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CONVERGENCE THEOREMS OF MODIFIED ISHIKAWA ITERATIVE SEQUENCES WITH MIXED ERRORS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT In this paper, we will discuss some sufficient and necessary conditions for modified Ishikawa iterative sequence with mixed errors to converge to fixed points for asymptotically quasi-nonexpansive mappings in Banach spaces The results presented in this paper extend, generalize and improve the corresponding results in Liu [4,5] and Ghosh-Debnath [2]

1. Introduction

Let E be a Banach space with $\|\cdot\|$, C be a nonempty subset of E, $\mathbb{N}$ be the set of all positive integers and $F(T)$ be the set of all fixed points of T.

DEFINITION 1.1 Let $T : C \to C$ be a mapping.

(1) The mapping T is said to be *asymptotically quasi-nonexpansive* if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such

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that

$$\|T^n x - p\| \le (1 + k_n)\|x - p\|,$$

for all $x \in C$ and $p \in F(T)$.

If $k_n = 0$ for $n = 1, 2, \cdots$, then the mapping T is said to be a *quasi-nonexpansive.*

(2) The mapping T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\|T^n x - T^n y\| \le (1 + k_n)\|x - y\|$$

for all $x, y \in C$ and $n = 1, 2, \cdots$.

If $k_n = 0$ for $n = 1, 2, \cdots$, then the mapping T is said to be a *nonexpansive.*

REMARK 1.1. From Definition 1.1, it is known that, if T is an asymptotically nonexpansive mapping and $F(T)$ is a nonempty set, then T is an asymptotically quasi-nonexpansive mapping But the converse is not true in general.

DEFINITION 1.2. Let C be a nonempty subset of E, $T : C \to C$ be a mapping and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three sequences in $[0, 1]$.

(1) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n + u_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n + v_n, \quad \forall\, n \in \mathbb{N} \end{cases} \tag{1.1}$$

is called the *modified Ishikawa iterative sequence with mixed errors,* where $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in C.

(2) In (1.1), if $\beta_n \equiv 0$, $v_n \equiv 0$ for all $n = 1, 2, \cdots$, then the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n + u_n, \quad \forall\, n \in \mathbb{N} \end{cases}$$

is called the *modified Mann iterative sequence with mixed errors.*

The concept of an asymptotically nonexpansive mapping was introduced by Gobel-Kirk [3] in 1972. It is well known that, if E is a uniformly convex Banach space, C is a nonempty closed bounded convex subset of E, then an asymptotically nonexpansive mapping defined on C has a fixed point in C (see, Browder [1]).

The iterative approximation problems for asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings were extensively studied by many authors. In particular, in 2001, Liu [4] obtained some sufficient and necessary conditions for Ishikawa iterative sequences to converge to fixed points for asymptotically quasi-nonexpansive mappings in Banach spaces. The later, Liu [5] generalized his result [4] to Ishikawa iterative sequence with errors for asymptotically quasi-nonexpansive mappings.

Motivated and inspired by Liu's results [4,5], in this paper, we will obtain some sufficient and necessary conditions for modified Ishikawa iterative sequences with mixed errors to converge to fixed points for asymptotically quasi-nonexpansive mappings in Banach spaces. The results presented in this paper extend, generalize and improve the corresponding results of Ghosh-Debnath [2], Gobel-Kirk [3] and Liu [4,5].

2. Main results

In order to prove the our main results, we will first prove the following lemma 2.1.

LEMMA 2.1. *Let E be a real Banach space, C be a nonempty convex subset of E, $T : C \to C$ be an asymptotically quasi-nonexpansive mapping satisfying $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T)$ be a nonempty set. Let $\{x_n\}$ be the modified Ishikawa iterative sequence with mixed errors defined in (1.1). Then*

(a) $\|x_{n+1} - p\| \leq (1 + k_n)^2 \|x_n - p\| + m_n, \quad \forall\, n \in \mathbb{N}, \forall\, p \in F(T)$, *where* $m_n = \alpha_n(1 + k_n)\|v_n\| + \|u_n\|$.

(b) *There exists a constant $M > 0$ such that*

$$\|x_{n+m}-p\| \leq M\|x_n-p\|+M\sum_{j=n}^{n+m-1} m_j, \quad \forall\, n,m \in \mathbb{N}, \;\; \forall\, p \in F(T).$$

Proof. (a) Since T is an asymptotically quasi-nonexpansive mappings, for all $p \in F(T)$, we have

(2.1)
$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\alpha_n)x_n+\alpha_n T^n y_n+u_n-p\|\\ &\leq (1-\alpha_n)\|x_n-p\|+\alpha_n\|T^n y_n-p\|+\|u_n\|\\ &\leq (1-\alpha_n)\|x_n-p\|+\alpha_n(1+k_n)\|y_n-p\|+\|u_n\|\end{aligned}$$

and

$$\begin{aligned}\|y_n-p\| &= \|(1-\beta_n)x_n+\beta_n T^n x_n+v_n-p\|\\ &\leq (1-\beta_n)\|x_n-p\|+\beta_n\|T^n x_n-p\|+\|v_n\|\\ &\leq (1-\beta_n)\|x_n-p\|+\beta_n(1+k_n)\|x_n-p\|+\|v_n\|.\end{aligned} \tag{2.2}$$

