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ON THE ČEBYŠEV'S INEQUALITY FOR UNWEIGHTED MEANS AND APPLICATIONS

S. S. DRAGOMIR

ABSTRACT. Some new sufficient conditions for the unweighted Čebyšev's inequality for real sequences to hold and related results are given. Applications for the moments of guessing mappings are also provided

1. Introduction

Let $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$, $\bar{\mathbf{y}} = (y_1, \ldots, y_n)$ be two $n-$tuples of real numbers. If $\bar{\mathbf{x}}, \bar{\mathbf{y}}$ are synchronous (asynchronous), this means that

$$(x_i - x_j)(y_i - y_j) \geq (\leq) 0 \text{ for each } i, j \in \{1, \ldots, n\}, \tag{1.1}$$

then the following well known Čebyšev's inequality

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i \geq (\leq) \frac{1}{n}\sum_{i=1}^{n} x_i \cdot \frac{1}{n}\sum_{i=1}^{n} y_i, \tag{1.2}$$

holds.

In [16], the following refinement of Čebyšev's inequality has been obtained

$$\begin{aligned} T_n(\bar{\mathbf{x}}, \bar{\mathbf{y}}) &:= \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \cdot \frac{1}{n}\sum_{i=1}^{n} y_i \\ &\geq \max\{|T_n(|\bar{\mathbf{x}}|, \bar{\mathbf{y}})|, |T_n(\bar{\mathbf{x}}, |\bar{\mathbf{y}}|)|, |T_n(|\bar{\mathbf{x}}|, |\bar{\mathbf{y}}|)|\} \\ &\geq 0, \end{aligned} \tag{1.3}$$

provided $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are synchronous

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In this paper, some new Čebyšev's type inequalities for unweighted means are obtained. Similar results for the weighted case are considered in [12]

2. Čebyšev's Type Inequalities

The following identities hold.

LEMMA 1. *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ *and* $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$, *be two sequences of real numbers Define* $A_k := \sum_{i=1}^{k} a_i$, $\bar{A}_k := A_n - A_k$, $k = 1, \ldots, n-1$. *Then*

$$T_n(\bar{\mathbf{x}}, \bar{\mathbf{a}}) = \frac{1}{n^2} \sum_{i=1}^{n-1} \det \begin{pmatrix} i & n \\ A_i & A_n \end{pmatrix} \Delta x_i = \frac{1}{n} \sum_{i=1}^{n-1} i \left(\frac{A_n}{n} - \frac{A_i}{i} \right) \Delta x_i = \frac{1}{n^2} \sum_{i=1}^{n-1} i(n-i) \left(\frac{\bar{A}_i}{n-i} - \frac{A_i}{i} \right) \Delta x_i, \tag{2.1}$$

where $\Delta x_i := x_{i+1} - x_i$ $(i = 0, \ldots, n-1)$ *is the forward difference.*

Proof. We use the following well known summation by parts formula

$$\sum_{\ell=p}^{q-1} b_\ell \Delta v_\ell = b_\ell v_\ell \Big|_p^q - \sum_{\ell=p}^{q-1} v_{\ell+1} \Delta b_\ell, \tag{2.2}$$

where $b_\ell, v_\ell \in \mathbb{R}$, $\ell = p, \dots, q$ $(q > p)$ If we choose in (2.2), $p = 1$, $q = n$, $b_i := iA_n - nA_i$, and $v_i = x_i$ $(i = 1, \dots, n)$, then we get

