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THE BESOV SPACES OF M-HARMONIC FUNCTIONS

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ABSTRACT. We extend the characterization for the analytic Besov space obtained by Nowak to the invariant harmonic Besov space

1. Introduction

Let $H(B)$ and $h(B)$ denote the spaces of holomorphic functions and of invariant harmonic functions on the unit ball B of C^n, respectively. For $0 < p < \infty$, the Bergman space $L^p_a(B)$, the Hardy space $H^p(B)$ and the Besov space $B_p(B)$ in the unit ball of C^n are defined respectively as

$$L^p_a(B) = \{f \in H(B) : ||f||^p_{L^p_a} = \int_B |f(z)|^p \, d\nu(z) < \infty\},$$

$$H^p(B) = \{f \in H(B) : ||f||^p_p = \sup_{0<r<1} \int_S |f(r\zeta)|^p \, d\sigma(\zeta) < \infty\}$$

and

$$B_p(B) = \{f \in H(B) : ||f||^p_{B_p} = \int_B (\hat{Q}f)^p(z) \, d\lambda(z) < \infty\}$$

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where ν is the normalized Lebesgue measure on B, S is the boundary of B, σ is the normalized Lebesgue measure on S, λ is the invariant measure on B and $\hat{Q}f$ is the maximal derivative of f with respect to the Bergman metric on B. We use the notations $L^p_m(B)$, $h^p(B)$ and $MB_p(B)$ in cases of invariant harmonic functions.

K. Stroethoff gave the following characterization for the Besov space on the unit disc D.

THEOREM A [6]. *If $2 < p < \infty$, then for an analytic function f on D,*

$$f \in B_p \Leftrightarrow \int_D \int_D \left| \frac{f(z) - f(w)}{z - w} \right|^p (1-|z|^2)^{\frac{p}{2}} (1-|w|^2)^{\frac{p}{2}} d\lambda(w) d\lambda(z) < \infty.$$

Recently M. Nowak ([3]) extended the theorem to $n \geq 2$ case. In this paper we shall give an M-harmonic version of the above characterization in the unit ball. The main result of the paper is as follows:

THEOREM B. *Assume that $f \in C^1(B)$ and $2n < p < \infty$. Then $f \in MB_p$ if and only if*

$$\int_B \int_{E(z,r)} \left| \frac{f(z) - f(w)}{1 - \langle z, w \rangle} \right|^p (1 - |z|^2)^{\frac{p}{2}} (1 - |w|^2)^{\frac{p}{2}} d\lambda(w) d\lambda(z) < \infty.$$

2. Notations and Preliminaries

We introduce a few facts that we need in the sequel, most of which are well known. See [1] and [5] for details. For each $a \in B$, the Möbius transformation $\varphi_a : B \to B$ is defined by

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle}, \ z \in B, \tag{2.1}$$

where $s_a = \sqrt{1 - |a|^2}$, P_a is the orthogonal projection from C^n onto the subspace generated by a and $Q_a = I - P_a$ i.e., $P_a z = \frac{\langle a, z \rangle}{|a|^2} a$.

For $a \in B$ and $z \in \bar{B}$,

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-<z,a>|^2}. \tag{2.2}$$

The determinant $J_R\varphi_a(z)$ of the real Jacobian matrix of φ_a satisfies the following :

(2.3)

$$J_R\varphi_a(z) = |J_C\varphi_a(z)|^2 = \Big(\frac{1-|a|^2}{|1-<z,a>|^2}\Big)^{n+1} = \Big(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\Big)^{n+1}.$$

The following transformation formula holds under $\varphi_a \in Aut(B)$

$$d\nu(\varphi_a(z)) = |J_C\varphi_a(z)|^2 d\nu(z). \tag{2.4}$$

For $0 < r < 1$, φ_a is a biholomorphic map from the ball $rB = B(0,r)$ onto the ellipsoid $E(a,r) := \{z \in B : |\varphi_a(z)| < r\}$. The invariant measure is given by

$$d\lambda(z) = \frac{1}{(1-|z|^2)^{n+1}} d\nu(z).$$

The invariant Laplacian $\tilde{\triangle}$ on B is given by

$$\begin{aligned}\tilde{\triangle} f(z) &= \frac{1}{n+1}\triangle(f\circ\varphi_z)(0)\\ &= \frac{4}{n+1}(1-|z|^2)\sum_{i,j=1}^{n}(\delta_{i,j}-z_i\bar{z}_j)\frac{\partial^2 f(z)}{\partial z_i\partial\bar{z}_j},\ f\in C^2(B),\end{aligned}$$