Substituting (2.2) into (2.1), it can be obtained that

$$\begin{aligned}\|x_{n+1}-p\| &\leq (1-\alpha_n)\|x_n-p\|\\ &\quad+\alpha_n(1+k_n)\{(1-\beta_n)\|x_n-p\|+\beta_n(1+k_n)\|x_n-p\|\\ &\quad+\|v_n\|\}+\|u_n\|\\ &= (1-\alpha_n)\|x_n-p\|+\alpha_n(1+k_n)(1-\beta_n)\|x_n-p\|\\ &\quad+\alpha_n\beta_n(1+k_n)^2\|x_n-p\|+\alpha_n(1+k_n)\|v_n\|+\|u_n\|\\ &\leq (1-\alpha_n)(1+k_n)^2\|x_n-p\|+\alpha_n(1-\beta_n)(1+k_n)^2\|x_n-p\|\\ &\quad+\alpha_n\beta_n(1+k_n)^2\|x_n-p\|+m_n\\ &= (1+k_n)^2\|x_n-p\|+m_n,\end{aligned}$$

where $m_n = \alpha_n(1+k_n)\|v_n\|+\|u_n\|$. This completes the proof of (a).

(b) If $a \geq 0$, then $1 + a \leq e^a$ and $(1+a)^2 \leq e^{2a}$. Therefore, from (a) we can obtain that

$$\begin{aligned}
\|x_{n+m} - p\| &\leq (1 + k_{n+m-1})^2 \|x_{n+m-1} - p\| + m_{n+m-1} \\
&\leq e^{2k_{n+m-1}} \|x_{n+m-1} - p\| + m_{n+m-1} \\
&\leq e^{2k_{n+m-1}} [(1 + k_{n+m-2})^2 \|x_{n+m-2} - p\| + m_{n+m-2}] \\
&\quad + m_{n+m-1} \\
&\leq e^{2(k_{n+m-1}+k_{n+m-2})} \|x_{n+m-2} - p\| \\
&\quad + e^{2k_{n+m-1}} m_{n+m-2} + m_{n+m-1} \\
&\leq e^{2(k_{n+m-1}+k_{n+m-2})} \|x_{n+m-2} - p\| \\
&\quad + e^{2k_{n+m-1}} (m_{n+m-1} + m_{n+m-2}) \\
&\leq \cdots \\
&\leq e^{2\sum_{j=n}^{n+m-1} k_j} \|x_n - p\| + e^{2\sum_{j=n}^{n+m-1} k_j} \sum_{j=n}^{n+m-1} m_j \\
&\leq M \|x_n - p\| + M \sum_{j=n}^{n+m-1} m_j,
\end{aligned}$$

where $M = e^{2\sum_{j=n}^{\infty} k_j}$ This completes the proof of (b). □

We also need the following lemma in the proof of our main results.

LEMMA 2.2 [5]. *Let $\{a_n\}$, $\{b_n\}$ and $\{\lambda_n\}$ be three nonnegative real sequences such that $a_{n+1} \leq (1+\lambda_n)a_n + b_n$, $\forall\, n \in \mathbb{N}$, $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$. Then*

(a) $\lim_{n\to\infty} a_n$ *exist.*
(b) *If* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Now, we are in a position to prove the our main theorems. $D(y, S)$ denotes the distance of y to set S, that is, $D(y, S) = \inf\{\|y - s\| : s \in S\}$.

THEOREM 2.1. *Let E be a real Banach space, C be a nonempty convex subset of E, $T : C \to C$ be an asymptotically quasi-nonexpansive mapping satisfying $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T)$ be a nonempty set. Let $\{x_n\}$ be the modified Ishikawa iterative sequence with mixed errors defined in (1.1), where $\{u_n\}$ and $\{v_n\}$ are two summable sequences in C, that is, $\sum_{n=1}^{\infty} \|u_n\| < \infty, \sum_{n=1}^{\infty} \|v_n\| < \infty$. Then the iterative sequence $\{x_n\}$ converges to a fixed point p if and only if*

$$\liminf_{n\to\infty} D(x_n, F(T)) = 0.$$

Proof. From Lemma 2.1 (a), we have

$$(2.3)\quad \|x_{n+1}-p\| \le (1+k_n)^2\|x_n-p\|+m_n, \quad \forall\, p \in F(T), \ \ \forall\, n \in \mathbb{N},$$

where $m_n = \alpha_n(1+k_n)\|v_n\|+\|u_n\|$. Since $0 \le \alpha_n \le 1, \sum_{n=1}^{\infty} k_n < \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\sum_{n=1}^{\infty} \|v_n\| < \infty$, we have $\sum_{n=1}^{\infty} m_n < \infty$. From (2.3), we have

$$D(x_{n+1}, F(T)) \le (1+k_n)^2 D(x_n, F(T)) + m_n.$$

Since $\liminf_{n\to\infty} D(x_n, F(T)) = 0$, by Lemma 2.2, we can obtain that

$$\lim_{n\to\infty} D(x_n, F(T)) = 0.$$

Now, we have to prove that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$. from Lemma 2.1, there exists a constant $M > 0$ such that

