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INTEGRAL INEQUALITIES OF GRÜSS TYPE VIA PÓLYA-SZEGÖ AND SHISHA-MOND RESULTS

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ABSTRACT Integral inequalities of Grüss type obtained via Pólya-Szegö and Shisha-Mond results are given Some applications for Taylor's generalized expansion are also provided

1. Introduction

For two measurable functions $f, g . [a, b] \to \mathbb{R}$, define the functional, which is known in the literature as Chebychev's functional

$$T(f,g;a,b) := \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \int_a^b g(x)\,dx,$$

provided that the involved integrals exist.

The following inequality is well known in the literature as the Grüss inequality [11]

$$|T(f,g;a,b)| \leq \frac{1}{4}(M-m)(N-n), \tag{1 1}$$

provided that $m \leq f \leq M$ and $n \leq g \leq N$ a.e. on $[a, b]$, where m, M, n, N are real numbers The constant $\frac{1}{4}$ in (1.1) is the best possible

Another inequality of this type is due to Chebychev (see for example [16, p. 207]). Namely, if f, g are absolutely continuous on $[a, b]$ and

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$f', g' \in L_\infty[a,b]$ and $\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)|$, then

$$|T(f,g;a,b)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2$$

and the constant $\frac{1}{12}$ is the best possible.

Finally, let us recall a result by Lupaş (see for example [16, p. 210]), which states that:

$$|T(f,g;a,b)| \le \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a),$$

provided f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible here.

For other Gruss type inequalities, see the books [16] and [13], and the papers [2]-[10], where further references are given.

The main aim of this paper is to establish some new Grüss type inequalities and apply them for the generalized Taylor's expansion.

2. Integral Inequalities of Grüss Type

The following Grüss type inequality holds.

THEOREM 1. *Let $f, g : [a,b] \to \mathbb{R}_+$ be two integrable functions so that*

$$0 < m \le f(x) \le M < \infty \text{ and } 0 < n \le g(x) \le N < \infty$$

for a.e. $x \in [a,b]$.

Then one has the inequality

$$\text{(2.1)} \quad |T(f,g;a,b)| \le \frac{1}{4}\cdot\frac{(M-m)(N-n)}{\sqrt{mnMN}}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx.$$

The constant $\frac{1}{4}$ is best possible in (2.1) in the sense that it cannot be replaced by a smaller constant

Proof. We have, by the Cauchy-Buniakowski-Schwartz inequality for double integrals, that

(2.2)

$$
\begin{aligned}
&|T(f,g,a,b)| \\
&= \left| \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f(x)-f(y))(g(x)-g(y))\,dxdy \right| \\
&\le \frac{1}{2(b-a)^2} \left[\int_a^b \int_a^b (f(x)-f(y))^2\,dxdy \cdot \int_a^b \int_a^b (g(x)-g(y))^2\,dxdy \right]^{\frac{1}{2}} \\
&= \frac{1}{2(b-a)^2} \left[4 \left[(b-a) \int_a^b f^2(x)\,dx - \left(\int_a^b f(x)\,dx \right)^2 \right] \right. \\
&\times \left. \left[(b-a) \int_a^b g^2(x)\,dx - \left(\int_a^b g(x)\,dx \right)^2 \right] \right]^{\frac{1}{2}} \\
&= \left[\frac{1}{b-a} \int_a^b f^2(x)\,dx - \left(\frac{1}{b-a} \int_a^b f(x)\,dx \right)^2 \right]^{\frac{1}{2}} \\
&\times \left[\frac{1}{b-a} \int_a^b g^2(x)\,dx - \left(\frac{1}{b-a} \int_a^b g(x)\,dx \right)^2 \right]^{\frac{1}{2}}
\end{aligned}
$$