where $\triangle = 4\sum_{j=1}^{n}\frac{\partial^2}{\partial z_j\partial\bar{z}_j}$ is the usual Laplacian. An invariant harmonic or simply M-harmonic function is a function in $C^2(B)$ which is annihilated by $\tilde{\triangle}$ in B. For a C^1-function f the invariant gradient $\tilde{\nabla}$ is the vector field on B defined by

$$\begin{aligned}\tilde{\nabla} f(z) &= \nabla(f\circ\varphi_z)(0)\\ &= \frac{2}{n+1}(1-|z|^2)\sum_{i,j=1}^{n}[\delta_{i,j}-z_i\bar{z}_j]\Big(\frac{\partial f}{\partial\bar{z}_i}\frac{\partial}{\partial z_j}+\frac{\partial f}{\partial z_j}\frac{\partial}{\partial\bar{z}_i}\Big)\end{aligned} \tag{2.5}$$

where ∇ is the real gradient in R^{2n}. Then

$$|\tilde{\nabla} f(z)|^2 = \frac{2}{n+1}(1-|z|^2)\sum_{i,j=1}^{n}[\delta_{i,j} - z_i\bar{z}_j]\Big(\frac{\partial f}{\partial \bar{z}_i}\frac{\partial \bar{f}}{\partial z_j} + \frac{\partial f}{\partial z_j}\frac{\partial \bar{f}}{\partial \bar{z}_i}\Big).$$

The Laplacian $\tilde{\Delta}$ and the gradient $\tilde{\nabla}$ are both invariant under the automorphisms of B. The Bergman metric $\beta : B \times C^n \to R$ is defined by the differential form :.

$$\beta^2(z,\xi) = \sum_{i,j=1}^{n} b_{ij}(z)\xi_i\bar{\xi}_j,$$

$$= \frac{(1-|z|^2)|\zeta|^2 + |< z,\zeta >|^2}{(1-|z|^2)^2},\ z,\zeta \in B \tag{2.6}$$

where

$$b_{ij}(z) = \frac{\partial^2 \log K(z,z)}{\partial z_i \partial \bar{z}_j},\ (i,j = 1,\cdots,n)$$

and

$$K(z,w) = \frac{1}{(1- < z,w >)^{n+1}},\ z,w \in B$$

denotes the Bergman kernel of B. It follows from (2.6) that

$$\frac{|\zeta|}{\sqrt{1-|z|^2}} \le \beta(z,\zeta) \le \frac{|\zeta|}{1-|z|^2},\ z,\zeta \in B. \tag{2.7}$$

DEFINITION. Let $f \in C^1(B)$ and $\xi \in C^n$. The maximal derivative of f with respect to the Bergman metric β on B is defined by

$$\hat{Q}f(z) = \sup_{|\xi|=1} \frac{|df(z)\cdot\xi|}{\beta(z,\xi)},\ z \in B$$

where

$$df(z)\cdot\xi = \sum_{i=1}^{n}\Big[\frac{\partial f}{\partial z_i}(z)\xi_i + \frac{\partial f}{\partial \bar{z}_i}(z)\bar{\xi}_i\Big]$$
$$= \partial f(z)\cdot\xi + \bar{\partial} f(z)\cdot\bar{\xi}.$$

The following identities are easily verified. For a C^1-function f in B and $\varphi \in Aut(B)$,

$$\hat{Q}(f \circ \varphi) = (\hat{Q}f) \circ \varphi,$$

$$\frac{1}{2}\sqrt{\tilde{\Delta}|f|^2} \le \hat{Q}f = 2|\tilde{\nabla} f| \le \sqrt{\tilde{\Delta}|f|^2}. \tag{2.8}$$

