East Asian Math. J 19 (2003), No 2, pp 273–289

LIE SEMIGROUPS IN O(2,2)

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ABSTRACT. We study Ol'shanskiĭ semigroups admitting triple decompositions in the Lie group O(2,2).

1. Introduction

An impotent class of subsemigroups of Lie groups is Ol'shanskiĭ semigroups that play the role of noncommutative analogue of tube domains in the harmonic analysis of hermitian semisimple Lie groups. In [4] the author gave some conditions for the existence of the Ol'shanskiĭ type semigroup (a semigroup variant of the Cartan decomposition) in a Lie group, and in [5] the authors investigate some conditions for the existence of a triple decomposition (a semigroup variant of the Harish-Chandra decomposition) from an Ol'shanskiĭ semigroup. The class of semigroups for which the triple decomposition obtains contains symplectic semigroups, or more generally the conformal compression semigroup of a symmetric cone in an euclidean Jordan algebra (see [5] and [7])

The Lie algebra $\mathfrak{so}(2,2)$ of the Lie group $O_o(2,2)$, the connected component of the identity in the group of linear transformations of $\mathbb{R}^4$ preserving a metric of signature $(2,2)$, is a symmetric algebra of Cayley type, $\mathfrak{q}^+ + \mathfrak{h} + \mathfrak{q}^-$, with $\dim \mathfrak{q}^{\pm} = 1$ In this paper we show that this symmetric algebra induces an Ol'shanskiĭ semigroup in $O(2,2)$ admitting a triple decomposition.

Received October 22, 2003
2000 Mathematics Subject Classification 22E15
Key words and phrases. Lie semigroup, Lie group

2. Lie semigroups with triple decompositions

Let G be a Lie group with Lie algebra $\mathcal{L}(G)$ and S be a closed subsemigroup of G with identity The *tangent wedge* of S is defined by

$$\mathcal{L}(S) = \{X \in \mathcal{L}(G) : \exp(tX) \in S \text{ for all } t \geq 0\}.$$

Then it is a closed convex cone containing zero and is a Lie wedge, i.e.,

$$e^{adX}\mathcal{L}(S) = \mathcal{L}(S),\ \forall X \in \mathcal{L}(S) \cap -\mathcal{L}(S)$$

The systematic groundwork for a Lie theory of semigroups was worked out by K H. Hofmann, J. Hilgert and J. D. Lawson (cf. [1]).

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\tau : G \to G$ be a differentiable involution of G The pair (G, τ) is called an *involutive group* Then the derivative of τ at the identity e, $d\tau(e) : \mathfrak{g} \to \mathfrak{g}$, is a Lie algebra involution and leads to a decomposition of $\mathfrak{g}$ into the $+1$-eigenspace $\mathfrak{h}$ and -1-eigenspace $\mathfrak{q}$, $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$, which satisfies

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, [\mathfrak{h}, \mathfrak{q}] \subseteq \mathfrak{q}, [\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{h}.$$

Lie algebras with a given decomposition with these properties are called *symmetric algebras*. Let H be a subgroup of $G_\tau := \{g \in G : \tau(g) = g\}$ containing the identity component of G_τ. If $\mathbf{C}$ is an $\mathrm{Ad}(H)$-invariant cone in $\mathfrak{q}$ and if $S := H(\exp \mathbf{C})$ is a subsemigroup of G for which the mapping

$$(h, X) \mapsto h(\exp \mathbf{C}) : H \times \mathbf{C} \to S$$

is a homeomorphism, then S is called an *Ol'shanskii semigroup*, and the factorization $s = h(\exp X)$ for $s \in S$ is called the *Ol'shanskii polar factorization*

The following appears at the Theorem 3 1 in [4].

THEOREM 2.1 Let (G, τ) be a finite dimensional involutive Lie group, and let $H \subset G_\tau$ be a closed subgroup containing the identity component of G_τ Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be the corresponding symmetric decomposition of the Lie algebra of G, and let $\mathfrak{z}$ denote the center of $\mathfrak{g}$. Let $\mathbf{C}$ be a closed convex cone in $\mathfrak{q}$ which is invariant under the adjoint action of H, and for which $\mathrm{ad}(X)$ has real spectrum for each $X \in \mathbf{C}$ Then the following conditions are equivalent

