0^{1,p}(\Omega))} \leq Ct^{-\frac{1}{2}}. \tag{4.2}$$

Proof. For $t > 0$ since $(t + A'_{p'})^{-1}$ is an isomorphism from $L^{p'}(\Omega)$ to $D(A'_{p'})$, the resolvent $((t + A'_{p'})^{-1})'$ is an isomorphism from $D(A'_{p'})^*$ onto $L^p(\Omega)$. It is not difficult to see that

$$((t+A'_{p'})^{-1})' = (t+(A'_{p'})')^{-1}$$

and

$$(t+(A'_{p'})')^{-1}|_{W^{-1,p}(\Omega)} = (t+A)^{-1}.$$

Therefore, we have

$$||(t+A)^{-1}||_{B(D(A'_{p'})^*,L^p(\Omega))} \leq C \tag{4.3}$$

where C is a constant. Combining (3.3) and (4.3) we obtain the inequality of (4.1). The proof of (4.2) is similar.

THEOREM 4.1. *For $1 < p < \infty$, the space $L^p(\Omega)$ belongs to the class $K_{1/2}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$.*

Proof. For any $u \in W_0^{1,p}(\Omega)$ and $t > 0$, from Lemma 4.1 and

$$u = A(t+A)^{-1}u + t(t+A)^{-1}u = (t+A)^{-1}Au + t(t+A)^{-1}u,$$

it follows

$$\begin{aligned} ||u||_{p,\Omega} \leq & ||(t+A)^{-1}||_{B(W^{-1,p}(\Omega),L^p(\Omega))}||Au||_{-1,p,\Omega} \\ & + t||(t+A)^{-1}||_{B(W^{-1,p}(\Omega),L^p(\Omega))}||u||_{-1,p,\Omega} \\ \leq & Ct^{-\frac{1}{2}}||u||_{1,p,\Omega} + Ct^{\frac{1}{2}}||u||_{-1,p,\Omega}. \end{aligned}$$

By choosing $t > 0$ such that $t^{-1/2}||u||_{1,p,\Omega} = t^{1/2}||u||_{-1,p,\Omega}$, we obtain

$$||u||_{p,\Omega} \leq C||u||^{\frac{1}{2}}_{1,p,\Omega}||u||^{\frac{1}{2}}_{-1,p,\Omega}.$$

Therefore, $L^P(\Omega)$ belongs to the class $\underline{K}_{1/2}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$. Put $u_0 = t(t+A)^{-1}u$ and $u_1 = A(t+A)^{-1}u$ for any $u \in L^p(\Omega)$. Then $u = u_0 + u_1$, and we obtain that

$$||u_0||_{1,p,\Omega} \leq t||(t+A)^{-1}u||_{B(L^p(\Omega), W_0^{1,p}(\Omega))}||u||_{p,\Omega} \leq Ct^{\frac{1}{2}}||u||_{p,\Omega}$$

$$||u_1||_{-1,p,\Omega} \leq C||(t+A)^{-1}u||_{1,p,\Omega} \leq Ct^{\frac{1}{2}}||u||_{p,\Omega}.$$

Therefore $L^P(\Omega)$ belongs to the class $\overline{K}_{1/2}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$, and hence, it belongs to $K_{1/2}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$.

THEOREM 4.2. *If $1 - 2\theta - 1/p \neq 0$ and $2\theta - 2 + 1/p \neq 0$ for $0 < \theta < 1$ and $1 < p,\ q < \infty$, then*

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} = \begin{cases} B^{1-2\theta}_{p,q}(\Omega) & \theta < \frac{1}{2}(1 - \frac{1}{p}), \\ B^{1-2\theta}_{p,q}(\Omega) & \theta > \frac{1}{2}(1 - \frac{1}{p}). \end{cases}$$

where $B^{1-2\theta}_{p,q}(\Omega) = \{u \in B^{1-2\theta}_{p,q}(\Omega) : u|_{\partial\Omega} = 0\}$. In particular, we obtain that

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1}{2},q} = B^0_{p,q}(\Omega).$$

Proof. Let $0 < \theta < 1/2$. Then from Proposition 4.1 we obtain that for any $0 < \theta < 1/2$

$$\begin{aligned}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} &= (W_0^{1,p}(\Omega), L^p(\Omega))_{2\theta,q} \\ &= (L^p(\Omega), W_0^{1,p}(\Omega))_{1-2\theta,q}\end{aligned}$$

