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ON TWO-DIMENSIONAL LANDSBERG SPACE OF A CUBIC FINSLER SPACE

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ABSTRACT In the present paper, we are to find the conditions that a cubic Finsler space is a Berwald space and a two-dimensional cubic Finsler space is a Landsberg space It is shown that if a two-dimensional cubic Finsler space is a Landsberg space, then it is a Berwald space.

1. Introduction

In the Cartan connection $C\Gamma$, a Finsler space is called *Landsberg space* if the covariant derivative $C_{hij|k}$ of the C-torsion tensor $C_{hij} = \dot{\partial}_h\partial_i\partial_j(L^2/4)$ satisfies $C_{hij|k}(x,y)y^k = 0$ A Berwald space is characterized by $C_{hij|k} = 0$. Berwald spaces are specially interesting and important because the connection is linear, and many examples of Berwald spaces have been known But any concrete example of a Landsberg space which is not a Berwald space is not known yet. If a Finsler space is a Landsberg space and satisfies some additional conditions, then it is merely a Berwald space [3] On the other hand, in a two-dimensional case, a general Finsler space is a Landsberg space if and only if its main scalar $I(x,y)$ satisfies $I_{|i}y^i = 0$ [7].

The purpose of the persent paper is devoted to finding a two-dimensional Landsberg space with a cubic metric $L^3 = c_1\alpha^2\beta + c_2\beta^3$, where c_1 and c_2 are constants. First we find the condition that a

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Finsler space with a cubic metric is a Berwald space (Theorem 3.2). Next we determine the difference vector and the main scalar of F^2 with the metric above. Finally we derive the condition that a two-dimensional Finsler space F^2 with a cubic metric is a Landsberg space (Theorem 4.1), and we show that if F^2 with the metric above is a Landsberg space, then it is a Berwald space (Theorem 4.2).

2. Preliminaries

Let $F^n = (M^n, L(\alpha, \beta))$ be an n-dimensional Finsler space with an (α, β)-metric and $R^n = (M^n, \alpha)$ the associated Riemannian space, where $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$. In the following the Riemannian metric α is *not supposed to be positive-definite* and we shall restrict our discussions to a domain of (x, y), where β does not vanish. The covariant differentiation in the Levi-Civita connection $\gamma_j{}^i{}_k(x)$ of R^n is denoted by the semi-colon. Let us list the symbols here for the late use:

$$2r_{ij} = b_{i;j} + b_{j;i}, \quad 2s_{ij} = b_{i;j} - b_{j;i}, \quad r^i{}_j = a^{it} r_{tj},$$
$$s^i{}_j = a^{it} s_{tj}, \quad r_i = b_t r^t{}_i, \quad s_i = b_t s^t{}_i, \quad b^i = a^{it} b_t, \quad b^2 = a^{ts} b_t b_s.$$
$$L_\alpha = \partial L/\partial\alpha, \quad L_\beta = \partial L/\partial\beta, \quad L_{\alpha\alpha} = \partial L_\alpha/\partial\alpha, \quad \text{and} \quad y_k = a_{kt} y^t.$$

The Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j)$ of F^n plays one of the leading roles in the present paper Denote by $B_j{}^i{}_k$ the difference tensor of $G_j{}^i{}_k$ from $\gamma_j{}^i{}_k$ as follows [8].

$$G_j{}^i{}_k(x, y) = \gamma_j{}^i{}_k(x) + B_j{}^i{}_k(x, y) \tag{2.1}$$

With the subscript 0, transvection by y^i, we have

$$G^i{}_j = \gamma_0{}^i{}_j + B^i{}_j, \quad 2G^i = \gamma_0{}^i{}_0 + 2B^i, \tag{2.2}$$

and then $B^i{}_j = \dot\partial_j B^i$, $B_j{}^i{}_k = \dot\partial_k B^i{}_j$, and $\dot\partial_j = \partial/\partial y^j$. On account of [8], the Berwald connection $B\Gamma$ of a Finsler space with