(1) $(h, X) \mapsto h(\exp X) : H \times \mathbf{C} \to H(\exp \mathbf{C})$ is a diffeomorphism onto a closed subset of G
(2) The mapping $\mathrm{Exp} : \mathfrak{h} \to G/H$ defined by $\mathrm{Exp}(X) = H(\exp X)$ restricted to $\mathbf{C}$ is a diffeomorphism onto a closed subset of G/H.
(3) The mapping exp restricted to $\mathbf{C}$ is a diffeomorphism onto a closed subset of G
(4) If $Z \in \mathfrak{z} \cap (\mathbf{C} - \mathbf{C})$ satisfies $\exp Z = e$, then $Z = 0$. For each non-zero $X \in \mathbf{C} \cap \mathfrak{z}$, the closure of $\exp(\mathbb{R}X)$ is not compact

If these conditions hold, then $S := H(\exp \mathbf{C})$ is a closed semigroup with the tangent wedge $\mathcal{L}(S) = \mathfrak{h} + \mathbf{C}$

Let $\mathfrak{g}$ be a symmetric algebra, $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ An element $X \in \mathfrak{g}$ is called *hyperbolic* if the spectrum of $\mathrm{ad}(X)$ is real and $\mathrm{ad}(X)$ is semisimple (i.e., diagonalizable) as a linear operator If a closed convex cone $\mathbf{C}$ in $\mathfrak{q}$ has dense interior in the vector space $\mathbf{C} - \mathbf{C}$, and if $\mathrm{ad}(X)$ is hyperbolic for each X in the interior of $\mathbf{C}$, then the cone is said to be *hyperbolic*

The symmetric Lie algebra $\mathfrak{g}$ is called a *symmetric algebra of Cayley type* if there exist abelian subalgebras $\mathfrak{q}_-$ and $\mathfrak{q}_+$ of $\mathfrak{g}$ contained in $\mathfrak{q}$ such that $\mathfrak{q} = \mathfrak{q}_- \oplus \mathfrak{q}_+$ Note that the triple $(\mathfrak{q}_-, \mathfrak{h}, \mathfrak{q}_+)$ is a $(-1, 0, 1)$-graded Lie algebra Let H be a closed subgroup of G with Lie algebra $\mathfrak{h}$. We define a smooth mapping by

$$\phi : \mathfrak{q}_- \times H \times \mathfrak{q}_+ \to G, \qquad \phi(X, h, Y) = (\exp X)h(\exp Y)$$

The following is the principal theorem of [5]

THEOREM 2 2. Let (G, τ) be a finite dimensional involutive Lie group such that the Lie algebra $\mathfrak{g} = \mathfrak{q}_- + \mathfrak{h} + \mathfrak{q}_+$ is a symmetric algebra of Cayley type Let H be a closed subgroup of G_τ with Lie algebra $\mathfrak{h}$ Suppose that $\mathbf{C}^-$ is a cone in $\mathfrak{q}_-$ and $\mathbf{C}^+$ is a cone in $\mathfrak{q}_+$ such that $\mathbf{C} := \mathbf{C}^+ + \mathbf{C}^-$ is a hyperbolic $\mathrm{Ad}(H)$-invariant cone. Set $S := (\exp \mathbf{C}^-)h(\exp \mathbf{C}^+)$ If any of the conditions (1)-(4) of Theorem 2.1 is satisfied, then the mapping

$$\phi : \mathbf{C}^- \times H \times \mathbf{C}^+ \to S, \qquad \phi(X, h, Y) = (\exp X)h(\exp Y)$$

is diffeomorphism If further the set S is closed, then S is semigroup and equal to the Ol'shanskiĭ semigroup $H(\exp \mathbf{C})$ The set S is closed and the conclusions follow in the case that $\mathbf{C}$ is pointed.

3. Groups with Lie algebra type D_r

Throughout we fix $n \in \mathbb{N}$. Let $\mathbb{R}^n$ be the Euclidean n-space with the usual inner product $\langle\cdot|\cdot\rangle$. We define a matrix $2n$ by $2n$ matrix J by $J = \begin{bmatrix} O & I \\ I & O \end{bmatrix}$, where $I = I_n$ denotes the $n \times n$ real identity matrix. Note that $J^2 = I_{2n}$ and $J^{-1} = J = J^t$ We define the symmetric bilinear form on $\mathbb{R}^{2n}$ by

$$(x|y) = \langle Jx|y\rangle, \ x, y \in \mathbb{R}^{2n}. \tag{1}$$

Let

$$G := \{M \in \mathrm{GL}(2n, \mathbb{R}) : (Mx|My) = (x|y) \text{ for all } x, y \in \mathbb{R}^{2n}\}.$$

Note that for $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G$, the inverse of M is given by $M^{-1} = \begin{bmatrix} D^t & B^t \\ C^t & A^t \end{bmatrix}$.