$$
\begin{aligned}
&\sum_{i=1}^{n-1} (iA_n - nA_i)\,\Delta x_i \\
&= (iA_n - nA_i)\,x_i\Big|_1^n - \sum_{i=1}^{n-1} \Delta\,(iA_n - nA_i)\,x_{i+1} \\
&= -\,(A_n - nA_1)\,x_1 - \sum_{i=1}^{n-1} \left[(i+1)\,A_n - nA_{i+1} - iA_n + nA_i\right] x_{i+1} \\
&= -A_n x_1 + nA_1 x_1 - \sum_{i=1}^{n-1} (A_n - na_{i+1})\,x_{i+1} \\
&= -A_n x_1 + nA_1 x_1 - A_n \sum_{i=1}^{n-1} x_{i+1} + n \sum_{i=1}^{n-1} a_{i+1} x_{i+1} \\
&= -A_n \sum_{i=1}^{n} x_i + n \sum_{i=1}^{n} a_i x_i
\end{aligned}
$$

and the first identity in (2 1) is proved. The others are obvious. □

The following theorem holds

THEOREM 1. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathbf{x}} = (x_1, \dots, x_n)$, *be two sequences so that either*

(i) $\bar{\mathbf{x}}$ *is increasing and*

$$\frac{1}{n} A_n - \frac{1}{i} A_i \geq 0$$

for each $i \in \{1, \dots, n-1\}$;
or

(ii) $\bar{\mathbf{x}}$ *is decreasing and*

$$\frac{1}{n} A_n - \frac{1}{i} A_i \leq 0$$

for each $i \in \{1, \dots, n-1\}$

Then one has the inequality

$$T_n(\bar{\mathbf{x}},\bar{\mathbf{a}}) \geq \max\{|A_n(\bar{\mathbf{x}},\bar{\mathbf{a}})|, |A_n(|\bar{\mathbf{x}}|,\bar{\mathbf{a}})|, |T_n(|\bar{\mathbf{x}}|,\bar{\mathbf{a}})|\} \geq 0, \tag{2.3}$$

where

$$A_n(\bar{\mathbf{x}},\bar{\mathbf{a}}) := \frac{1}{n}\sum_{i=1}^{n-1}|A_i|\,\Delta x_i - \frac{|A_n|}{n}\cdot\frac{1}{n}\sum_{i=1}^{n-1} i\Delta x_i.$$

Proof. If either (i) or (ii) hold, then

$$\left(\frac{A_n}{n}-\frac{A_i}{i}\right)(x_{i+1}-x_i) = \left|\left(\frac{A_n}{n}-\frac{A_i}{i}\right)(x_{i+1}-x_i)\right| \geq \begin{cases} \left|\left(\frac{|A_n|}{n}-\frac{|A_i|}{i}\right)(x_{i+1}-x_i)\right| \geq 0 \\ \left|\left(\frac{|A_n|}{n}-\frac{|A_i|}{i}\right)(|x_{i+1}|-|x_i|)\right| \geq 0 \\ \left|\left(\frac{A_n}{n}-\frac{A_i}{i}\right)(|x_{i+1}|-|x_i|)\right| \geq 0 \end{cases}$$

for each $i \in \{1, \ldots, n-1\}$.

Multiplying by i, summing over i, and using the generalized triangle inequality, we get

$$T_n(\bar{\mathbf{x}},\bar{\mathbf{a}}) = \frac{1}{n}\sum_{i=1}^{n-1} i\left|\left(\frac{A_n}{n}-\frac{A_i}{i}\right)(x_{i+1}-x_i)\right| \geq \frac{1}{n}\times\begin{cases} \left|\sum_{i=1}^{n-1} i\left(\frac{|A_n|}{n}-\frac{|A_i|}{i}\right)(x_{i+1}-x_i)\right| \\ \left|\sum_{i=1}^{n-1} i\left(\frac{|A_n|}{n}-\frac{|A_i|}{i}\right)(|x_{i+1}|-|x_i|)\right| \\ \left|\sum_{i=1}^{n-1} i\left(\frac{A_n}{n}-\frac{A_i}{i}\right)(|x_{i+1}|-|x_i|)\right| \end{cases}$$

from where we easily deduce the desired inequality (2.3). □

REMARK 1. *We observe that if* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *is monotonic increasing in mean, i.e.,*

$$\frac{1}{i} A_i \leq \frac{1}{i+1} A_{i+1} \quad \text{for } i = 1, \dots, n-1,$$

then obviously

$$\frac{1}{i} A_i \leq \frac{1}{n} A_n$$

for each $i \in \{1, \dots, n-1\}$. *The converse is not true*

We also note that if $\bar{\mathbf{a}}$ *is monotonic nondecreasing, then it is increasing in mean and thus*

$$\frac{1}{i} A_i \leq \frac{1}{n} A_n$$

for each $i \in \{1, \dots, n-1\}$.