Therefore, in view of the result of Grisvard theorem in Triebel [14; p. 321],

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} = \begin{cases} B^{1-2\theta}_{p,q}(\Omega) & 1 - 2\theta > \frac{1}{p}, \\ B^{1-2\theta}_{p,q}(\Omega) & 1 - 2\theta < \frac{1}{p}. \end{cases}$$

Let $1/2 < \theta < 1$. Then from Proposition 4.1 it follows

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} = (L^p(\Omega), W^{-1,p}(\Omega))_{2\theta-1,q}$$
$$= ((L^{p'}(\Omega), W_0^{-1,p'}(\Omega))_{2\theta-1,q'})^*$$

where $p' = p/(p-1)$. In view of Grisvard theorem if $2\theta - 1 - 1/p' \neq 0$ then

$$(L^{p'}(\Omega), W_0^{1,p'}(\Omega))_{\theta,q} = \begin{cases} B^{2\theta-1}_{p',q'}(\Omega) & 2\theta - 1 > \frac{1}{p'}, \\ B^{2\theta-1}_{p',q'}(\Omega) & 2\theta - 1 < \frac{1}{p'}. \end{cases}$$

From Theorem 4.8.2 in H. Triebel [14; p. 332], we obtain that

$$(B^{2\theta-1}_{p',q'}(\Omega))^* = B^{1-2\theta}_{p,q}(\Omega)$$

if $2\theta - 1 - 1/p' \neq 0$. Since $2\theta - 1 - 1/p' \neq 0$ i.e. $2\theta - 2 + 1p \neq 0$, if $1/2 < \theta < 1$ and $2\theta - 2 + 1/p \neq 0$ then

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} = B^{1-2\theta}_{p,q}(\Omega).$$

Consequently, we obtain that

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1-\theta}{2},q} = B^{\theta}_{p,q}(\Omega), \qquad \text{if} \quad 0 < \theta < \frac{1}{p}$$

and

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1+\theta}{2},q} = B^{-\theta}_{p,q}(\Omega) \qquad \text{if} \quad 0 < \theta < 1 - \frac{1}{p}.$$

Hence, if $0 < \theta < \min\{1/p, 1 - 1/p\}$

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1}{2},q}$$
$$= ((W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1-\theta}{2},q}, (W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1+\theta}{2},q})_{\frac{1}{2},q}$$
$$= (B^{\theta}_{p,q}(\Omega), B^{-\theta}_{p,q}(\Omega))_{\frac{1}{2},q} = B^{0}_{p,q}(\Omega).$$

The last equality is obtained from Theorem 1 of section 4.3.1 in H. Treibel [14; p. 317]. Hence the proof is complete.

THEOREM 4.3. *Let $1 < p,\ q < \infty$.*
(i) If $2/q - 2 + 1/p \neq 0$ then

$$H_{p,q} = \begin{cases} B_{p,q}^{1-\frac{2}{q}}(\Omega) & \text{if} \quad \frac{1}{q} < \frac{1}{2}(1 - \frac{1}{p}), \\ B_{p,q}^{1-\frac{2}{q}}(\Omega) & \text{if} \quad \frac{1}{q} > \frac{1}{2}(1 - \frac{1}{p}). \end{cases}$$

(ii) If $\frac{1}{p'} < 1/n(1 - 2/q')$ then

$$H_{p',q'} \subset C_0(\overline{\Omega}) \subset L^\infty.$$

Proof. The relation (i) follows directly from Theorem 4.2. Let $\frac{1}{p'} < 1/n(1 - 2/q')$. Then from (i)

$$H_{p',q'} = (W_0^{1,p'}(\Omega), W^{-1,p'}(\Omega))_{\frac{1}{q'},q'} = B_{p',q'}^{1-\frac{2}{q'}}(\Omega)$$

and from Sobolev-Besov's and Sobolev's embedding theorems we obtain that

$$B_{p',q'}^{1-\frac{2}{q'}}(\Omega) \subset W_{p'}^{1-\frac{2}{q'}} \subset C_0(\overline{\Omega})$$

Hence, the first inclusion in (ii) follows.

