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EMBEDDING OF WEIGHTED L^p SPACES AND THE $\bar\partial$-PROBLEM

HONG RAE CHO

ABSTRACT. Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. In this paper, we prove the following inequality

$$\|u\|_{p_2,\alpha_2} \lesssim \|u\|_{p_1,\alpha_1} + \|\bar\partial u\|_{p_1,\alpha_1+p_1/2},$$

where $1 \le p_1 \le p_2 < \infty$, $\alpha_j > 0, (n+\alpha_1)/p_1 = (n+\alpha_2)/p_2$, and $1/p_2 \ge 1/p_1 - 1/2n$

1. Introduction and statement of results

Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. For $z \in D$ let $\delta(z)$ denote the distance from z to ∂D. For $\alpha > 0$, we define a weighted measure dV_α on D by $dV_\alpha = C_\alpha \delta^{\alpha-1} dV$ where dV is the volume element and C_α is chosen so that dV_α is a probability measure. As $\alpha \to 0^+$ the measures dV_α converges as measures on ∂D to the normalized surface measure on ∂D which we denote dV_0 (or sometimes $d\sigma$). We will denote the L^p space with respect to dV_α by L^p_α, and the associated norm by $\|\cdot\|_{p,\alpha}$. We will denote by $A^p_\alpha(D) = L^p_\alpha(D) \cap \mathcal{O}(D)$ the subspace of $L^p_\alpha(D)$ consisting of

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functions which are holomorphic on D. In particular, $A^p_0(D)$ is the Hardy class usually denoted by $H^p(D)$, which we identify in the usual way with a subspace of $L^p_0(D) = L^p(\partial D; d\sigma)$ (see [16]). Beatrous [6] proved the following embedding theorem.

THEOREM 1.1 ([6]). *Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary and assume that $0 < p_1 \leq p_2 < \infty$, $\alpha_j > 0$, and $(n+\alpha_1)/p_1 = (n+\alpha_2)/p_2$. Then $A^{p_1}_{\alpha_1}(D) \subset A^{p_2}_{\alpha_2}(D)$ and the inclusion is continuous.*

In the case $\alpha_1 = 0$ the embedding in Theorem 1.1 will be

$$H^{p_1}(D) \subset A^{p_2}_{\alpha_2}(D), \quad \text{where} \quad n/p_1 = (n+\alpha_1)/p_2.$$

This is a generalization of a well-known result of Hardy-Littlewood in the unit disc (see [13, p.87]). Beatrous [7] proved that the case $\alpha_1 = 0$ holds if D is strictly pseudoconvex domains. Recently, the author proved the case $\alpha_1 = 0$ in convex domains of finite type (see [9]). Moreover, it is proved that the case $\alpha_1 = 0$ holds in bounded domains with C^2 boundary ([10], [11]).

In this paper, we extend Theorem 1.1 for $L^p_\alpha(D)$ functions u with some growth condition of $\bar\partial u$, and give some consequences for the $\bar\partial$-problem.

THEOREM 1.2. *Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary and assume that $1 \leq p_1 \leq p_2 < \infty$, $\alpha_j > 0, (n+\alpha_1)/p_1 = (n+\alpha_2)/p_2$, and $1/p_2 \geq 1/p_1 - 1/2n$. Let $u \in L^{p_1}_{\alpha_1}(D)$. Then u belongs to $L^{p_2}_{\alpha_2}(D)$ under the extra condition that $\bar\partial u \in L^{p_1}_{\alpha_1+p_1/2}(D)$.*

In condition of $\bar\partial u$, one recognizes the gain for the solution of the $\bar\partial$-equation in strictly pseudoconvex domains. Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. Let $f \in L^p_{\alpha+p/2}(D)$ be a $\overline{\partial}$-closed (0,1) form on D. In ([2], [12]) it was proved that there is a solution u for $\overline{\partial} u = f$ such that

$$\|u\|_{p,\alpha} \leq C_{p,\alpha}\|f\|_{p,\alpha+p/2} \quad \text{for} \quad 1 \leq p < \infty, \alpha > 0.$$

Thus we get the following result.