REMARK 2. *We observe, since,*

$$\frac{A_n}{n} - \frac{A_i}{i} = \frac{n-i}{n} \left(\frac{\bar{A}_i}{n-i} - \frac{A_i}{i} \right)$$

for each $i \in \{1, \dots, n-1\}$, *that*

$$\frac{A_n}{n} \geq \frac{A_i}{i} \quad \text{for each } i \in \{1, \dots, n-1\}$$

if and only if

$$\frac{\bar{A}_i}{n-i} \geq \frac{A_i}{i} \quad \text{for each } i \in \{1, \dots, n-1\}.$$

Here $\bar{A}_i := A_n - A_i$ *for* $i \in \{1, \dots, n-1\}$.

Using the second identity in (2.1), we may prove the following refinement of Čebyšev's inequality as well.

THEOREM 2. *Assume that* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{x}}$ *satisfy the hypothesis (i) or (ii) in Theorem 1 Then one has the inequality:*

$$T_n(\bar{\mathbf{x}}, \bar{\mathbf{a}}) \geq \max\{|D_n(\bar{\mathbf{x}}, \bar{\mathbf{a}})|, |D_n(|\bar{\mathbf{x}}|, \bar{\mathbf{a}})|\} \geq 0, \tag{2.4}$$

where

$$D_n(\bar{\mathbf{x}}, \bar{\mathbf{a}}) := \frac{1}{n^2} \sum_{i=1}^{n-1} (n-i) |A_i| \Delta x_i - \frac{1}{n^2} \sum_{i=1}^{n-1} i \left|\bar{A}_i\right| \Delta x_i.$$

Proof. If either (i) or (ii) holds, then

$$\left(\frac{\bar{A}_i}{n-i}-\frac{A_i}{i}\right)(x_{i+1}-x_i)=\left|\left(\frac{\bar{A}_i}{n-i}-\frac{A_i}{i}\right)(x_{i+1}-x_i)\right| \geq \begin{cases} \left|\left(\frac{|\bar{A}_i|}{n-i}-\frac{|A_i|}{i}\right)(x_{i+1}-x_i)\right| \\ \left|\left(\frac{|\bar{A}_i|}{n-i}-\frac{|A_i|}{i}\right)(|x_{i+1}|-|x_i|)\right| \end{cases}$$

for each $i \in \{1,\ldots,n-1\}$.

Multiplying by $i(n-i)$, summing over i and using the generalized triangle inequality, we have

$$T_n(\bar{\mathbf{x}},\bar{\mathbf{a}})=\frac{1}{n^2}\sum_{i=1}^{n-1} i(n-i)\left|\left(\frac{\bar{A}_i}{n-i}-\frac{A_i}{i}\right)\Delta x_i\right| \geq \frac{1}{n^2}\begin{cases} \left|\sum_{i=1}^{n-1} i(n-i)\left(\frac{|\bar{A}_i|}{n-i}-\frac{|A_i|}{i}\right)(x_{i+1}-x_i)\right| \\ \left|\sum_{i=1}^{n-1} i(n-i)\left(\frac{|\bar{A}_i|}{n-i}-\frac{|A_i|}{i}\right)(|x_{i+1}|-|x_i|)\right| \end{cases}$$

from where we easily deduce the desired inequality (2.4). □

3. Other Related Results

The following result holds.

