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A NOTE ON DEFINING IDENTITIES OF DISTRIBUTIVE LATTICES

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Abstract. There are many conditions or identities for a lattice to be distributive In this paper, we study some identities on algebras of type (2,2) and find another set of identities defining distributive lattices. We also study certain identities which define algebras of type (2,2) whose subalgebras generated by two elements are all distributive lattices with at most 4 elements.

It is known that the conditions $(x+y)z = xz + yz$, $xy + z = (x+z)(y+z)$ and $(x+y)(y+z)(z+x) = xy + yz + zx$ are equivalent for a lattice ([5]). A lattice satisfying any of the above identities is called a distributive lattice. A distributive lattice is one of the most interesting and important algebras in every area of mathematics and its application In this reason, the characterization of distributive lattices has been a very important research topic for a long time. There are many useful characterizations of distributive lattices in many different ways.

Padmanabhan ([10]) characterized lattices as algebras $(A, +, \cdot)$ of type (2,2) by two identities $(x+y)z = z(x+x) + z(y+x)$ and $(zx + zy) + (zz + zz) = z$. An interesting fact of this result is that lattices can be characterized by identities in three variables. Thus, by adding the identity $(x+y)z = xz + yz$ or $xy + z = (x+z)(y+z)$ to these two identities, we can define distributive lattices by identities in three variables.

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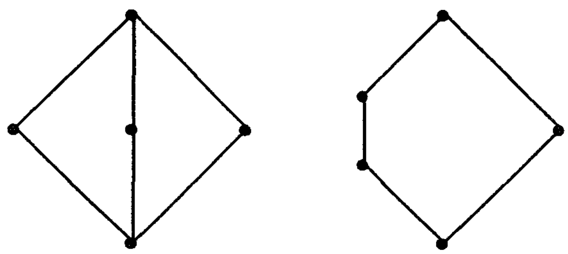
Birkhoff ([1]) characterized distribtive lattices by prohibiting sublattices isomorphic to one of the lattices M_5 and N_5 on the right. Birkhoff ([2]) and Stone ([11]) showed that a lattice is distributive if and only if it is isomorphic to a ring of sets.



M_5 N_5

More recently, approaching from another point of view, Dudek ([4]) showed that a commutative idempotent algebra of type (2,2) with $x + y \neq x \cdot y$ is term equivalent to a distributive lattice if and only if $(A, +, \cdot)$ has 9 essentially ternary term functions In connection with this characterization, Dudek raised the question: is it true that a commutative idempotent algebra of type (2,2) with $x + y \neq x \cdot y$ is a distributive lattice if and only if $(A, +, \cdot)$ has 114 essentially 4-ary term functions? ([5], [8]) Dudek ([5]) also showed that an algebra (A, Ω) is term equivalent to a distributive lattice if and only if it has l_n essentially n-ary term functions for all integer n, where l_n is the number of essentially n-ary term functions of the two-element lattice. It is known that the number l_n for $n \geq 8$ is unknown This is called a Dedekind's problem ([3]) and more detailed discussion can be found in [8] and [9]. Some other identities related to distributive lattices are discussed in [6]

In this paper, we study some identities on algebras of type (2,2) and find another set of identities defining distributive lattices in Theorem 1. In Theorem 2, we look in certain identities which define algebras of type (2,2) whose subalgebras generated by two elements are all distributive lattices with at most 4 elements.

For further definitions and terminology, we refer readers to [6] and [7].

Consider the following identities on an algebra $(A, +, \cdot)$ of type (2,2).

(1) $x + x = x$;

(2) $(x + y)y = y$;

(3) $x + y = y + x$, $xy = yx$;

(1)′ $xx = x$,

(2)′ $xy + y = y$.

Lemma 1. *If $(A, +, \cdot)$ is an algebra of type (2,2) satisfying the identities (1) and (2), then it also satisfies the identity (1)′. Dually, if $(A, +, \cdot)$ satisfies the identities (1)′ and (2)′, then it also satisfies the identity (1)*

Proof. Putting y for x in (2), we have $(x + x)x = x$, which implies (1)′ by (1). The proof of the dual part is similar.