COROLLARY 1.3. *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. Let $1 \le p_1 \le p_2 < \infty$, $\alpha_j > 0$, $(n+\alpha_1)/p_1 = (n+\alpha_2)/p_2$, and $1/p_2 \ge 1/p_1 - 1/2n$. Let $f \in L^{p_1}_{\alpha_1+p_1/2}(D)$ be a $\bar\partial$-closed $(0,1)$ form on D. Then there is a solution $u \in L^{p_2}_{\alpha_2}(D)$ for $\bar\partial u = f$.*

REMARK 1.4. When $1 \le p_1 < 2$, if we take $p = p_1$, $\alpha_1 = 1 - p/2$, and $\alpha_2 = 1$, then Corollary 1.3 implies that for a $\bar\partial$-closed $(0,1)$ form $f \in L^p(D)$ there is a solution $u \in L^q(D)$ for $\bar\partial u = f$, where $1/q = 1/p - 1/(2n+2)$. The result is the optimal L^p-estimate for $\bar\partial$ proved in [14] when $1 \le p < 2$. For more recent results about estimates for $\bar\partial$ and $\bar\partial_b$ by means of integral kernels we can refer ([1], [3], [4], [5]).

2. Proof of Theorem 1.2

We shall rely on Bonami-Sibony's ideas [8] for the proof of Theorem 1.2. Before proceeding with the proof, we give the key lemma.

LEMMA 2.1 ([8]). *Let B be the unit ball, $\tilde B$ its homothetic of radius $R_0 > 1$, let $1 \le p \le r < \infty$. Then there exists a constant $C > 0$ such that for any $f \in L^p(\tilde B)$ for which $\bar\partial f$ belongs to $L^t(\tilde B)$ with $t \ge 1$ and $1/r \ge 1/t - 1/2n$:*

$$\left(\int_B |f|^r \, dV\right)^{1/r} \le C\left(\int_{\tilde B} |f|^p \, dV\right)^{1/p} + C\left(\int_{\tilde B} |\bar\partial f|^t \, dV\right)^{1/t}.$$

Proof of Theorem 1.2. It is enough to prove the inequality

$$\int_D |u|^{p_2}\delta^{\alpha_2-1} dV \lesssim \int_D |u|^{p_1}\delta^{\alpha_1-1} dV + \int_D |\bar\partial u|^{p_1}\delta^{\alpha_1+p_1/2-1} dV.$$

For $p_0 \in D$ sufficiently near ∂D, we translate and rotate the coordinate system so that $z(p_0) = 0$ and the Im z_1 axis is perpendicular to ∂D. Let $\mathcal{B}_\epsilon(p_0)$ denote the non-isotropic ball

$$\mathcal{B}_\epsilon(p_0) = \left\{ \frac{|z_1|^2}{(\epsilon\delta(p_0))^2} + \sum_2^n \frac{|z_j|^2}{\epsilon\delta(p_0)} < 1 \right\}.$$

Since ∂D is C^2, it follows that there is an $\epsilon_0 > 0$ such that for p_0 sufficiently near ∂D and $z \in \mathcal{B}_{\epsilon_0}(p_0)$ we have $z \in D$ and

$$\frac{\delta(p_0)}{2} \le \delta(z) \le 2\delta(p_0). \tag{2.1}$$

There is a compact subset K of D, a sequence $\{p_j\}$ in $D \setminus K$, and a positive integer N such that

(2.2)

the family $\{ \mathcal{B}_{\epsilon_0/2}(p_j) \}$ covers $D \setminus K$, and

(2.3)

each point of D lies in at most N of the sets $\mathcal{B}_{\epsilon_0}(p_j)$.