THEOREM 3. *Let* $\bar{\mathbf{a}}=(a_1,\ldots,a_n)$ *and* $\bar{\mathbf{x}}=(x_1,\ldots,x_n)$, *be two sequences of real numbers. If* $\bar{\mathbf{a}}$ *is monotonic decreasing (increasing) in mean, i.e.,*

$$\frac{1}{i}A_i \leq (\geq) \frac{1}{i+1}A_{i+1} \quad \textit{for each } i \in \{1,\ldots,n-1\},$$

and $\bar{\mathbf{x}}$ is convex (concave), i.e.,

$$\frac{x_{i+2}+x_i}{2} \geq (\leq)\, x_{i+1} \quad \textit{for each} \quad i \in \{1,\ldots,n-2\},$$

then we have the inequality

$$(3.1)\quad T_n(\bar{\mathbf{x}},\bar{\mathbf{a}}) \geq \left[\frac{A_n}{n} - \frac{2}{n(n-1)}\sum_{i=1}^{n-1}(n-i)\,a_i\right]\left(x_n - \frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Proof. Define the sequences

$$p_i = i,\ z_i := \frac{A_n}{n} - \frac{A_i}{i},$$
$$y_i := \Delta x_i = x_{i+1} - x_i,\ i = 1,\ldots,n-1.$$

Then $p_i > 0$,

$$z_{i+1} - z_i = \frac{A_{i+1}}{i+1} - \frac{A_i}{i} \geq 0 \quad \text{for each} \quad i \in \{1,\ldots,n-2\},$$

and

$y_{i+1} - y_i = \Delta x_{i+1} - \Delta x_i = x_{i+2} + x_i - 2x_{i+1} \geq 0$ for each $i \in \{1,\ldots,$ $n-2$. Applying the weighted Čebyšev's inequality for monotonic sequences, we have

$$P_{n-1}\sum_{i=1}^{n-1} p_i z_i y_i \geq \sum_{i=1}^{n-1} p_i z_i \cdot \sum_{i=1}^{n-1} p_i y_i$$

giving

$$(3.2)\quad \frac{n(n-1)}{2}\sum_{i=1}^{n-1} i\left(\frac{A_n}{n} - \frac{A_i}{i}\right)\Delta x_i \geq \sum_{i=1}^{n-1} i\left(\frac{A_n}{n} - \frac{A_i}{i}\right)\sum_{i=1}^{n-1} i\Delta x_i.$$

However, by (2.1) we have

$$\sum_{i=1}^{n-1} i\left(\frac{A_n}{n} - \frac{A_i}{i}\right)\Delta x_i = n\left[\frac{1}{n}\sum_{i=1}^{n} a_i x_i - \frac{1}{n}\sum_{i=1}^{n} a_i \cdot \frac{1}{n}\sum_{i=1}^{n} x_i\right],$$

$$\sum_{i=1}^{n-1} i\left(\frac{A_n}{n} - \frac{A_i}{i}\right) = \frac{(n-1)\,n}{2n}A_n - \sum_{i=1}^{n-1} A_i = \frac{n-1}{2}A_n - \sum_{i=1}^{n-1}(n-1)\,a_i,$$

$$\begin{aligned}\sum_{i=1}^{n-1} i\Delta x_i &= \sum_{i=1}^{n-1} i\left(x_{i+1}-x_i\right)\\ &= x_2+2x_3+\cdots+(n-2)\,x_{n-1}+(n-1)\,x_n\\ &\quad -x_1-2x_2-\cdots-(n-1)\,x_{n-1}\\ &= nx_n-(x_1+\cdots+x_n)=n\left(x_n-\frac{1}{n}X_n\right).\end{aligned}$$