(α,β)-metric $L(\alpha,\beta)$ is given by (2.1) and (2.2), where $B_j{}^i{}_k$ are the components of a Finsler tensor of (1,2)-type which is determined by

$$L_\alpha B_j{}^k{}_i y^j y_k = \alpha L_\beta (b_{j,i} - B_j{}^k{}_i b_k) y^j. \tag{2.3}$$

According to [8], $B^i(x,y)$ is called the *difference vector.* If $\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$, where $\gamma^2 = b^2\alpha^2 - \beta^2$, then B^i is written as follows:

$$B^i = \frac{E}{\alpha} y^i + \frac{\alpha L_\beta}{L_\alpha} s^i{}_0 - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} C^* \left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i \right), \tag{2.4}$$

where $E = \beta L_\beta C^*/L$ and

$$C^* = \{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)\}/2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}).$$

Furthermore, by means of [4] we have

(2.5)
$$\alpha_{|i} = -\frac{L_\beta}{L_\alpha}\beta_{|i},$$
(2.6)
$$\beta_{|i} y^i = r_{00} - 2b_r B^r,$$
(2.7)
$$b^2{}_{|i} y^i = 2(r_0 + s_0),$$
(2.8)
$$\gamma^2{}_{|i} y^i = 2(r_0 + s_0)\alpha^2 - 2\left(\frac{L_\beta}{L_\alpha} b^2\alpha + \beta\right)(r_{00} - 2b_r B^r).$$

The following Lemmas have been shown as follows.

LEMMA 2.1. *([2]) If $\alpha^2 \equiv 0 \pmod \beta$, that is, $a_{ij}(x)y^i y^j$ contains $b_i(x)y^i$ as a factor, then the dimension n is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

LEMMA 2.2. *([4]) We consider the two-dimensional case.*

(1) *If $b^2 \neq 0$, then there exist a sign $\varepsilon = \pm 1$ and $\delta = d_i(x)y^i$ such that $\alpha^2 = \beta^2/b^2 + \varepsilon\delta^2$ and $d_i b^i = 0$.*

(2) *If $b^2 = 0$, then there exists $\delta = d_i(x)y^i$ such that $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

If there are two functions $f(x)$ and $g(x)$ satisfying $f\alpha^2 + g\beta^2 = 0$, then $f = g = 0$ is obvious, because $f \neq 0$ implies a contradition $\alpha^2 = (-g/f)\beta^2$.

We shall state one more remark : Throughout the paper, we shall say "homogeneous polynomial(s) in (y^i) of degree r"as $hp(r)$ for brevity. Thus $\gamma_0{}^i{}_0$ are $hp(2)$.

3. Berwald spaces

In the present paper, we treat a condition that a Finsler space with a cubic metric is a Berwlad space (cf.[11]). Then the so-called *cubic metric* on a differentiable manifold with the local coordinates x^i is defined by

$$L(x,y) = (a_{ijk}(x)y^i y^j y^k)^{1/3} \quad (y^i = \dot{x}^i), \tag{3.1}$$

where $a_{ijk}(x)$ are components of a symmetric tensor of (0,3)-type, depending on the position x alone, and a Finsler space with a cubic metric is called the *cubic Finsler space*. The Finsler metric given by (3.1) was considered by Wegener (1935) [14] and by Kropina [6], and was also studied by M. Matsumoto [9], H. Shimada [13] and S. Numata [9]. It is regarded as a direct generalization of Riemannian metric in a sense. We quote from the proposition as follows:

PROPOSITION 3.1. *([9]) Let F^n be a Finsler space with a cubic metric $L(x,y)$.*

(1) *In case of $n > 2$, if L is an (α,β)-metric where α is non-degenerate, then L^3 can be written in the form $L^3 = c_1\alpha^2\beta + c_2\beta^3$ with two constants c_1 and c_2.*