PROPOSITION 3.1. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GL}(2n, \mathbb{R})$. Then the following are equivalent:

(1) $M \in G$, i e., M preserves $(\cdot|\cdot)$.
(2) $M^t J M = J$.
(3) $A^t C$, $B^t D$ are skew-symmetric and $A^t D + C^t B = I$.
(4) DC^t, BA^t are skew-symmetric and $DA^t + CB^t = I$.

Proof. Straightforward. □

REMARK 3 2. Let $\mathrm{O}(n, n)$ be the group of all pseudo-orthogonal real matrices of signature (n, n). Then this group can be expressed as

$$\mathrm{O}(n, n) = \left\{M \in \mathrm{GL}(2n, \mathbb{R}) : M^t I_{n,n} M = I_{n,n}\right\},$$

where $I_{n,n} := \begin{bmatrix} I & O \\ O & -I \end{bmatrix}$ and I is the identity matrix of size $n \times n$. The group $\mathrm{O}(n, n)$ has four connected components and the identity component of $\mathrm{O}(n, n)$ is equal to the identity component of $\mathrm{SO}(n, n) =$

$\mathrm{O}(n,n) \cap \mathrm{SL}(2n,\mathbb{R})$ (cf. [2]). Observe that $RI_{n,n}R^{-1} = J$, where R is the real Cayley transform, i.e.,

$$R = \begin{bmatrix} I & -I \\ I & I \end{bmatrix}$$

Thus we have $G = R\mathrm{O}(n,n)R^{-1}$

We define an involution $\tau : G \to G$ by

$$\tau\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \begin{bmatrix} A & -B \\ -C & D \end{bmatrix}.$$

Then (G,τ) is an involutive Lie group with the fixed group

$$G_\tau = \left\{\begin{bmatrix} A & O \\ O & (A^t)^{-1} \end{bmatrix} \cdot A \in \mathrm{GL}(n,\mathbb{R})\right\}.$$

Let $\mathfrak{g}$ be the Lie algebra of G Then obviously, we have

$$\begin{aligned} \mathfrak{g} &= \{M \in M_{2n}(\mathbb{R}) \cdot JM^tJ = -M\} \\ &= \left\{\begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} \quad B^t = -B, C^t = -C, A \in M_n(\mathbb{R})\right\} \\ &\cong \mathfrak{so}(n,n) \end{aligned}$$

and hence $\mathfrak{g}$ is the classical Lie algebra type D_n with its dimension $2n^2 - n$. If $n \geq 3$, then D_n is simple.

Furthermore, the differential $d\tau(e) \cdot \mathfrak{g} \to \mathfrak{g}$ at the identity e is given by

$$d\tau(e)\left(\begin{bmatrix} A & B \\ C & -A^t \end{bmatrix}\right) = \begin{bmatrix} A & -B \\ -C & -A^t \end{bmatrix}$$

and it defines a symmetric algebra

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{q}.$$

Furthermore, by setting

$$\begin{aligned} \mathfrak{q}^+ &= \left\{\begin{bmatrix} O & X \\ O & O \end{bmatrix} \cdot X^t = -X\right\}, \\ \mathfrak{q}^- &= \left\{\begin{bmatrix} O & O \\ X & O \end{bmatrix} \quad X^t = -X\right\}, \end{aligned}$$

$\mathfrak{g} = \mathfrak{q}^+ + \mathfrak{h} + \mathfrak{q}^-$ becomes a symmetric algebra of Cayley type. Let

$$Q^+ = \left\{ \begin{bmatrix} I & X \\ O & I \end{bmatrix} : X^t = -X \right\} = \exp \mathfrak{q}^+$$

$$Q^- = \left\{ \begin{bmatrix} I & O \\ X & I \end{bmatrix} : X^t = -X \right\} = \exp \mathfrak{q}^-$$

$$H = G_\tau = \left\{ \begin{bmatrix} D & O \\ O & (D^{-1})^t \end{bmatrix} : D \in \mathrm{GL}(n, \mathbb{R}) \right\}.$$

Then $Q^\pm$ is a Lie subgroup of G with its Lie algebra $\mathfrak{q}^\pm$, respectively

PROPOSITION 3.3. Let $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G$ Then the following are equivalent:

(1) $g \in Q^+HQ^-$
(2) D is invertible

Proof. If $g \in Q^+HQ^-$, then the element g is of the form

$$\begin{bmatrix} * & * \\ * & (F^{-1})^t \end{bmatrix}$$

for some $F \in \mathrm{GL}(n, \mathbb{R})$ and hence D is invertible. Conversely, we note that if D is invertible, then

$$(2) \qquad g = \begin{bmatrix} I & BD^{-1} \\ O & I \end{bmatrix} \begin{bmatrix} (D^{-1})^t & O \\ O & D \end{bmatrix} \begin{bmatrix} I & O \\ D^{-1}C & I \end{bmatrix}$$

By Proposition 3.1, $B^t = -D^tBD^{-1}$ and $C^t = -D^{-1}CD^t$. It follows that

$$(BD^{-1})^t = (D^{-1})^t B^t = -(D^{-1})^t D^t BD^{-1} = -BD^{-1}$$

and

$$(D^{-1}C)^t = C^t(D^{-1})^t = -D^{-1}CD^t(D^{-1})^t = -D^{-1}C$$

Thus $g \in Q^+HQ^-$ $\square$

Note: The factorization in the Proposition 3.3 is uniquely determined (see Theorem 5.2 in [5]).

REMARK 3.4 The set $P := HQ^-$ is a closed subgroup of G. Let $\mathcal{M} := G/P$. Then there is a canonical imbedding

$$\text{(3)} \qquad \mathrm{Skew}(n,\mathbb{R}) \hookrightarrow \mathfrak{q}^+ \hookrightarrow \mathcal{M}, \quad X \mapsto \begin{bmatrix} O & X \\ O & O \end{bmatrix} \mapsto \begin{bmatrix} I & X \\ O & I \end{bmatrix} P$$

If $n = 2$, then $\mathrm{Skew}(2,\mathbb{R})$ is one-dimensional and it contains the positive cone $\Omega := \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} : x > 0 \right\}$ In the last section of this paper, we will devote to show that the compression semigroup of Ω defined by $\Gamma_\Omega = \{g \in G \mid g \cdot \Omega \subset \Omega\}$ has an Ol'shanskiĭ and triple factorizations

4. Factorizations of Γ_Ω

Throughout this section we denote by G the group of 4 by 4 real matrices M satisfying the condition

$$M^t J M = J, \quad \text{where } J = \begin{bmatrix} O & I \\ I & O \end{bmatrix}.$$

Note that G is the group $R\,\mathrm{O}(2,2)R^{-1}$

Let $\mathfrak{g}$ be the Lie algebra of G Then $\mathfrak{g}$ is the classical Lie algebra type D_2. It is well-known that

$$D_2 \cong \mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$$

LEMMA 4 1. Let $A \in \mathrm{GL}(2,\mathbb{R})$ and let $X \in \mathrm{Skew}(2,\mathbb{R})$. Then $AXA^t = A^tXA = \det(A)X \in \mathrm{Skew}(2,\mathbb{R})$.

Proof For $A \in \mathrm{GL}(2,\mathbb{R})$ and $X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \in \mathrm{Skew}(2,\mathbb{R})$, we have

$$\begin{aligned} AXA^t = A^tXA &= \begin{bmatrix} 0 & \det(A)x \\ -\det(A)x & 0 \end{bmatrix} \\ &= \det(A)X \in \mathrm{Skew}(2,\mathbb{R}). \end{aligned}$$

□

For convenience, we let

$$\mathrm{Skew}^+(2,\mathbb{R}) := \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} : x \geq 0 \right\},$$

$$\mathrm{Skew}^-(2,\mathbb{R}) := \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} : x \leq 0 \right\},$$

and let

$$\begin{aligned} \mathbf{C}^+ &:= \left\{ \begin{bmatrix} O & X \\ O & O \end{bmatrix} : X \in \mathrm{Skew}^+(2,\mathbb{R}) \right\}, \\ \mathbf{C}^- &:= \left\{ \begin{bmatrix} O & O \\ Y & O \end{bmatrix} : Y \in \mathrm{Skew}^-(2,\mathbb{R}) \right\} \\ \mathbf{C} &:= \mathbf{C}^+ + \mathbf{C}^-. \end{aligned}$$

Then $\mathbf{C}^-$ is a cone in $\mathfrak{q}_-$, $\mathbf{C}^+$ is a cone in $\mathfrak{q}_+$, and we can easily show that $\mathbf{C}$ is a pointed closed convex cone in the Lie algebra $\mathfrak{g}$ of G.