Using (3.2), we get,

$$\begin{aligned}& n\left[\frac{1}{n}\sum_{i=1}^{n} a_i x_i-\frac{1}{n}\sum_{i=1}^{n} a_i\cdot\frac{1}{n}\sum_{i=1}^{n} x_i\right]\\ \geq\ & \frac{2}{(n-1)\,n}\left[\frac{n-1}{2}A_n-\sum_{i=1}^{n-1}(n-i)\,a_i\right]\left[n\left(x_n-\frac{1}{n}X_n\right)\right]\\ =\ & \left[A_n-\frac{2}{n-1}\sum_{i=1}^{n-1}(n-i)\,a_i\right]\left[x_n-\frac{1}{n}\sum_{i=1}^{n}x_i\right]\end{aligned}$$

and the inequality (3 1) is obtained. □

REMARK 3. *If $\bar{\mathbf{a}}$ is monotonic decreasing (increasing) in mean but $\bar{\mathbf{x}}$ is concave (convex), the reverse inequality in (3.1) holds.*

The following result also holds

THEOREM 4. *Let $\bar{\mathbf{a}}=(a_1,\ ..,a_n)$ and $\bar{\mathbf{x}}=(x_1,\ ..,x_n)$, be two sequences of real numbers If $\bar{\mathbf{a}}$ is monotonic decreasing (increasing) in mean and $\bar{\mathbf{x}}$ is monotonic increasing with $x_n>x_1$, then one has the inequality*

$$T_n(\bar{\mathbf{x}},\bar{\mathbf{a}})\geq(\leq)\left(x_n-\frac{1}{n}\sum_{i=1}^{n}x_i\right)\left(\frac{A_n}{n}-\frac{1}{x_n-x_1}\sum_{i=1}^{n-1}\frac{A_i}{i}\Delta x\right). \tag{3.3}$$

Proof Define the sequence

$$\begin{aligned}&p_i=\Delta x_i,\, i\in\{1,\ldots,n-1\},\\ &z_i:=\frac{A_n}{n}-\frac{A_i}{i}\text{ and}\\ &y_i:=i,\, i\in\{1,\ldots,n-1\}.\end{aligned}$$

Then

$$p_i \geq 0, \quad i \in \{1, \ldots, n-1\} \quad \text{with} \quad \sum_{i=1}^{n-1} p_i > 0,$$

$$z_{i+1} - z_i = \frac{A_{i+1}}{i+1} - \frac{A_i}{i} \underset{(\leq)}{\geq} 0 \text{ for each } i \in \{1, \ldots, n-2\},$$

and y_i is increasing.

Applying the weighted Čebyšev's inequality for monotonic sequences, we have

(3.4)

$$\sum_{i=1}^{n-1} \Delta x_i \sum_{i=1}^{n-1} i \left(\frac{A_n}{n} - \frac{A_i}{i} \right) \Delta x_i \geq (\leq) \sum_{i=1}^{n-1} i \Delta x_i \sum_{i=1}^{n-1} \left(\frac{A_n}{n} - \frac{A_i}{i} \right) \Delta x_i.$$

However,

$$\sum_{i=1}^{n-1} \Delta x_i = x_n - x_1 > 0,$$

$$\sum_{i=1}^{n-1} i \left(\frac{A_n}{n} - \frac{A_i}{i} \right) \Delta x_i = n \left(\frac{1}{n} \sum_{i=1}^{n} a_i x_i - \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} x_i \right),$$

$$\sum_{i=1}^{n-1} i \Delta x_i = n \left(x_n - \frac{1}{n} X_n \right)$$

and

$$\sum_{i=1}^{n-1} \left(\frac{A_n}{n} - \frac{A_i}{i} \right) \Delta x_i = \frac{A_n}{n} (x_n - x_1) - \sum_{i=1}^{n-1} \frac{A_i}{i} \Delta x_i$$

Using (3.4), we have

$$(x_n - x_1)\, n \left(\frac{1}{n} \sum_{i=1}^{n} a_i x_i - \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} x_i \right)$$
$$\geq (\leq) \left(n x_n - \frac{1}{n} \sum_{i=1}^{n} x_i \right) \left[\frac{A_n}{n} (x_n - x_1) - \sum_{i=1}^{n-1} \frac{A_i}{i} \Delta x_i \right]$$

from where we get (3.3). □

REMARK 4. *A similar result may be stated if one uses the weighted Čebyšev's inequality for:*