For brevity we denote by $\mathcal{B}_j$ the ball $\mathcal{B}_{\epsilon_0/2}(p_j)$ and by $\tilde{\mathcal{B}}_j$ the ball $\mathcal{B}_{\epsilon_0}(p_j)$.

By homogeneity, it follows from Lemma 2.1 that

$$\begin{aligned} &\left(\int_{\mathcal{B}_j} |u|^{p_2}\, dV\right) \delta(p_j)^{-(n+1)} \\ &\lesssim \left(\int_{\tilde{\mathcal{B}}_j} |u|^{p_1}\right)^{p_2/p_1} \delta(p_j)^{-(n+1)p_2/p_1} \\ &\quad + \left(\int_{\tilde{\mathcal{B}}_j} |\bar\partial u|^{p_1}\, dV\right)^{p_2/p_1} \delta(p_j)^{p_2/2-(n+1)p_2/p_1}. \end{aligned} \tag{2.4}$$

By (2.1), (2.2), and (2.3), we have to give a bound to

$$\begin{aligned} \int_D |u|^{p_2}\delta^{\alpha_2-1}\, dV &\sim \sum_j \int_{\mathcal{B}_j} |u|^{p_2}\delta^{\alpha_2-1}\, dV \\ &\sim \sum_j \left(\int_{\mathcal{B}_j} |u|^{p_2}\, dV\right) \delta(p_j)^{\alpha_2-1}, \end{aligned}$$

where the summation is a finite sum.

Using (2.4), it is enough to show that

$$(2.5) \quad \sum \left(\int_{\tilde{B}_j} |u|^{p_1} \, dV \right) \delta(p_j)^{-(n+1)(1-p_1/p_2)+(\alpha_2-1)p_1/p_2} < \infty$$

and

(2.6)

$$\sum \left(\int_{\tilde{B}_j} |\bar{\partial} u|^{p_1} \, dV \right) \delta(p_j)^{p_1/2-(n+1)(1-p_1/p_2)+(\alpha_2-1)p_1/p_2} < \infty.$$

We note that $-(n+1)(1-p_1/p_2)+(\alpha_2-1)p_1/p_2 = \alpha_1 - 1$. Hence the inequalities (2.5) and (2.6) follows from (2.3) and growth conditions of u and $\bar{\partial} u$. □

3. An example

In this section we give an example to show that the embedding in Theorem 1.2 is the optimal result in some sense for strictly pseudoconvex domains We restrict ourselves to the unit ball B_2 in $\mathbb{C}^2$

LEMMA 3.1 ([15]). *For $z \in B_n$, c real, $\eta > -1$, define*

$$J_{c,\eta}(z) = \int_{B_n} \frac{(1-|\zeta|^2)^\eta}{|1-\bar{\zeta} \cdot z|^{n+1+\eta+c}} dV(\zeta).$$

When $c < 0$, then $J_{c,\eta}$ is bounded in B_n. When $c > 0$, then $J_{c,\eta}(z) \approx (1-|z|^2)^{-c}$. Finally, $J_{0,\eta} \approx -\log(1-|z|^2)$.

THEOREM 3.2. *Let $1 \le p_1 \le p_2 < \infty$, $\alpha_j > 0$, and $(2+\alpha_1)/p_1 = (2+\alpha_2)/p_2$. For any $\epsilon > 0$ there is $u_{p_1,\alpha_1,\epsilon} \in L^{p_1}_{\alpha_1}(B_2)$ such that $u_{p_1,\alpha_1,\epsilon}$ does not belong to $L^{p_2+\epsilon}_{\alpha_2}(B_2)$ or $L^{p_2}_{\alpha_2-\epsilon}(B_2)$, while $\bar{\partial} u_{p_1,\alpha_1,\epsilon}$ belongs to $L^{p_1}_{\alpha_1+p_1/2}(B_2)$.*

Proof. If $dp_1-\alpha_1-p_1/2=2-\mu$ then $dp_2-\alpha_2-p_2/2+\mu p_2/p_1=2$. So, we can choose $d>0$ such that $dp_1-\alpha_1-p_1/2<2$, $dp_2-\alpha_2-p_2/2+\epsilon>2$, and $dp_2-\alpha_2-p_2/2+\epsilon(d-1/2)>2$.