Utilizing the Pólya-Szegö inequality for integrals [15], i.e.,

$$
1 \le \frac{\int_a^b h^2(x)\,dx \int_a^b l^2(x)\,dx}{\left(\int_a^b h(x)\,l(x)\,dx \right)^2} \le \frac{1}{4} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2,
$$

provided $0 < m_1 \le h(x) \le M_1 < \infty$, $0 < m_2 \le l(x) \le M_2 < \infty$ for a.e $x \in [a,b]$, we may state that

$$
\frac{(b-a)\int_a^b f^2(x)\,dx}{\left(\int_a^b f(x)\,dx \right)^2} \le \frac{1}{4} \left(\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}} \right)^2 = \frac{1}{4} \cdot \frac{(M+m)^2}{mM},
$$

giving

$$\frac{(b-a)\int_a^b f^2(x)\,dx-\left(\int_a^b f(x)\,dx\right)^2}{\left(\int_a^b f(x)\,dx\right)^2}\leq\frac{1}{4}\cdot\frac{(M+m)^2}{mM}-1=\frac{(M-m)^2}{4mM},$$

that is,

$$(b-a)\int_a^b f^2(x)\,dx-\left(\int_a^b f(x)\,dx\right)^2\leq\frac{(M-m)^2}{4mM}\left(\int_a^b f(x)\,dx\right)^2. \tag{2.3}$$

In a similar fashion, we obtain

$$(b-a)\int_a^b g^2(x)\,dx-\left(\int_a^b g(x)\,dx\right)^2\leq\frac{(N-n)^2}{4nN}\left(\int_a^b g(x)\,dx\right)^2 \tag{2.4}$$

Using (2.2), (2.3) and (2 4), we deduce the desired inequality (2.1).

Now, assume that the inequality in (2.1) holds with a constant $c>0$, i.e.,

$$\begin{aligned}&|T(f,g;a,b)|\\&\leq c\cdot\frac{(M-m)(N-n)}{\sqrt{mnMN}}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx.\end{aligned} \tag{2.5}$$

We choose the functions $f=g$ with

$$f(x)=\begin{cases}m, & x\in\left[a,\frac{a+b}{2}\right]\\[2mm] M, & x\in\left(\frac{a+b}{2},b\right]\end{cases},\ 0<m<M<\infty.$$

Then

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f^2(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2\\&=\frac{m^2+M^2}{2}-\left(\frac{m+M}{2}\right)^2\\&=\frac{1}{4}(M-m)^2,\end{aligned}$$

and by (2.5) we deduce

$$\frac{1}{4}(M-m)^2 \le c \cdot \frac{(M-m)^2}{mM} \cdot \left(\frac{m+M}{2}\right)^2$$

from where we get

$$mM \le c\,(M-m)^2 \tag{2.6}$$

for any $0 < m < M < \infty$.

If in (2.6) we consider $m = 1-\varepsilon$, $M = 1+\varepsilon$, $\varepsilon \in (0,1)$, then we get $1-\varepsilon^2 \le 4c$ for any $\varepsilon \in (0,1)$, which shows that $c \ge \frac{1}{4}$. □

The second result of Grüss type is embodied in the following theorem.

THEOREM 2. *Assume that f and g are as in Theorem 1. Then one has the inequality:*

$$\begin{aligned}&|T(f,g;a,b)| \\ &\le \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{N}-\sqrt{n}\right)\sqrt{\frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx}.\end{aligned} \tag{2.7}$$

The constant $c = 1$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. We shall use the Shisha-Mond inequality [17] (see also [13, p. 121])

$$\frac{\sum_{i=1}^n z_i^2}{\sum_{i=1}^n z_i y_i} - \frac{\sum_{i=1}^n z_i y_i}{\sum_{i=1}^n y_i^2} \le \left(\sqrt{\frac{M_1}{m_2}} - \sqrt{\frac{m_1}{M_2}}\right)^2, \tag{2.8}$$

provided $0 < m_1 \le z_i \le M_1 < \infty$ and $0 < m_2 \le y_i \le M_2 < \infty$ for all $i \in \{1,\ldots,n\}$.