(2) *In case of $n = 2$, L is always written in a generalized $(-1/3)$-Kropina type $L = \alpha^{2/3}\beta^{1/3}$, where α may be degenerate.*

Now the cubic metric $L(\alpha, \beta)$ of Finsler space F^n is given by

$$L^3(\alpha, \beta) = c_1\alpha^2\beta + c_2\beta^3, \tag{3.2}$$

where c_1 and c_2 are constants. In this case we have

$$\begin{gathered} 3L^2L_\alpha = 2c_1\alpha\beta, \quad 3L^2L_\beta = c_1\alpha^2 + 3c_2\beta^2 \\ 9L^5L_{\alpha\alpha} = 2c_1\beta^2(3c_2\beta^2 - c_1\alpha^2), \quad w = 2c_1(3c_2\beta^2 - c_1\alpha^2), \\ C^* = \frac{3L^3\{c_1\beta r_{00} - (c_1\alpha^2 + 3c_2\beta^2)s_0\}}{2c_1\alpha\beta\{3(c_2b^2 + c_1)\beta^2 - c_1\gamma^2\}}. \end{gathered} \tag{3.3}$$

Substituting (3.3) into (2.3), we obtain

$$2c_1\beta B_{jki}y^j y^k = (c_1\alpha^2 + 3c_2\beta^2)(b_{j;i} - B_{jki}b^k)y^j, \tag{3.4}$$

where $B_{jki} = a_{kr}B_j{}^r{}_i$.

Suppose that the cubic Finsler space F^n is a Berwald space, that is, $B_j{}^i{}_k = B_j{}^i{}_k(x)$ and hence $b_{i;j}$ do not depend on y^i. Thus (3.4) leads to

$$B_{00i} = (c_1\alpha^2 + 3c_2\beta^2)p_i, \tag{3.5}$$

$$(b_{j;i} - B_{jki}b^k)y^j = 2c_1\beta p_i, \tag{3.6}$$

where we put $p_i = p_i(x)$. Making use of (3.5), we have

$$B_{jki} + B_{kji} = 2p_i(c_1a_{jk} + 3c_2b_jb_k) \tag{3.7}$$

Since B_{jki} is symmetric in (j, i), (3.7) gives rise to

$$B_{jki} = c_1(p_ia_{jk} + p_ja_{ki} - p_ka_{ij}) + 3c_2(p_ib_jb_k + p_jb_kb_i - p_kb_ib_j). \tag{3.8}$$

Therefore substitution of (3.8) in (3.6) yields

$$b_{j;i} = 3ap_ib_j + (3a - 2c_1)p_jb_i - p_b(c_1a_{ij} + 3c_2b_ib_j), \tag{3.9}$$

where we put $a = c_2b^2 + c_1$ and $p_b = p_kb^k$.

If $\alpha^2 \equiv 0 \pmod{\beta}$, then Lemma 2.1 shows that $n = 2$, $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$, $b^2 = 0$ and $b^i d_i = 2$. Thus (3.4) is of the form

$$(3.10) \qquad 2c_1 B_{00i} = (c_1\delta + 3c_2\beta)(b_{j,i} - B_{jki}b^k)y^j,$$

which leads to

$$(3.11) \qquad B_{00i} = (c_1\delta + 3c_2\beta)u_{1i},$$

$$(3.12) \qquad (b_{j,i} - B_{jki}b^k)y^j = 2c_1 u_{1i},$$

where $u_{1i} = q_i u_k y^k$ are $hp(1)$ and $q_i = q_i(x)$. Then we have

$$B_{jki} + B_{kji} = q_i(c_1 d_j + 3c_2 b_j)u_k + q_i(c_1 d_k + 3c_2 b_k)u_j,$$

which lead to

$$2B_{jki} = c_1\{q_i(d_j u_k + d_k u_j) + q_j(d_k u_i + d_i u_k) - q_k(d_i u_j + d_j u_i)\} + 3c_2\{q_i(b_j u_k + b_k u_j) + q_j(b_k u_i + b_i u_k) - q_k(b_i u_j + b_j u_i)\}.$$