Let $u_{p_1,\alpha_1,\epsilon}(z_1,z_2)=\bar{z}_2/(1-z_1)^d$. For simplicity of notation, we write $u=u_{p_1,\alpha_1,\epsilon}$ and $r_{z_1}=\sqrt{1-|z_1|^2}$. Then we have

(3.1)

$$\|u\|_{p_1,\alpha_1}^{p_1} \lesssim \int_{B_2} \frac{|z_2|^{p_1}(1-|z_1|^2-|z_2|^2)^{\alpha_1-1}}{|1-z_1|^{dp_1}}dV$$
$$\lesssim \int_{|z_1|<1} \frac{dA(z_1)}{|1-z_1|^{dp_1}} \int_{|z_2|<r_{z_1}} |z_2|^{p_1}(1-|z_1|^2-|z_2|^2)^{\alpha_1-1}dA(z_2).$$

By the polar coordinate change $|z_2|^2=re^{i\theta}$, we have

(3.2)

$$\begin{aligned} I(z_1) &= \int_{|z_2|<r_{z_1}} |z_2|^{p_1}(1-|z_1|^2-|z_2|^2)^{\alpha_1-1}dA(z_2) \\ &= 2\pi(1-|z_1|^2)^{\alpha_1-1}\int_0^{r_{z_1}} r^{p_1+1}\left(1-\frac{r^2}{1-|z_1|^2}\right)^{\alpha_1-1}dr \\ &= 2\pi(1-|z_1|^2)^{\alpha_1+p_1/2}\int_0^1 (1-s^2)^{\alpha_1-1}s^{p_1+1}ds, \end{aligned}$$

where we set $s=r/\sqrt{1-|z_1|^2}$. Note that

$$\int_0^1 (1-s^2)^{\alpha_1-1}s^{p_1+1}ds=\frac{1}{2}B\left(\frac{p_1}{2}+1,\alpha_1\right),$$

where $B(\cdot,\cdot)$ is the beta function. By (3.1), (3.2) and Lemma 3.2, it follows that

$$\begin{aligned} \|u\|_{p_1,\alpha_1}^{p_1} &\lesssim \int_{|z_1|<1} \frac{dA(z_1)}{|1-z_1|^{dp_1-\alpha_1-p_1/2}} \\ &= \lim_{r\to 1^-}\int_{|z_1|<1} \frac{dA(z_1)}{|1-z_1 r|^{dp_1-\alpha_1-p_1/2}} \\ &\lesssim 1, \quad \text{since} \quad dp_1-\alpha_1-\frac{p_1}{2}<2. \end{aligned}$$

Similarly, we can prove that $\|\bar{\partial}u\|_{p_1,\alpha_1+p_1/2} \lesssim 1$.

Now we have

$$\begin{aligned}
&\|u\|_{p_2+\epsilon,\alpha_2}^{p_2+\epsilon} \\
&= \int_{|z_1|<1} \frac{dA(z_1)}{|1-z_1|^{d(p_2+\epsilon)}} \int_{|z_2|<r_{z_1}} |z_2|^{p_2+\epsilon}(1-|z_1|^2-|z_2|^2)^{\alpha_2-1}dA(z_2) \\
&= 2\pi \int_{|z_1|<1} \frac{(1-|z_1|^2)^{\alpha_2+(p_2+\epsilon)/2}}{|1-z_1|^{d(p_2+\epsilon)}} \int_0^1 (1-s^2)^{\alpha_2-1}s^{p_2+1+\epsilon}ds \\
&\gtrsim \int_{|z_1|<1} \frac{(1-|z_1|^2)^{\alpha_2+(p_2+\epsilon)/2}}{|1-z_1|^{d(p_2+\epsilon)}} dA(z_1) \\
&= \lim_{r\to 1^-} \int_{|z_1|<1} \frac{(1-|z_1|^2)^{\alpha_2+(p_2+\epsilon)/2}}{|1-z_1 r|^{d(p_2+\epsilon)}} dA(z_1) \\
&\approx \lim_{r\to 1^-} \frac{1}{(1-r^2)^{dp_2-\alpha_2-p_2/2+\epsilon(d-1/2)-2}} = \infty,
\end{aligned}$$