3. Proof of Theorem B

Proof. [Proof of the necessity of Theorem B]. The mean value property of $f \in h(B)$ implies that

$$f(z) = \frac{1}{r^{2n}} \int_{rB} f(w) K_B(\frac{z}{r}, \frac{w}{r}) d\nu(w), \ z \in rB$$

for the reproducing kernel $K_{rB}(z,w) = \frac{1}{r^{2n}} K_B(\frac{z}{r}, \frac{w}{r})$ of rB. Using similar arguments as in the proof of (1.6a) [1], we have that

$$|\nabla f(0)| \le C||f||_{L^p(rB,\nu)}$$

where C is a positive constant independent of f. Replacing f by $f \circ \varphi_a - f(a)$ yields

$$|\nabla(f \circ \varphi_a)(0)| \le C\Big(\int_{rB} |(f \circ \varphi_a)(w) - f(a)|^p d\nu(w)\Big)^{1/p}.$$

By (2.5) and (2.8),

$$\begin{aligned}
(\hat{Q}f)(a) &= C\Big(\int_{rB} |(f \circ \varphi_a)(w) - f(a)|^p d\nu(w)\Big)^{1/p} \\
&\qquad \text{(a change of variable)} \\
&= C\Big(\int_{E(a,r)} |f(w) - f(a)|^p d\nu(\varphi_a(w))\Big)^{1/p} \\
&\qquad \text{(by (2.3) and (2.4))} \\
&= C\Big(\int_{E(a,r)} |f(w) - f(a)|^p \frac{(1-|a|^2)^{n+1}}{|1- < w,a > |^{2n+2}} d\nu(w)\Big)^{1/p} \\
&\le C\Big(\int_{E(a,r)} \frac{|f(w) - f(a)|^p}{|1- < w,a > |^{n+1}} d\nu(w)\Big)^{1/p}.
\end{aligned}$$

Then we have

$$(\hat{Q}f)^p(a) \le C \int_{E(a,r)} \frac{|f(w) - f(a)|^p}{|1 - <w, a>|^{n+1}} d\nu(w).$$

Integrating both sides of the inequality and using the fact that for $w \in E(a, r)$

$$\frac{(1 - |w|^2)^{n+1}}{|1 - <w, a>|^{n+1-p}} \approx (1 - |w|^2)^{\frac{p}{2}} (1 - |a|^2)^{\frac{p}{2}},$$

$$\begin{aligned}
&\int_B (\hat{Q}f)^p(a) d\lambda(a) \\
&\le C \int_B \int_{E(a,r)} \frac{|f(w) - f(a)|^p}{|1 - <w, a>|^{n+1}} d\nu(w) d\lambda(a) \\
&= C \int_B \int_{E(a,r)} \left| \frac{f(w) - f(a)}{1 - <w, a>} \right|^p \cdot \frac{(1 - |w|^2)^{n+1}}{|1 - <w, a>|^{n+1-p}} d\lambda(w) d\lambda(a) \\
&\le \int_B \int_{E(a,r)} \left| \frac{f(w) - f(a)}{1 - <w, a>} \right|^p (1 - |w|^2)^{\frac{p}{2}} (1 - |a|^2)^{\frac{p}{2}} d\lambda(w) d\lambda(a).
\end{aligned}$$

□

The following lemmas will be needed to prove the converse part of Theorem B.

LEMMA 3.1. *For $f \in C^1(B)$, $1 < p < \infty$*

$$\begin{aligned}
&\int_B \frac{|f(z) - f(0)|^p}{|z|^p} (1 - |z|)^{\frac{p}{2} - n - 1} d\nu(z) \\
&\lesssim \int_B \frac{(\hat{Q}f)^p(z)(1 - |z|)^{\frac{p}{2} - n - 1}}{|z|^{n+p-2}} d\nu(z).
\end{aligned}$$

Proof. For $z \in B$ and a C^1-function f, we have by the proof of Lemma 1.1 in [1]

$$\begin{aligned} |f(z) - f(0)| &= \Big| \int_0^1 \frac{df(tz)}{dt} dt \Big| \\ &= \Big| \int_0^1 \frac{df(tz) \cdot z}{\beta(tz, z)} \, \beta(tz, z) dt \Big| \\ &\le \int_0^1 (\hat{Q}f)(tz) \cdot \beta(tz, z) dt \\ \text{(by (2.7))} \quad &\le \int_0^1 (\hat{Q}f)(tz) \frac{|z|}{1 - |tz|} dt. \end{aligned}$$