THEOREM 4 2. We have $H_\circ(\exp \mathbf{C})$ is an Ol'shanskiĭ semigroup with the following triple decomposition,

$$H_\circ(\exp \mathbf{C}) = (\exp \mathbf{C}^+)H_\circ(\exp \mathbf{C}^-),$$

where $H_\circ$ is the identity component of H.

Proof. In order to prove this theorem, we will show that our setting satisfies the conditions given in Theorem 2 2.

Step 1: *The cone* $\mathbf{C}$ *is invariant under the adjoint action of the identity component* $H_\circ$ *of* $H = G_\tau$.

We note that the identity component of H is equal to

$$H_\circ = \left\{ \begin{bmatrix} A & O \\ O & (A^{-1})^t \end{bmatrix} \in H \quad \det(A) > 0 \right\}.$$

Let $K = \begin{bmatrix} A & O \\ O & (A^{-1})^t \end{bmatrix} \in H_\circ$ and let $C = \begin{bmatrix} O & X \\ Y & O \end{bmatrix} \in \mathbf{C}$. Then

$$KCK^{-1} = \begin{bmatrix} O & AXA^t \\ (A^{-1})^t Y A^{-1} & O \end{bmatrix}$$

and by Lemma 4.1,

$$AXA^t = \det(A)X \in \mathrm{Skew}^+(2,\mathbb{R}),$$
$$(A^{-1})^t Y A^{-1} = \det(A^{-1})Y = \frac{1}{\det(A)}Y \in \mathrm{Skew}^-(2,\mathbb{R}).$$

Thus we have the cone $\mathbf{C}$ is $\mathrm{Ad}(H_\circ)$-invariant

Step 2: *For any* $C \in \mathbf{C}$, $\mathrm{ad}(C)$ *has a real spectrum.*

Let

$$C = \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & -x & 0 \\ 0 & -y & 0 & 0 \\ y & 0 & 0 & 0 \end{bmatrix} \in \mathbf{C}$$

Then the characteristic polynomial of C is

$$p(\lambda) = \det(\lambda I - C) = (\lambda^2 - xy)^2$$

and hence C has a real spectrum. By Lemma 4.1 in [4], $\mathrm{ad}(C)$ has a real spectrum.

Step 3 *The cone* $\mathbf{C}$ *is hyperbolic.*

We note that $\mathbf{C}$ is a closed convex cone in $\mathfrak{q}$ and $\mathbf{C}$ has dense interior in the vector space $\mathbf{C} - \mathbf{C}$ By step 2, the spectrum of $\mathrm{ad}(C)$ is real for each $C \in \mathbf{C}$ To show that $\mathrm{ad}(C)$ is semisimple for each C in the interior of $\mathbf{C}$, it is sufficient to show that C is semisimple ([3]) For $x, y > 0$, let

$$C = \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & -x & 0 \\ 0 & -y & 0 & 0 \\ y & 0 & 0 & 0 \end{bmatrix} \in \mathbf{C}.$$

Then we can easily show that the matrix

$$P = \begin{bmatrix} x & 0 & -x & 0 \\ 0 & -x & 0 & x \\ 0 & \sqrt{xy} & 0 & \sqrt{xy} \\ \sqrt{xy} & 0 & \sqrt{xy} & 0 \end{bmatrix}$$

diagonalizes C, i e.,

$$P^{-1}CP = \mathrm{diag}(\sqrt{xy}, \sqrt{xy}, -\sqrt{xy}, -\sqrt{xy}).$$

Thus $\mathbf{C}$ is a hyperbolic cone

Obviously, the Lie algebra $\mathfrak{g}$ is semisimple and hence the center $\mathfrak{z}$ of $\mathfrak{g}$ is trivial. By Theorem 2.1 and Theorem 2.2, the proof is now complete. $\square$

Set

$$\Omega := \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} : x > 0 \right\} = \mathrm{Skew}^+(2, \mathbb{R}) \setminus \{O\},$$

$$\overline{\Omega} := \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} : x \geq 0 \right\} = \mathrm{Skew}^+(2, \mathbb{R})$$

By (3), the group G naturally acts on $\mathcal{M} = G/P$ We define a compression semigroup with respect to $\overline{\Omega}$

$$\Gamma_\Omega := \{g \in G \quad g \cdot \overline{\Omega} \subset \overline{\Omega}\}.$$