since $d(p_2+\epsilon)-\alpha_2-(p_2+\epsilon)/2 = dp_2-\alpha_2-p_2/2+d\epsilon-\epsilon/2 > 2$.

Similarly, we can show that $\|u\|_{p_2,\alpha_2-\epsilon}$ is divergent since $dp_2 - \alpha_2 - p_2/2 + \epsilon > 2$. Thus we get the result. □

REFERENCES

[1] K. Adachi and H. R. Cho, *$L^p(1 \le p \le \infty)$ estimates for $\bar{\partial}$ on a certain pseudoconvex domain in $\mathbb{C}^n$*, Nagoya Math. J. **148** (1997), 127–136

[2] H. Ahn and H. R. Cho, *Optimal non-isotropic L^p estimates with weights for $\bar{\partial}$ in strictly pseudoconvex domains*, Kyushu J. Math. **56** (2002), 447–457

[3] H. Ahn and H. R. Cho, *Zero sets of holomorphic functions in the Nevanlinna type class on convex domains in $\mathbb{C}^2$*, Japanese J. Math. **28** (2002), 245–260

[4] H. Ahn and H. R. Cho, *Optimal Sobolev estimates for $\bar{\partial}$ on convex domains of finite type*, Math. Z. (to appear)

[5] H. Ahn and H. R. Cho, *Optimal Holder and L^p estimates for $\bar{\partial}_b$ on the boundaries of convex domains of finite type*, submitted

[6] F. Beatrous, *L^p estimates for extensions of holomorphic functions*, Michigan Math. J. **32** (1985), 361–380

[7] ———, *Estimates for derivatives of holomorphic functions in pseudoconvex domains*, Math. Z. **191** (1986), 91–116

[8] A. Bonami and N. Sibony, *Sobolev embedding in $\mathbb{C}^n$ and the $\bar{\partial}$-equation*, The Journal of Geometric Analysis **1-4** (1991), 307–327.
[9] H. R. Cho, *Estimates on the mean growth of H^p functions in convex domains of finite type*, Proc Amer Math Soc. (to appear).
[10] H. R Cho and E. G Kwon, *Embedding of Hardy spaces into weighted Bergman spaces in bounded domains with C^2 boundary*, Illinois J. Math (to appear)
[11] H. R. Cho and E. G. Kwon, *Growth rate of the functions in Bergman type spaces*, Journal Math. Anal. Appl. (to appear).
[12] A. Dautov and G. M Henkin, *Zeros of Holomorphic functions of finite order and weighted estimates for solutions of the $\bar{\partial}$-equation*, Math. USSR Sb. **35** (1979), 449–459.
[13] P. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
[14] S. G Krantz, *Optimal Lipschitz and L^p regularity for the equation $\bar{\partial}u = f$ on strongly pseudoconvex domains*, Math Ann **219** (1976), 233–260
[15] W. Rudin, *Function theory in the unit ball of $\mathbb{C}^n$*, Spinger-Verlag, New York, 1980
[16] E M. Stein, *Boundary behavior of holomorphic functions of Several Complex Variables*, Princeton Univ Press, Princeton, N J , 1972

Department of Mathematics
Pusan National University, Pusan 609-735
South Korea
E-mail: chohr@pusan.ac.kr