5. Control problem for $L^1(\Omega)$-valued controller

As is seen in section 4, if $1/p' - 1/n(1-2/q') < 0$ then we obtained that

$$H_{p',q'} \subset C_0(\overline{\Omega}) \subset L^\infty(\Omega).$$

Thus, since

$$H_{p,q} = H_{p',q'}^* \supset C_0(\overline{\Omega})^* \supset L^1(\Omega)$$

we consider Φ_0 as an operator in $B(U, H_{p,q})$. Hence it is possible to investigate the control problem for (3.11) and (3.12) in $H_{p,q}$. In what follows in this section we fix p and q so that $1/p' - 1/n(1-2/q') < 0$. Then it immediately implies that $1 < p < n/(n-1)$. Let A be the infinitesimal generator of $S(t)$ as in section 3. Then the equation (3.11) and (3.12) can be transformed into an abstract equation as follows

$$z(t) = Az(t) + \Phi w(t),$$
$$z(0) = g$$

where $g = (g^0, g^1) \in Z_{p,q}$ and the controller operator is defined $\Phi w = (\Phi_0 w, 0)$. In this case for the solution u of (3.11) and (3.12)the equation satisfied by $(u(t), u_t(\cdot))$ is an equation in $Z_{p,q}$ since Φ_0 is an operator into $H_{p,q}$. Since the dual Φ_0^* of Φ_0 is the operator from $L^\infty(\Omega)$ into U^*, the operator Φ_0^* may be considered as an operator from $H_{p',q'}$ into U^*, Hence with the aid of Theorem 4.3 we remarked that the condition that $\Phi_0^*\phi = 0$ almost everywhere can be rewrite to the fact that $\Phi_0^*\phi \equiv 0$ for $\phi \in Z_{p,q}$. We define the attainable set by

$$R = \{\int_0^t S(t-\tau)\Phi w(\tau)d\tau : w \in L^q(0,t;U), \quad t \geq 0\}.$$

DEFINITION 5.1. *(1) The system (3.11), (3.12) is approximately controllable if $\overline{R} = Z_{p,q}$, where $\overline{R}$ is the closure of R in $Z_{p,q}$*

(2) The system (3.14), (3.15) is observable if for $\phi \in Z_{p',q'}$, $\Phi_0^*[S_T(t)\phi]^0 \equiv 0$ implies $\phi = 0$.

THEOREM 5.1. *Let the structural operator F is an isomorphism. Then the system (3.11) and (3.12) is approximately controllable if and only if the system (3.14) and (3.15) is observable.*

Proof. Let the system (3.11) and (3.12) is approximately controllabe. Then for $f \in Z_{p,q}^*$

$$(f, \int_0^t S(t-\tau)\Phi w(\tau)d\tau) = 0$$

for $w \in L^q(0,t;U)$ and $t > 0$ implies $f = 0$. By duality theorem it is equivalent to the fact that for any $f \in Z_{p,q}^*$, $\Phi^* S^*(t) f \equiv 0$ implies

$f = 0$. Since the operator F^* is an isomorphism by assumption, there exists $\phi \in Z_{p',q'}$ such that $f = F^*\phi$. From (3.16) we obtain that

$$\Phi_0^*[S^*(t)f]^0 = \Phi_0^*[FS_T(t)\phi]^0 = \Phi_0^*[S_T(t)\phi]^0.$$

Hence, the system (3.11) and (3.12) is approximately controllable iff for any $\phi \in Z_{p',q'}$, $\Phi_0^*[S_T(t)\phi]^0 \equiv 0$ implies $\phi = 0$. Therefore, the statement is equivalent that the system (3.14) and (3.14) is observable.

Remark. When we deal with the control problem of (3.11) and (3.12) in negative space $W^{-1,p}(\Omega)$, we needed a assumption that the kernel $a(\cdot)$ is Hölder continuous for using of the properties of fundamental solution since Φ_0 is not operator into $H_{p,q}$. If we assume that $a(\cdot)$ is Hölder continuous then the fundamental solution of (3.11) and (3.12) exists (see in [13]). By fixing p, q so that $1 < p < n/(n-1)$ and $1/p' < 1/n(1-1/q')$, we can obtain the Theorem 5.1 using the solution semigroup without requirement of fundamental solution of (3.11) and (3.12). Hence, the kernel $a(\cdot)$ need not be Holder continuous but has only to belong to $L^{q'}(-h, 0)$ for wellposedness and regularity for equation (3.11) and (3.12).

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Division of Mathematical Sciences
Pukyong National University
Pusan 608-737, Korea
E-mail: e-mail: jmjeong@pknu.ac.kr

Department of Mathematical Education
Kyungnam University
Masan 631-701, Korea