$p_i := \frac{A_n}{n} - \frac{A_i}{i} \geq 0$ *(assumed for each $i \in \{1, \dots, n-1\}$)*

$y_i := i,$ *(monotonic increasing)*

$z_i := \Delta x_i$ *(assumed monotonically increasing (decreasing))*

(or equivalently, $\bar{\mathbf{x}}$ is convex (concave))

4. Some Applications for Moments of Guessing Mappings

In 1994, J L Massey [18] considered the problem of guessing the value taken on by a discrete random variable X in one trial of a random experiment by asking questions of the form "Did X take on its i^{th} possible value?" until the answer is in the affirmative.

This problem arises for instance when a cryptologist must try different possible secret keys one at a time *after* minimizing the possibilities by some cryptoanalysis

Consider a random variable X with finite range $X = \{x_1, \dots, x_n\}$ and distribution $P_X(x_k) = p_k$ for $k = 1, 2, \dots, n$.

A one-to-one function $G \ \chi \to \{1, \dots, n\}$ is a guessing function for X Thus

$$E(G^m) .= \sum_{k=1}^{n} k^m p_k$$

is the m^{th} moment of this function, provided we renumber the x_i such that x_k is always the k^{th} guess.

In [18], Massey observed that, $E(G)$, the average number of guesses, *is minimized by a guessing strategy that guesses the possible values of X in decreasing order of probability.*

In the same paper [18], Massey proved that for an optimal guessing strategy

$$E(G) \geq \frac{1}{4} 2^{H(X)} + 1 \text{ provided } H(X) \geq 2 \text{ bits},$$

where $H(X)$ is the Shannon entropy

$$H(X) = -\sum_{i=1}^{n} p_i \log_2(p_i)$$

He also showed that $E(G)$ may be arbitrarily large when $H(X)$ is an arbitrarily small positive number so that there is no interesting upper bound on $E(G)$ in terms of $H(X)$.

In 1996, Arikan [2] proved that any guessing algorithm for X obeys the lower bound

$$E(G^\rho) \geq \frac{\left[\sum_{k=1}^{n} p_k^{\frac{1}{1+\rho}}\right]^{1+\rho}}{[1+\ln n]^\rho}, \quad \rho \geq 0$$

where an optimal guessing algorithm for X satisfies

$$E(G^\rho) \leq \left[\sum_{k=1}^{n} p_k^{\frac{1}{1+\rho}}\right]^{1+\rho}, \quad \rho \geq 0.$$

In 1997, Boztaş [4] proved that for $m \geq 1$, and integer

$$E(G^m) \leq \frac{1}{m+1}\left[\sum_{k=1}^{n} p_k^{\frac{1}{1+m}}\right]^{1+m}$$
$$+\frac{1}{m+1}\left\{(m+1\,2)\,E\left(G^{m-1}\right) - (m+1\,3)\,E\left(G^{m-2}\right) + \cdots + (-1)^{m+1}\right\}$$

provided the guessing strategy satisfies the relation:

$$p_{k+1}^{\frac{1}{1+m}} \leq \frac{1}{k}\left(p_1^{\frac{1}{1+m}} + \cdots + p_k^{\frac{1}{1+m}}\right), \quad k = 1, \ldots, n-1.$$

In 1997, Dragomir and Boztaş [14] obtained, for any guessing sequence, the following bounds for the expectation:

$$\left|E(G) - \frac{n+1}{2}\right| \leq \frac{(n-1)(n+1)}{6} \max_{1\leq i<j\leq n} |p_i - p_j|,$$

$$\left|E(G) - \frac{n+1}{2}\right| \leq \sqrt{\frac{(n-1)(n+1)\left(n\|p\|_2^2 - 1\right)}{12}},$$

where $\|p\|_2^2 = \sum_{i=1}^n p_i^2$ and

$$\left| E(G) - \frac{n+1}{2} \right| \le \left[\frac{n+1}{2} \right] \left(n - \left[\frac{n+1}{2} \right] \right) \max_{1 \le k \le n} \left| p_k - \frac{1}{n} \right|,$$

with $[x]$ representing the integer part of x.