Since the dimension is equal to two and (b_i, d_i) are independent pairs, we can put $u_i = hb_i + kd_i$ and then $u_i b^i = 2k$. Then we have

(3.13)

$$B_{jki}b^k = c_1\left[q_i(hb_j + 2kd_j) + q_j(hb_i + 2kd_i) - \frac{1}{2}q_b\{h(b_i d_j + b_j d_i) + 2kd_i d_j\}\right] + 3c_2\left[k(q_i b_j + q_j b_i) - \frac{1}{2}q_b\{2hb_i b_j + k(b_i d_j + b_j d_i)\}\right],$$

where $q_b = q_k b^k$. Substituting (3.13) into (3.12), we have

(3.14)

$$b_{i,j} = c_1\{q_i(3hb_j + 4kd_j) + q_j(hb_i + 2kd_i)\} + 3c_2 k(q_i b_j + q_j b_i) - \frac{1}{2}q_b\{(c_1 h + 3c_2 k)(b_i d_j + b_j d_i) + 2(3c_2 h b_i b_j + c_1 k d_i d_j)\}.$$

In case of $L^3 = \alpha^2\beta$, substitution of $c_1 = 1$ and $c_2 = 0$ in (3.14) leads to

$$b_{i,j} = h(b_i q_j + 3b_j q_i) + 2k(d_i q_j + 2d_j q_i) - \frac{1}{2}q_b\{h(b_i d_j + b_j d_i) + 2kd_i d_j)\}.$$

Summarizing up all the above, we have the following

THEOREM 3.2. *A cubic Finsler space with $L = c_1\alpha^2\beta + c_2\beta^3$, where c_1 and c_2 are constants, is a Berwald space if and only if*

(1) $\alpha^2 \not\equiv 0 \pmod{\beta}$: $b_{i,j}$ is of the form (3.9), where $a = c_2 b^2 + c_1$ and $p_b = p_k b^k$.

(2) $\alpha^2 \equiv 0 \pmod{\beta}$: $n = 2$, $b^2 = 0$ and $b_{i,j}$ is of the form (3.14), where $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$ and (h, k) are functions of (x^i).

4. Two-dimensional Landsberg spaces

Now we are to find the necessary and sufficient conditions that a two-dimensional Finsler space with a cubic metric (3 2) is a Landsberg space (cf.[10]).

Because the difference vector B^i and the main scalar εI^2 play the leading roles, we have to determine the difference vector B^i. The difference vector B^i of the Finsler space has been first given by [12]. Here, by means of (2.4) and reference of (3 3), we have

$$(4.1) \qquad 2B^i = \frac{2A}{\Omega}\left\{y^i + \frac{(3c_2\beta^2 - c_1\alpha^2)}{2c_1\beta}b^i\right\} + \frac{(c_1\alpha^2 + 3c_2\beta^2)}{c_1\beta}s^i{}_0,$$

where

$$A = c_1\beta r_{00} - (c_1\alpha^2 + 3c_2\beta^2)s_0,$$
$$\Omega = 3a\beta^2 - c_1\gamma^2.$$

It follows from (4 1) that

$$(4.2) \qquad r_{00} - 2b_r B^r = \frac{2\beta A}{\Omega}$$

Now we deal with the necessary and sufficient conditions that a two-dimensional Finsler space F^2 with a cubic metric (3.2) is a Landsberg space. It is well known that in the two-dimensional case, a general Finsler space is a Landsberg space if and only if its main scalar $I_{|i}y^i = 0$. Owing to [1], [5], the main scalar I of a two-dimensional Finsler space F^2 with a cubic metric (3.2) and (3.3) is obtained as follows.

$$(4.3) \qquad \varepsilon I^2 = \frac{c_1\gamma^2 Z^2}{2\Omega^3},$$

where $Z = 9a\beta^2 + c_1\gamma^2$.