Lemma 2. *If $(A, +, \cdot)$ is an algebra of type (2,2) satisfying the identities (1)–(3) and $(x + y)z = xz + yz$, then the operation '+' is associative. Dually, if $(A, +, \cdot)$ satisfies the identities (1)′, (2)′, (3) and $xy + z = (x + z)(y + z)$, then the operation '$\cdot$' is associative*

Proof. Auume the identities (1)–(3) By Lemma 1, (1)′ holds and so we have

$$((x + y) + z)y = (x + y)y + zy = y + zy = yy + zy = (y + z)y = y.$$

Then it follows that

$$\begin{aligned} ((x + y) + z)((z + y) + x) &= ((x + y) + z)(z + y) + ((x + y) + z)x \\ &= [((x + y) + z)z + ((x + y) + z)y] + x \\ &= (z + y) + x \end{aligned}$$

Therefore,

$$\begin{aligned} x + (y + z) &= (z + y) + x = ((x + y) + z)((z + y) + x) \\ &= ((z + y) + x)((x + y) + z) = (x + y) + z. \end{aligned}$$

The dual part can be proved analoguously.

Lemma 3. *If $(A,+,\cdot)$ is an algebra of type (2,2) satisfying the identities (1)–(3), then the identities $(x+y)z = xz+yz$ and $xy+z = (x+z)(y+z)$ are equivalent, and $(A,+,\cdot)$ is a distributive lattice if one of these identities holds. Dually, if $(A,+,\cdot)$ satisfies the identities (1)′, (2)′, (3), we obtain the same conclusion.*

Proof. By Lemma 1, we also have the identity (1)′. Suppose $(x+y)z = xz+yz$. By Lemma 2, '+' is associative. Thus $(A,+,\cdot)$ is a distributive lattice.

$$\begin{aligned}(x+z)(y+z) &= (x+z)y+(x+z)z = (xy+zy)+z = xy+(yz+z)\\ &= xy+(yz+zz) = xy+(y+z)z = xy+z.\end{aligned}$$

Conversely, suppose $xy+z = (x+z)(y+z)$. Then, we also have (2)′ since

$$xy+y = (x+y)(y+y) = (x+y)y = y,$$

and so (2)′ holds Using the dual part of Lemma 2, we infer that the operation '$\cdot$' is associative. Thus,

$$\begin{aligned}xz+yz &= (x+yz)(z+yz) = (yz+x)z\\ &= ((y+x)(z+x))z = (y+x)((z+x)z) = (x+y)z,\end{aligned}$$

as required. We note that The dual part can be proved analoguously.

Theorem 1. *Let $(A,+,\cdot)$ be an algebra of type (2,2) satisfying the identities (1)–(3) Then the following conditions are equivalent.*

(i) *$(A,+,\cdot)$ is a distributive lattice;*

(ii) *$(A,+,\cdot)$ satisfies $(x+y)z = xz+yz$,*

(iii) *$(A,+,\cdot)$ satisfies $xy+z = (x+z)(y+z)$*

Dually, we obtain the same conclusion if $(A,+,\cdot)$ satisfies the identities (1)′, (2)′, (3).

Proof By Lemmas 1, 2 and 3.

In connection with this theorem, we are interested in algebras $(A,+,\cdot)$ of type (2,2) satisfying the identities (1)–(3) and

(4) $\quad (xy+yz)+zx = ((x+y)(y+z))(z+x)$.

Note that these algebras are very close to distributive lattices. The identity $xy+y = y$ follows from the identities (1)–(4) but it is not a consequence of (1)–(3) only, as the following Lemma and Example show.

Lemma 4. *If* $(A, +, \cdot)$ *is an algebra of type (2,2) satisfying the identities (1)–(4), then it also satisfies the identities (1)′ and (2)′.*

Proof. By Lemma 1, we have $xx = x$. Letting $z = x$ in (4), we have

$$(xy + yx) + xx = ((x + y)(y + x))(x + x)$$

and so $xy + x = (x + y)x$. By (3), we have $xy + x = x$, which is the same identity as (2)′.

Example 1. Consider the algebra $(\mathbb{N}, +, \cdot)$, where $\mathbb{N}$ is the set of positive integers and the operations are defined by $x + y = [x, y]$, the least common multiple of x and y, and $xy = \min\{x, y\}$. It is easy to check that $(\mathbb{N}, +, \cdot)$ satisfies (1)–(3) but it does not satisfy $xy + y = y$. By Lemma 4, $(\mathbb{N}, +, \cdot)$ does not satisfies (4)

It is still aproblem that if there is an algebra $(A, +, \cdot)$ of type (2,2) which is a model of the identities (1)–(4) but not a lattice.