For other results on $E(G^p)$, $p > 0$ see also [15]. We highlight only the following result which uses the Grüss inequality, giving for $p, q > 0$ that

$$\left| E\left(G^{p+q}\right) - E(G^p) E(G^q) \right| \le \frac{1}{4} (n^q - 1)(n^p - 1). \tag{4.1}$$

The result (4 1) may be complemented in the following way (see for example [11]).

THEOREM 5. *With the above assumptions, we have the inequality*

$$\left| E\left(G^{p+q}\right) - \frac{1+n^q}{2} E(G^p) - \frac{1+n^p}{2} E(G^q) + \frac{1+n^q}{2} \cdot \frac{1+n^p}{2} \right|$$
$$\le \frac{1}{4} (n^q - 1)(n^p - 1),$$

for any $p, q > 0$

Applications for different particular instances of $p, q > 0$ may be provided, but we omit the details.

To obtain other inequalities for the moments of guessing mappings, we use the following Čebyšev type inequality [6]

$$D_n(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \ge (\le) 0 \tag{4 2}$$

provided

$$(x_i - x_M)(y_i - y_M) \ge (\le) 0 \text{ for each } i \in \{1, \ldots, n\}$$

with a subscript M denoting the arithmetic mean

The following result holds [6].

THEOREM 6. *Assume that* $S_n(p), p > 0$ *denotes the sum of* p^{th}-*power of the first* n *natural numbers, that is*

$$S_n(p) := \sum_{k=1}^n i^p.$$

If

$$p_i \begin{cases} \leq (\geq) \frac{1}{n}, & for\ i \leq \left\lfloor \frac{S_n(p)}{n} \right\rfloor^{1/p} \\ \geq (\leq) \frac{1}{n}, & otherwise \end{cases}$$

where $\lfloor x \rfloor$ *represents the integer part of* x, *then we have the inequality*

$$E\left(G^p\right) \geq (\leq) \frac{1}{n} S_n(p).$$

The proof follows by the inequality (4.2) on choosing $x_i = p_i$ and $y_i = i^p$, but we omit the details.

For particular values of p, one may produce some interesting particular inequalities.

If $p = 1$, then we have the inequality

$$E(G) \geq (\leq) \frac{n+1}{2}$$

provided

$$p_i \begin{cases} \leq (\geq) \frac{1}{n}, & i \leq \left\lfloor \frac{n+1}{2} \right\rfloor \\ \geq (\leq) \frac{1}{n}, & \text{otherwise} \end{cases}$$

For $p = 2$, then

$$E(G) \geq (\leq) \frac{1}{6} (n+1)(2n+1)$$

provided

$$p_i \begin{cases} \leq (\geq) \frac{1}{n}, & i \leq \left\lfloor \frac{1}{6} (n+1)(2n+1) \right\rfloor^{1/2} \\ \geq (\leq) \frac{1}{n}, & \text{otherwise} \end{cases}$$

Using Theorem 1, (i), we are able to point out the following result that complements Theorem 6 above

THEOREM 7. *With the above notations for* $S_n(p)$ *and* $E\left(G^p\right)$, $p > 0$, *we have the inequality*

$$E\left(G^p\right) \geq \frac{1}{n} S_n(p) \tag{4.3}$$

provided the probability distribution p_i $(i = 1, \ldots, n)$ *satisfy*

$$\frac{p_1 + \cdots + p_i}{i} \leq \frac{1}{n}, \quad i = 1, \ldots, n-1. \tag{4.4}$$

If the sign of the inequality in (4.4) is reversed, then (4.3) holds with "$\leq$".