Applying a standard procedure for Riemann sums instead of z_i, y_i, i.e.,

$$\frac{\frac{b-a}{n}\sum_{i=0}^{n} h^2\left(a+\frac{i}{n}(b-a)\right)}{\frac{b-a}{n}\sum_{i=0}^{n} h\left(a+\frac{i}{n}(b-a)\right) l\left(a+\frac{i}{n}(b-a)\right)} - \frac{\frac{b-a}{n}\sum_{i=0}^{n} h\left(a+\frac{i}{n}(b-a)\right) l\left(a+\frac{i}{n}(b-a)\right)}{\frac{b-a}{n}\sum_{i=0}^{n} l^2\left(a+\frac{i}{n}\cdot(b-a)\right)} \le \left(\sqrt{\frac{M_1}{m_2}}-\sqrt{\frac{m_1}{M_2}}\right)^2,$$

provided h, l are Riemann integrable on $[a,b]$ and $0 < m_1 \le h(x) \le M_1 < \infty$, $0 < m_2 \le l(x) \le M_2 < \infty$, we may deduce, by letting $n \to \infty$, the integral inequality

$$\text{(2.9)} \qquad \frac{\int_a^b h^2(x)\,dx}{\int_a^b h(x)\,l(x)\,dx} - \frac{\int_a^b h(x)\,l(x)\,dx}{\int_a^b l^2(x)\,dx} \le \left(\sqrt{\frac{M_1}{m_2}}-\sqrt{\frac{m_1}{M_2}}\right)^2,$$

which is the integral version of the Shisha-Mond inequality (2.8).

From (2.9) we may easily deduce

$$\text{(2.10)} \qquad \begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b f^2(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2 \\ &\le \left(\sqrt{M}-\sqrt{m}\right)^2 \frac{1}{b-a}\int_a^b f(x)\,dx \end{aligned}$$

and

$$\text{(2.11)} \qquad \begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b g^2(x)\,dx - \left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)^2 \\ &\le \left(\sqrt{N}-\sqrt{n}\right)^2 \frac{1}{b-a}\int_a^b g(x)\,dx. \end{aligned}$$

Finally, by making use of (2.2), (2.10) and (2.11), we obtain the desired inequality (2.7).

To prove the sharpness of the constant, assume that (2.7) holds with a constant $c > 0$, i.e ,

(2.12) $|T(f,g;a,b)| \leq$

$$c\left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{N}-\sqrt{n}\right)\sqrt{\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx}.$$

Now, let us choose $f = g$ and

$$f(x)=\begin{cases} m, & \text{if } x\in\left[a,\frac{a+b}{2}\right], \\ M, & \text{if } x\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

Then from (2.12) we deduce (see also Theorem 1) that

$$\frac{1}{4}(M-m)^2 \leq c\left(\sqrt{M}-\sqrt{m}\right)^2\frac{m+M}{2},\quad 0<m<M<\infty$$

that is,

$$\frac{1}{4}\left(\sqrt{M}-\sqrt{m}\right)^2\left(\sqrt{M}+\sqrt{m}\right)^2 \leq c\left(\sqrt{M}-\sqrt{m}\right)^2\frac{m+M}{2},$$

giving for any $0 < m < M < \infty$ that

$$\left(\sqrt{M}+\sqrt{m}\right)^2 \leq 2c(m+M) \tag{2.13}$$

If in (2 13) we choose $m = 1-\varepsilon$, $M = 1+\varepsilon$, $\varepsilon \in (0,1)$, we get $\left(\sqrt{1-\varepsilon}+\sqrt{1+\varepsilon}\right)^2 \leq 4c$. Letting $\varepsilon \to 0+$, we deduce $c \geq 1$, and the theorem is proved □

By the classical Grüss' inequality, we obviously have

$$|T(f,g,a,b)| \leq \frac{1}{4}(M-m)(N-n). \tag{2.14}$$

It is natural to compare the bounds provided by (2.1), (2.7) and (2.14).