A commutative idempotent groupoid is called a *near-semilattice* if it satisfies the identity $(xy)y = xy$. An algebra $(A, +, \cdot)$ of type (2,2) is called a *bi-near-semilattice* if both $(A, +)$ and $(A, \cdot)$ are near-semilattices.

Lemma 5. *If* $(A, +, \cdot)$ *is an algebra of type (2,2) satisfying the identities (1)–(4), then* $(A, +, \cdot)$ *is a bi-near-semilattice*

Proof. By Lemma 4, we also have $xy + y = y$. Putting xy for x in (4), we obtain

$$\begin{aligned} ((xy)y + yz) + z(xy) &= ((xy + y)(y + z))(z + xy) \\ &= (y(y + z))(z + xy) \\ &= y(z + xy). \end{aligned}$$

Put xy for z in the above identity, we have

$$((xy)y + y(xy)) + (xy)(xy) = y(xy + xy)$$

and hence $(xy)y + xy = (xy)y$ Since $(xy)y + xy = xy$ by Lemma 4, we have $xy = (xy)y$. Thus $(A, \cdot)$ is a near-semilattice. Now, putting

$x+y$ for x in (4), we obtain that

$$\begin{aligned}(((x+y)+y)(y+z))((x+y)+z) &= ((x+y)y+(yz))+(x+y)z \\ &= (y+yz)+(x+y)z \\ &= y+(x+y)z\end{aligned}$$

Then, putting $x+y$ for z in this identity, we have

$$(((x+y)+y)(y+(x+y)))((x+y)+(x+y)) = y+(x+y)(x+y)$$

and hence $((x+y)+y)(x+y) = (x+y)+y$. Since $((x+y)+y)(x+y) = x+y$ by Lemma 4, we have $x+y = (x+y)+y$. This shows that $(A,+)$ is a near-semilattice.

Lemma 6. *Let $(A,+,\cdot)$ be an algebra of type (2,2) satisfying the identities (1)–(4), then we have the following statements.*

(i) *The term $m(x,y,z) = ((x+y)(y+z))(z+x)$ is a majority function, i e , $m(x,y,y) = m(y,x,y) = m(y,y,x) = y$;*

(ii) *$(A,+,\cdot)$ satisfies the identities $(x+y)(xy) = xy$ and $(x+y)+xy = x+y$.*

(iii) *If '+' is associative then $(A,+,\cdot)$ also satisfies the identity $(xy+z)x = xy+xz$*

Proof (i) Using Lemma 4, we have $m(x,y,y) = ((x+y)(y+y))(y+x) = ((x+y)y)(x+y) = y(x+y) = y$. Similarly, we can show $m(y,x,y) = m(y,y,x) = y$.

(ii) Put xz for y in (4), then we obtain

$$((x+xz)(xz+z))(z+x) = (x(xz)+(xz)z)+xz.$$

The left hand side of the identity is equal to $(xz)(z+x)$ by Lemma 4, while the right hand side is equal to $(xz+xz)+xz$ by Lemma 5. Thus, we have that $(xz)(z+x) = (xz+xz)+xz = xz$, which implies $(x+y)(xy) = xy$ Using this identity and Lemma 4, we get

$$(x+y)+xy = (x+y)+(x+y)(xy) = x+y.$$

(iii) In the proof of Lemma 5, we have $(xy+z)y = (xy+zy)+(xy)z$ and hence $(xy+zy)+(xy)z = ((xy)z+xy)+zy = xy+zy$, as required.

Theorem 2. *Let $(A,+,\cdot)$ be an algebra of type (2,2) satisfying the identities (1)–(4), then each subalgebra of $(A,+,\cdot)$ generated by two elements has at most four elements and it is a distributive lattice.*

Proof. Let $a, b \in A$ and let $S(a,b)$ denote the subalgebra generated by $\{a,b\}$. By Lemmas 5 and 6, we infer that any term in the two variables x and y is equal to one of the terms $x,\ y,\ xy, x+y$ on $(A,+,\cdot)$. Thus $S(a,b) = \{a, b, a+b, ab\}$. Now it is routine to check that $S(a,b)$ satisfies the identities (1)–(3) and the identity $(x+y)z = xz+yz$. Then, by Theorem 1, $S(a,b)$ is a distributive lattice.

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