Before discussing our problem, we have to check the assumption $\Omega \neq 0$ ans $c_1 \neq 0$ in the two-dimensional case becsuse Ω appears in the denominators in (4.1), (4.2) and (4.3), and c_1 also appears (4.1). Lemma 2.2 shows that $\Omega = 0$ if and only if $c_1 = 0$ and $b^2 = 0$, $c_1 = 0$ and $c_2 = 0$, namely, the space is $L^3 = c_2\beta^3$ and $b^2 = 0$, $L = 0$. Consequently, $\Omega \neq 0$, $c_1c_2 \neq 0$ and $b^2 \neq 0$ are a proper assumption in the present section.

The covariant differentiation of (4.3) yields

$$\varepsilon I^2{}_{|i} = \frac{c_1Z}{2\Omega^4}(\Omega Z\gamma^2{}_{|i} + 2\gamma^2\Omega Z_{|i} - 3\gamma^2 Z\Omega_{|i}). \tag{4.4}$$

Transvecting (4.4) by y^i, we get

(4.5)

$$\varepsilon I^2{}_{|i}y^i = \frac{27c_1b^2\beta(c_1\alpha^2 + c_2\beta^2)Z}{2\Omega^4}(a\beta\gamma^2{}_{|i}y^i - 2a\gamma^2\beta_{|i}y^i - c_2\beta\gamma^2b^2{}_{|i}y^i).$$

Consequently, the cubic Finsler space is a Landsberg space if and only if

$$27c_1b^2\beta(c_1\alpha^2 + c_2\beta^2)Z(a\beta\gamma^2_{|i}y^i - 2a\gamma^2\beta_{|i}y^i - c_2\beta\gamma^2b^2_{|i}y^i) = 0,$$

which imply

$$(c_1\alpha^2 + c_2\beta^2)Z(a\beta\gamma^2_{|i}y^i - 2a\gamma^2\beta_{|i}y^i - c_2\beta\gamma^2b^2_{|i}y^i) = 0$$

because of $c_1 \neq 0$ and $b^2 \neq 0$. Thus the following three cases should be considered to find the condition:

(1°) $c_1\alpha^2 + c_2\beta^2 = 0$: Lemma 2.2 shows a contradiction immediately, that is, we obtain $c_1 \neq 0$ and $c_2 \neq 0$.

(2°) $Z = 9a\beta^2 + c_1\gamma^2 = 0$: This implies $c_1 = 0$ and $c_2 = 0$, which is a contradiction by Lemma 2 2, that is, $Z \neq 0$.

(3°) $a\beta\gamma^2{}_{|i}y^i - 2a\gamma^2\beta_{|i}y^i - c_2\beta\gamma^2b^2{}_{|i}y^i = 0$: By means of (2.6), (2.7) and (2.8) this equation is written as

$$2c_1\beta(r_0 + s_0) - 3ab^2(r_{00} - 2b_rB^r) = 0.$$

Substituting (4.2) in the above, we obtain

$$\beta[c_1(3a+c_1)\beta r_0 + \{3(3a-2c_1)a + c_1^2\}\beta s_0 - 3c_1 ab^2 r_{00}] + c_1 b^2\alpha^2\{(3a-c_1)s_0 - c_1 r_0\} = 0. \tag{4.6}$$

First, this gives a condition $c_1 b^2\alpha^2\{(3a-c_1)s_0 - c_1 r_0\} = 0 \pmod{\beta}$. Since $b^2 \neq 0$ may be supposed in this case, Lemma 2.1 shows $\alpha^2 \not\equiv 0 \pmod{\beta}$ and so there exists a function $g(x)$ satisfying

$$(3a-c_1)s_0 - c_1 r_0 = g\beta. \tag{4.7}$$

Then (4.6) is reduced to

$$[c_1(3a+c_1)r_0 + \{3(3a-2c_1)a + c_1^2\}s_0]\beta + c_1 b^2(g\alpha^2 - 3ar_{00}) = 0.$$