PROPOSITION 1. *The bounds provided by (2.1), (2 7) and (2.14) are not related. This means that one is better than the others depending on the different choices of functions f and g*

Proof. 1. With the assumptions in Theorem 2, consider, for $f = g$, $n = m$, $N = M$, the quantity

$$U := \frac{\left(\int_a^b f(x)\,dx\right)^2}{(b-a)^2 mM} > 0.$$

We want to compare this quantity with 1.

Choose $a = 0$, $b = 3$ and

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,2], \\ k & \text{if } x \in (2,3], \quad k \geq 1. \end{cases}$$

Then $\int_a^b f(x)\,dx = 1 + k$, $m = 1$, $M = k$ and thus

$$U(k) = U = \frac{(k+2)^2}{9k}.$$

We observe that

$$U(k) - 1 = \frac{(k-1)(k-4)}{9k},$$

showing that if $k \in (0,1] \cup [4,\infty)$, $U(k) \geq 1$ while for $k \in (1,4)$, $U(k) < 1$.

In conclusion, for the above choice, if $k \in (1,4)$, the bound provided by (2 1) is better than the bound provided by (2.14), while for $k \in (4,\infty)$ this bound is worse than that provided by the Grüss inequality

2. With the assumptions in Theorem 2, consider, for $f = g$, $n = m$, $N = M$, the quantity

$$I_1 := \frac{1}{4}(M-m)^2, \quad I_2 := \left(\sqrt{M} - \sqrt{m}\right)^2 \frac{1}{b-a}\int_a^b f(x)\,dx.$$

If we assume that $m = 0$, $M = 1$, then $I_1 = \frac{1}{4}$, $I_2 = \frac{1}{b-a}\int_a^b f(x)\,dx$, provided $0 \leq f(x) \leq 1$, $x \in [a,b]$.

Now, if we choose f so that $\frac{1}{b-a}\int_a^b f(x)\,dx < \frac{1}{4}$, then the bound provided by (2.7) is better than the one provided by (2.14). If $\frac{1}{b-a}\int_a^b f(x)\,dx > \frac{1}{4}$, then Grüss' inequality provides a better bound.

3. With the assumptions in Theorem 2, consider, for $f = g$, $n = m$, $N = M$, the quantities

$$J_1 \ : \ = \frac{1}{4}\frac{(M-m)^2}{mM} \cdot \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2,$$
$$J_2 \ : \ = \left(\sqrt{M}-\sqrt{m}\right)^2 \cdot \frac{1}{b-a}\int_a^b f(x)\,dx.$$

If we choose $m = 1$, $M = 4$, we get

$$J_1 = \frac{9}{16}y^2, \quad J_2 = y \text{ where } y := \frac{1}{b-a}\int_a^b f(x)\,dx \in [1,4].$$

Now, observe that

$$J_1 - J_2 = \frac{y(9y-16)}{16},$$

showing that for $y \in \left[1, \frac{16}{9}\right]$ the bound provided by (2.1) is better than the bound provided by (2.7) while for $y \in \left(\frac{16}{9}, 4\right]$, the conclusion is the other way around.

□

3. Some Pre-Grüss Type Inequalities and Applications

If there is no information available about the upper and lower bounds of the function g, but the integrals

$$\int_a^b g^2(x)\,dx \quad \text{and} \quad \int_a^b g(x)\,dx$$

can be exactly computed, then the following pre-Grüss type result may be stated.

THEOREM 3. *Let $f, g : [a,b] \to \mathbb{R}$ be two integrable functions such that there exist $m, M > 0$ with*

$$0 < m \le f(x) \le M < \infty$$

and $g \in L_2[a,b]$. *Then one has the inequality*

$$|T(f,g;a,b)| \le \frac{1}{2} \cdot \frac{(M-m)}{\sqrt{mM}} \cdot \frac{1}{b-a}\int_a^b f(x)\,dx \times \left[\frac{1}{b-a}\int_a^b g^2(x)\,dx - \left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)^2\right]^{\frac{1}{2}}.$$

The constant $\frac{1}{2}$ *is best possible.*

The proof is similar to the one incorporated in Theorem 1 and we omit the details.