Let q be the conjugate exponent of p. By Hölder inequality, for $1/p + 1/q = 1$, we have

$$\begin{aligned} \frac{|f(z) - f(0)|}{|z|} &\le \int_0^1 (\hat{Q}f)(tz) \frac{1}{1 - |tz|} dt \\ &\le \Big\{ \int_0^1 (\hat{Q}f)^p(tz) \Big(\frac{1}{1 - |tz|} \Big)^{\frac{p}{2}} dt \Big\}^{\frac{1}{p}} \times \Big\{ \int_0^1 \Big(\frac{1}{1 - |tz|} \Big)^{\frac{q}{2}} dt \Big\}^{\frac{1}{q}}. \end{aligned}$$

Thus

$$\frac{|f(z) - f(0)|^p}{|z|^p} \le \int_0^1 (\hat{Q}f)^p(tz) \Big(\frac{1}{1 - |tz|} \Big)^{\frac{p}{2}} dt \times \Big\{ \int_0^1 \Big(\frac{1}{1 - |tz|} \Big)^{\frac{q}{2}} dt \Big\}^{\frac{p}{q}}.$$

By an elementary calculation in the second integral,

$$\Big\{ \int_0^1 \Big(\frac{1}{1 - |tz|} \Big)^{\frac{q}{2}} dt \Big\}^{\frac{p}{q}} \le C \Big(\frac{1}{|z|} \Big)^{\frac{p}{q}} (1 - |z|)^{\frac{-q+2}{2} \cdot \frac{p}{q}}.$$

Then

$$\frac{|f(z) - f(0)|^p}{|z|^p} \le C \Big(\frac{1}{|z|} \Big)^{\frac{p}{q}} (1 - |z|)^{\frac{-q+2}{2} \cdot \frac{p}{q}} \int_0^1 (\hat{Q}f)^p(tz) \Big(\frac{1}{1 - |tz|} \Big)^{\frac{p}{2}} dt.$$

In integrating both sides on the unit ball,

$$\begin{aligned}
&\int_B \frac{|f(z)-f(0)|^p}{|z|^p}(1-|z|)^{\frac{p}{2}-n-1}d\nu(z) \\
&\le C\int_B\int_0^1 \frac{1}{|z|^{\frac{p}{q}}(1-|z|)^{\frac{q-2}{2}\cdot\frac{p}{q}}}(\hat{Q}f)^p(tz)\Big(\frac{1}{1-|tz|}\Big)^{\frac{p}{2}} \\
&\qquad (1-|z|)^{\frac{p}{2}-n-1}dtd\nu(z) \\
&= \int_0^1\int_{B_t} \frac{(\hat{Q}f)^p(z)(1-|\frac{z}{t}|)^{\frac{p}{2}-n-1}}{|\frac{z}{t}|^{\frac{p}{q}}(1-|\frac{z}{t}|)^{\frac{q-2}{2}\cdot\frac{p}{q}}\cdot(1-|z|)^{\frac{p}{2}}}d\nu(z)(\frac{1}{t})^{2n}dt \\
&= \int_B \frac{(\hat{Q}f)^p(z)}{|z|^{\frac{p}{q}}(1-|z|)^{\frac{p}{2}}}d\nu(z)\int_{|z|}^1 \frac{(\frac{1}{t})^{2n}(1-|\frac{z}{t}|)^{\frac{p}{2}-n-1-\frac{q-2}{2}\cdot\frac{p}{q}}}{(\frac{1}{t})^{\frac{p}{q}}}dt.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{|z|}^1 \frac{(\frac{1}{t})^{2n}(1-|\frac{z}{t}|)^{\frac{p}{2}-n-1-\frac{q-2}{2}\cdot\frac{p}{q}}}{(\frac{1}{t})^{\frac{p}{q}}}dt \\
&\le \Big(\frac{1}{|z|}\Big)^{n-1}\cdot\int_{|z|}^1\Big(t-|z|\Big)^{\frac{p}{q}-n-1}dt \\
&= C\Big(\frac{1}{|z|}\Big)^{n-1}\Big(1-|z|\Big)^{\frac{p}{q}-n},
\end{aligned}$$