REFERENCES

[1] D. Andrica and C Badea, *Grüss' inequality for positive linear functionals,* Periodica Math Hungarica, **19** (1988), 155-167

[2] E Arikan, *An inequality on guessing and its application to sequential decoding,* IEEE Tran. Inf Th., **42** (1988), 99-105.

[3] M Biernacki, H Pidek, and C. Ryll-Nardzewski, *Sur une inégalité entre des intégrales definies,* Ann. Univ. Mariae Curie-Skolodowska, **A4** (1950), 1-4.

[4] S Boztaş, *Comments on "An Inequality of Guessing and Its Applications to Sequential Decoding",* IEEE Tran Inf Th , **43** (1997), 2062-2063

[5] P. Cerone and S S Dragomir, *A refinement of Grüss' inequality and applications,* RGMIA Res Rep Coll , **5**(2002), No 2, Article 15 [ONLINE http://rgmia.vu.edu.au/v5n2.html]

[6] P. Cerone and S S. Dragomir, *New inequalities for the Čebyšev functional involving two n−tuples of real numbers and applications,* RGMIA Res. Rep. Coll., **5** (2002), Article 4 [ONLINE. http://rgmia.vu.edu.au/v5n3.html]

[7] S S Dragomir, *A generalization of Grüss's inequality in inner product spaces and applications,* J Math. Anal. Appl **237** (1999), 74–82

[8] S. S. Dragomir, *Integral Grüss inequality for mappings with values in Hilbert spaces and applications,* J Korean Math. Soc. **38** (2001), 1261–1273.

[9] S. S. Dragomir, *Another Gruss type inequality for sequences of vectors in normed linear spaces and applications,* J. Comp Analysis & Appl., **4** (2002), 157-172

[10] S. S Dragomir, *A Grüss type inequality for sequences of vectors in normed linear spaces,* RGMIA Res Rep. Coll., **5** (2002), Article 9 [ONLINE http://rgmia.vu.edu.au/v5n2.html]

[11] S S. Dragomir, *A companion of the Grüss inequality and applications,* RGMIA Res. Rep Coll , **5** (2002), Supplement, Article 13 [ON LINE: http://rgmia.vu.edu.au/v5(E).html]

[12] S S Dragomir, *On the Cebyshev inequality for weighted means and applications,* in preparation(2002)

[13] S S. Dragomir and G L Booth, *On a Grüss-Lupaş type inequality and its application for the estimation of p−moments of guessing mappings,* Math. Comm., **5** (2000), 117-126

[14] S. S Dragomir and S Boztaş, *Some estimates of the average number of guesses to determine a random variable,* Proc 1997 IEEE Int Symp. on Inf Th , (Ulm, Germany, 1997), p 159.

[15] S. S. Dragomir and S. Boztaş, *Estimation of arithmetic means and their applications in guessing theory,* Math. Comput. Modelling, **28** (1998), 31-43.

[16] S S Dragomir and J Pečarıc, *Refinements of some ınequalitıes for ısotonıc functıonals*, Anal Num. Theor. Approx , **18** (1989), 61-65
[17] A M. Fink, *A treatıse on Gruss' inequalıty* Analytıc and Geometric Inequalitıes and Applicatıons, 93–113, Math Appl , **478** (1999), Kluwer Acad Publ., Dordrecht,
[18] J L Massey, *Guessıng and entropy,* Proc 1994 IEEE Int Symp. on Inf. Th., (Trondheım, Norway, 1994), p. 204
[19] J. Pečarıc, *On some ınequalıtıes analogous to Grüss inequalıty*, Mat. Vesnık, **4** (1980), 197–202.

School of Computer Science & Mathematics
Victorıa Unıversity of Technology
PO Box 14428
MCMC 8001, Vıctorıa, Australia
E-maıl sever@matılda vu.edu.au
URL: http·//rgmıa.vu.edu.au/SSDragomirWeb.html