Similarly, we may state the corresponding pre-Grüss inequality that may be deduced from Shisha-Mond's result

THEOREM 4. *With the assumption of Theorem 3, we have*

$$|T(f,g;a,b)| \le \left(\sqrt{M} - \sqrt{m}\right)\sqrt{\frac{1}{b-a}\int_a^b f(x)\,dx} \times \left[\frac{1}{b-a}\int_a^b g^2(x)\,dx - \left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)^2\right]^{\frac{1}{2}}$$

The constant $c = 1$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

Following Matić et al [12], we may say that the sequence of polynomials $\{P_n(x)\}_{n\in\mathbb{N}}$ is a *harmonic sequence* if

$$P_n'(x) = P_{n-1}(x) \quad \text{for } n \ge 1 \text{ and } P_0(x) = 1.$$

In the above mentioned paper [12], the authors considered the following particular instances of harmonic polynomials:

$$P_n(t) = \frac{(t-x)^n}{n!}, \quad n \ge 0;$$

$$P_n(t) = \frac{1}{n!}\left(t - \frac{a+x}{2}\right)^n, \quad n \ge 0,$$

$$P_n(t) = \frac{(x-a)^n}{n!} B_n\left(\frac{t-a}{x-a}\right), \quad P_0(t) = 1, \ n \ge 2;$$

where $B_n(t)$ are the well known Bernoulli polynomials, and

$$P_n(t) = \frac{(x-a)^n}{n!} E_n\left(\frac{t-a}{x-a}\right), \quad P_0(t) = 1, \ n \geq 1,$$

where $E_n(t)$ are the Euler polynomials.

The following perturbed version of the generalized Taylor's formula was obtained in [12].

THEOREM 5. *Let $\{P_n(x)\}_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials. Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. Suppose that $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous. Then for any $x \in I$ we have the generalized Taylor's formula:*

(3.1)

$$f(x) = \tilde{T}_n(f;a,x) + (-1)^n [P_{n+1}(x) - P_{n+1}(a)] \left[f^{(n)};a,x\right] + \tilde{G}_n(f;a,x),$$

where

$$\tilde{T}_n(f;a,x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a)\right]$$

and

$$\left[f^{(n)};a,x\right] = \frac{f^{(k)}(x) - f^{(k)}(a)}{x-a}.$$

For $x \geq a$, the remainder $\tilde{G}(f;a,x)$ satisfies the estimation

$$\left|\tilde{G}_n(f,a,x)\right| \leq \frac{x-a}{2} (\Gamma(x) - \gamma(x)) \left[T(P_n,P_n)\right]^{\frac{1}{2}}, \tag{3.2}$$

where

$$T(P_n,P_n;a,x) := \frac{1}{x-a}\int_a^x P_n^2(t)\,dt - \left(\frac{1}{x-a}\int_a^x P_n(t)\,dt\right)^2$$

and

$$\gamma(x) = \inf_{t\in[a,x]} f^{(n+1)}(t), \quad \Gamma(x) = \sup_{t\in[a,x]} f^{(n+1)}(t)$$

Using Theorems 3 and 4, we may point out the following bounds for the remainder $\tilde{G}(f;a,x)$ as well.