(2.4)

$$\|x_{n+m}-p\| \le M\|x_n-p\|+M \sum_{j=n}^{n+m-1} m_j, \quad \forall\, p \in F(T), \ \ \forall\, n, m \in \mathbb{N}.$$

Since $\lim_{n\to\infty} D(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} m_n < \infty$, there exists a constant N_1 such that for all $n \ge N_1$,

$$D(x_n, F(T)) < \frac{\epsilon}{4M} \quad \text{and} \quad \sum_{j=N_1}^{\infty} m_j < \frac{\epsilon}{6M}.$$

So, $D(x_{N_1}, F(T)) < \frac{\epsilon}{4M}$.

We note that there exists $p_1 \in F(T)$ such that $\|x_{N_1} - p_1\| < \frac{\epsilon}{3M}$. From (2.4), we can obtain that for all $n \geq N_1$,

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\
&\leq M\|x_{N_1} - p_1\| + M\sum_{j=N_1}^{N_1+m-1} m_j + M\|x_{N_1} - p_1\| \\
&\quad + M\sum_{j=N_1}^{N_1+m-1} m_j \\
&< M\frac{\epsilon}{3M} + M\frac{\epsilon}{6M} + M\frac{\epsilon}{3M} + M\frac{\epsilon}{6M} \\
&= \epsilon.
\end{aligned}$$

Since ϵ is an arbitrary positive number, this implies that $\{x_n\}$ is a Cauchy sequence. therefore, $\lim_{n\to\infty} x_n$ exists. Let $\lim_{n\to\infty} x_n = p$. It will be proven that p is a fixed point, that is, $p \in F(T)$. Let $\bar{\epsilon} > 0$. Since $\lim_{n\to\infty} x_n = p$, there exists a natural number N_2 such that for all $n \geq N_2$,

$$\|x_n - p\| < \frac{\bar{\epsilon}}{2(2 + k_1)}. \tag{2.5}$$

$\lim_{n\to\infty} D(x_n, F(T)) = 0$ implies that there exists a natural number $N_3 \geq N_2$ such that for all $n \geq N_3$,

$$D(x_n, F(T)) < \frac{\bar{\epsilon}}{3(4 + 3k_1)}.$$

Therefore, there exists a $\bar{p} \in F(T)$ such that

$$\|x_{N_3} - \bar{p}\| < \frac{\bar{\epsilon}}{2(4 + 3k_1)}. \tag{2.6}$$

From (2.5) and (2.6), we have

$$\begin{aligned}\|Tp-p\| &\le \|Tp-\bar{p}+\bar{p}-Tx_{N_3}+Tx_{N_3}-\bar{p}+\bar{p}-x_{N_3}+x_{N_3}-p\| \\ &\le \|Tp-\bar{p}\|+2\|Tx_{N_3}-\bar{p}\|+\|x_{N_3}-\bar{p}\|+\|x_{N_3}-p\| \\ &\le (1+k_1)\|p-\bar{p}\|+2(1+k_1)\|x_{N_3}-\bar{p}\|+\|x_{N_3}-\bar{p}\| \\ &\quad +\|x_{N_3}-p\| \\ &\le (1+k_1)\|x_{N_3}-p\|+(1+k_1)\|x_{N_3}-\bar{p}\| \\ &\quad +2(1+k_1)\|x_{N_3}-\bar{p}\|+\|x_{N_3}-\bar{p}\|+\|x_{N_3}-p\| \\ &= (2+k_1)\|x_{N_3}-p\|+(4+3k_1)\|x_{N_3}-\bar{p}\| \\ &< (2+k_1)\frac{\bar{\epsilon}}{2(2+k_1)}+(4+3k_1)\frac{\bar{\epsilon}}{2(4+3k_1)} \\ &= \bar{\epsilon}.\end{aligned}$$

Since $\bar{\epsilon}$ is an arbitrary positive number, we can obtain that $Tp = p$, that is, p is a fixed point of T. This completes the proof of Theorem 2.1. □

THEOREM 2.2. *Let E be a real Banach space, C be a nonempty convex subset of E, $T : C \to C$ be an quasi-nonexpansive mapping satisfying $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T)$ be a nonempty set. Let $\{x_n\}$ be the modified Ishikawa iterative sequence with mixed errors defined in (1.1), where $\{u_n\}$and $\{v_n\}$ are two summable sequences in C. Then the iterative sequence $\{x_n\}$ converges to a fixed point p if and only if*

$$\liminf_{n\to\infty} D(x_n, F(T)) = 0.$$

Proof. From the Definition 1.1(1), a quasi-nonexpansive mapping is asymptotically quasi-nonexpansive mapping. Therefore, Theorem 2.2 can be proven from Theorem 2.1 immediately. □

REMARK 2.1. (1) Theorem 2.1 extends Theorem 1 in Liu [5], in terms of mixed errors as more general errors.

(2) Theorem 2.2 generalizes and improves the corresponding results in Liu [4,5] and Ghosh-Debnath [2].

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