we have

$$\begin{aligned}
&\int_B \frac{|f(z)-f(0)|^p}{|z|^p}(1-|z|)^{\frac{p}{2}-n-1}d\nu(z) \\
&\lesssim \int_B \frac{(\hat{Q}f)^p(z)}{|z|^{\frac{p}{q}}(1-|z|)^{\frac{p}{2}}}\Big(\frac{1}{|z|}\Big)^{n-1}\Big(1-|z|\Big)^{\frac{p}{q}-n}d\nu(z) \\
&= \int_B \frac{(\hat{Q}f)^p(z)\Big(1-|z|\Big)^{\frac{p}{q}-n-\frac{p}{2}}}{|z|^{\frac{p}{q}+n-1}}d\nu(z) \\
&= \int_B \frac{(\hat{Q}f)^p(z)(1-|z|)^{\frac{p}{2}-n-1}}{|z|^{n+p-2}}d\nu(z).
\end{aligned}$$

□

Let $2n < p < \infty$ and $-1 < \alpha < \infty$ and let γ be a positive number such that

$$\max\Big(1-\frac{1}{p}, 1+\frac{\alpha-1}{p}\Big) < \gamma < 1+\frac{\alpha}{p}.$$

LEMMA 3.2 [3]. *For the Bergman kernel $K(z,w)$, there exists a constant $C > 0$ such that for each $z \in B$*

$$\int_B \frac{|K(z,w)|^2\Big(1-|\varphi_w(z)|\Big)^{\alpha}}{|\varphi_w(z)|^{2n+\alpha+p(1-\gamma)-1}} d\nu(w) = CK(z,z).$$

LEMMA 3.3. *Let $2n < p < \infty$. Then there exists a positive constant C such that for every $f \in C^1(B)$*

$$\int_B \int_B \frac{|(f\circ\varphi_z)(w) - f(z)|^p}{|w|^p} \Big(1-|w|\Big)^{\frac{p}{2}-n-1} d\nu(w) d\lambda(z)$$
$$\leq C \int_B (\hat{Q}f)^p(w) d\lambda(w).$$

Proof. Let $f \in C^1(B)$. Lemma 3.1 with f replaced by $f\circ\varphi_z$ implies

$$\int_B \frac{|(f\circ\varphi_z)(w) - f(z)|^p}{|w|^p} \Big(1-|w|\Big)^{\frac{p}{2}-n-1} d\nu(w)$$
$$\lesssim \int_B \frac{\hat{Q}(f\circ\varphi_z)^p(w)\Big(1-|w|\Big)^{\frac{p}{2}-n-1}}{|w|^{n+p-2}} d\nu(w).$$

By Lemma 3.2, we can proceed analogously to the proof of Lemma 3.6 in [2]. □

Proof. [Completion of the proof of Theorem B]. Let $f \in MB_p$. Then

$$\int_B \int_{E(z,r)} \left|\frac{f(z)-f(w)}{1-<z,w>}\right|^p (1-|z|^2)^{\frac{p}{2}}(1-|w|^2)^{\frac{p}{2}} d\lambda(w)d\lambda(z)$$

$$\leq \int_B \int_B \left|\frac{f(z)-f(w)}{w-P_w z - s_w Q_w z}\right|^p (1-|z|^2)^{\frac{p}{2}}(1-|w|^2)^{\frac{p}{2}} d\lambda(w)d\lambda(z)$$

(by (2.1))

$$= \int_B \int_B \frac{|f(z)-f(w)|^p}{|\varphi_w(z)|^p \cdot |1-<z,w>|^p}(1-|z|^2)^{\frac{p}{2}}(1-|w|^2)^{\frac{p}{2}} d\lambda(w)d\lambda(z)$$

(by (2.2))

$$= \int_B \int_B \frac{|f(z)-f(w)|^p}{|\varphi_w(z)|^p}(1-|\varphi_w(z)|^2)^{\frac{p}{2}} d\lambda(w)d\lambda(z)$$

$$= \int_B \int_B \frac{|(f\circ\varphi_w)(u)-f(w)|^p}{|u|^p}(1-|u|^2)^{\frac{p}{2}} d\lambda(u)d\lambda(w)$$

$$= \int_B \int_B \frac{|(f\circ\varphi_w)(u)-f(w)|^p}{|u|^p}(1-|u|^2)^{\frac{p}{2}-n-1} d\nu(u)d\lambda(w)$$

$$\leq C \int_B (\hat{Q}f)^p(w)d\lambda(w).$$

□

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