THEOREM 6. *Assume that $\{P_n(x)\}_{n\in\mathbb{N}}$ and f are as in Theorem 5. Moreover, if $\gamma(x) > 0$, then we have the representation (3.1) and the remainder $\tilde{G}(f;a,x)$ satisfies the bounds*

$$\left|\tilde{G}_n(f;a,x)\right| \leq \begin{cases} \frac{1}{2} \cdot \frac{\Gamma(x)-\gamma(x)}{\sqrt{\gamma(x)\Gamma(x)}}\left[f^{(n)};a,x\right]\left[T(P_n,P_n;a,x)\right]^{\frac{1}{2}}(x-a) \\ \cdot \\ \left(\sqrt{\Gamma(x)}-\sqrt{\gamma(x)}\right)\sqrt{\left[f^{(n)};a,x\right]}\left[T(P_n,P_n;a,x)\right]^{\frac{1}{2}}(x-a) \end{cases} \tag{3.3}$$

for any $x \geq a$.

The proof is similar to the one in Theorem 3, [12] and we omit the details.

REMARK 1. *If we choose the above particular instances of harmonic polynomials, then we may produce a number of particular Taylor-like formulae whose remainder will obey similar bounds to those incorporated in (3.3). We omit the details.*

REMARK 2. *As shown by Proposition 1, the bounds provided by (3 2) and (3.3) cannot be compared in general.*

REFERENCES

[1] P. Cerone, S. S. Dragomir, J. Roumeliotis and J. Šunde, *A new generalisation of the trapezoid formula for n−time differentiable mappings and applications,* Demonstratio Mathematica, **33**(4) (2000), 719-736.

[2] S S Dragomir, *Grüss inequality in inner product spaces,* The Australian Math Soc Gazette, **26** (1999), No. 2, 66-70

[3] S S. Dragomir, *A generalization of Grüss' inequality in inner product spaces and applications,* J Math Anal. Appl., **237** (1999), 74-82.

[4] S. S. Dragomir, *A Gruss type integral inequality for mappings of r-Hölder's type and applications for trapezoid formula,* Tamkang J of Math., **31**(1) (2000), 43-47

[5] S. S. Dragomir, *Some integral inequalities of Grüss type,* Indian J. of Pure and Appl. Math , **31**(4) (2000), 397-415.

[6] S. S. Dragomir, *Better bounds in some Ostrowski-Grüss type inequalities,* RGMIA Res. Rep. Coll., **3** (1), (2000), Article 3

[7] S S. Dragomir and G L Booth, *On a Grüss-Lupaş type inequality and its applications for the estimation of p-moments of guessing mappings,* Mathematical Communications, **5** (2000), 117-126

[8] S. S Dragomir and I. Fedotov, *An inequality of Grüss' type for Riemann-Stieltjes integral and applications for special means,* Tamkang J. of Math., **29**(4) (1998), 286-292.

[9] S. S. Dragomir, J. Šunde and C. BuUşe, *Some new inequalities for Jeffreys divergence measure in information theory,* RGMIA Res. Rep. Coll., **3** (2) (2000), Article 5.

[10] A. M Fink, *A treatise on Gruss' inequality, Analytic and Geometric Inequalities and Applications, 93-113,* Math. Appl , **478**, Kluwer Academic Publishers, Dordrecht, 1999.

[11] G Grüss, *Uber das Maximum des absoluten Betrages von* $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, Math. Z. , **39**(1935), 215-226.

[12] M Matić, J Pečarić and N Ujević, *On new estimations of the remainder in generalised Taylor's formula,* Math Ineq Appl., **2**(2) (1999), 343-361.

[13] D. S Mitrinović, J E Pečarić and A. M Fink, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht, 1993.

[14] D S Mitrinović, J Pečarić and A. M Fink, *Inequalities for Functions and their Integrals and Derivatives,* Kluwer Academic Publishers, 1994

[15] G. Pólya and G Szego, *Aufgaben und Lehrsotse ans der Analysis,* Vol. 1, Berlin 1925, pp 57 and 213-214

[16] J Pečarić, F Prochan and Y Tong, *Convex Functions, Partial Orderings and Statistical Applications,* Academic Press, San Diego, 1992

[17] O Shisha and B Mond, *Bounds on difference of means, Inequalities,* Academic Press, New York, 1967, pp. 293-308

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