This implies that there exists a 1-form $\mu = m_i(x)y^i$ such that

$$3ar_{00} = g\alpha^2 - \beta\mu, \tag{4.8}$$

and the above is reduced to

$$c_1(3a+c_1)r_0 + \{3(3a-2c_1)a + c_1^2\}s_0 + c_1 b^2\mu = 0.$$

Thus the above and (4.7) yield

$$s_0 = \frac{(3a+c_1)g\beta - c_1 b^2\mu}{6a(3a-c_1)}, \tag{4.9}$$

$$r_0 = -\frac{\{(3a-c_1)g\beta + c_1 b^2\mu\}}{6c_1 a}. \tag{4.10}$$

Consequently, (4.8), (4.9) and (4.10) are attained from (4.6). Since (4.8) is written in the form

$$6ar_{ij} = 2ga_{ij} - (b_i m_j + b_j m_i),$$

the transvection by $b^i y^j$ yields

$$6ar_0 = 2g\beta - (b^2\mu + m_b\beta), \tag{4.11}$$

where $m_b = m_k b^k$.

Comparing (4.11) with (4.10), we have

$$c_1 m_b = (3a + c_1) g.$$

Thus we get the condition in the form

$$(4.8') \qquad r_{00} = \frac{c_1 m_b}{3a(3a + c_1)}\alpha^2 - \frac{1}{3a}\beta\mu,$$

$$(4.9') \qquad s_0 = \frac{c_1(m_b\beta - b^2\mu)}{6a(3a - c_1)}.$$

Eliminating μ from (4.8′) and (4.9′), we have

$$(4.12) \qquad r_{00} = \frac{c_1 f}{3a(3a + c_1)}\alpha^2 - \frac{f}{3b^2 a}\beta^2 + \frac{2(3a - c_1)}{c_1 b^2}\beta s_0,$$

where $f(x) = m_b$.

Thus we have the following

THEOREM 4.1. *The necessary and sufficient condition for a two-dimensional cubic Finsler space with $c_1 c_2 \neq 0$ and $b^2 \neq 0$ to be a Landsberg space is that (4.12) is satisfied.*

Now we shall prove the reduction theorem:

THEOREM 4.2. *Let F^2 be a two-dimensional cubic Finsler space with $c_1 c_2 \neq 0$ and $b^2 \neq 0$. If F^2 is a Landsberg space, then F^2 is a Berwald space.*

Proof. The condition that (3.9) be a Berwald space may be rewritten in the form

$$(4.13) \qquad \begin{aligned} &(1) \quad r_{ij} = (3a - c_1)(b_i p_j + b_j p_i) - p_b(c_1 a_{ij} + 3c_2 b_i b_j), \\ &(2) \quad s_{ij} = c_1(b_i p_j - b_j p_i). \end{aligned}$$

Now let F^2 be a Landsberg space, that is, suppose that (4.12) holds. Then the system of linear equations

$$b^1 p_1 + b^2 p_2 = -\frac{f}{3a(3a + c_1)}, \quad -b_2 p_1 + b_1 p_2 = \frac{s_{12}}{c_1}$$

for (p_1, p_2), where f is the one in (4.12), determines uniqne solution (p_1, p_2) because of $c_1 b^2 \neq 0$. The above are written as

$$f + 3a(3a + c_1)p_b = 0, \quad s_{ij} = c_1(b_i p_j - b_j p_i).$$

The latter is nothing but (2) of (4.13). Then we obtain $s_0 = c_1(b^2\phi - p_b\beta)$, $\phi = p_i(x)y^i$, and (4.12) is now written in the form

$$r_{00} = 2(3a - c_1)\beta\phi - p_b(c_1\alpha^2 + 3c_2\beta^2),$$

which is nothing but (1) of (4.13). Thus the proof is completed. □

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