THEOREM 4.3. We have $\Gamma_\Omega = \Gamma^+ H_\circ \Gamma^- = H_\circ(\exp \mathbf{C})$, where $\Gamma^+ = \exp \mathbf{C}^+$ and $\Gamma^- = \exp \mathbf{C}^-$

Proof. Obviously, the sets Γ^+ and $H_\circ$ are contained in Γ_Ω. To show that $\Gamma^+ H_\circ \Gamma^- \subset \Gamma_\Omega$, it is sufficient to show that $\Gamma^- \subset \Gamma_\Omega$. Let

$$g = \begin{bmatrix} I & O \\ Y & I \end{bmatrix} \in \Gamma^-, \text{ where } Y = \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} \text{ for some } y \leq 0$$

Note that $I + YX$ is invertible for all $X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \in \overline{\Omega}$ By Proposition 3.3, we have

$$\begin{bmatrix} I & O \\ Y & I \end{bmatrix} \begin{bmatrix} I & X \\ O & I \end{bmatrix} = \begin{bmatrix} I & X \\ Y & I + YX \end{bmatrix} \in \begin{bmatrix} I & X(I + YX)^{-1} \\ O & I \end{bmatrix} P.$$

Since

$$X(I + YX)^{-1} = \begin{bmatrix} 0 & \dfrac{x}{1 - xy} \\ \dfrac{-x}{1 - xy} & 0 \end{bmatrix}$$

and $\dfrac{x}{1-xy} \geq 0$ for $x \geq 0, y \leq 0$, we have

$$g \cdot X = X(I+YX)^{-1} \in \overline{\Omega}.$$

Now suppose that

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_{\Omega}.$$

Then since $g \cdot \mathbf{0} \in \overline{\Omega}$, $g \in Q^+P$. By Proposition 3 3, D is invertible and we have the following factorization

$$g = \begin{bmatrix} I & BD^{-1} \\ O & I \end{bmatrix} \begin{bmatrix} (D^{-1})^t & O \\ O & D \end{bmatrix} \begin{bmatrix} I & O \\ D^{-1}C & I \end{bmatrix} \tag{4}$$

with $BD^{-1} = g \cdot \mathbf{0} \in \overline{\Omega}$. Thus the first term in the righthand of (4) belongs to Γ^+, i.e ,

$$\begin{bmatrix} I & BD^{-1} \\ O & I \end{bmatrix} \in \Gamma^+. \tag{5}$$

For convenience, we let $Y = D^{-1}C = \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix}$. Since $g \cdot \overline{\Omega} \subset \overline{\Omega}$,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & X \\ O & I \end{bmatrix} = \begin{bmatrix} * & * \\ * & CX+D \end{bmatrix} \in Q^+P$$

for all $X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \in \overline{\Omega}$ By Proposition 3.3, $CX+D$ is invertible and hence

$$I + YX = I + D^{-1}CX = D^{-1}(D+CX)$$

is invertible We note that

$$\begin{aligned} & I + YX \text{ is invertible for all } X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \in \overline{\Omega} \\ \iff & \begin{bmatrix} 1-xy & 0 \\ 0 & 1-xy \end{bmatrix} \text{ is invertible for all } x \geq 0 \\ \iff & y \leq 0 \end{aligned}$$

Thus we have that the last term in the righthand of (4) belongs to Γ^-, i.e.,

$$\begin{bmatrix} I & O \\ D^{-1}C & I \end{bmatrix} \in \Gamma^-. \tag{6}$$

Finally, to show that the middle term in the righthand of (4) is contained in $H_\circ$, we have to prove that $\det(D) > 0$.
By (4.1), $g \cdot \overline{\Omega} \subset \overline{\Omega}$ implies that

$$g \cdot X = BD^{-1} + (D^t)^{-1} X (I + YX)^{-1} D^{-1} \in \overline{\Omega}$$

for all $X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \in \overline{\Omega}$ Since $BD^{-1} \in \mathrm{Skew}^+(2, \mathbb{R})$, we can write

$$BD^{-1} = \begin{bmatrix} 0 & z \\ -z & 0 \end{bmatrix} \text{ with } z \geq 0.$$

Let $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the matrix $BD^{-1} + (D^t)^{-1} X (I + YX)^{-1} D^{-1}$ is of the form

$$\begin{bmatrix} 0 & z + \dfrac{x(ad - bc)}{1 - xy} \\ -z - \dfrac{x(ad - bc)}{1 - xy} & 0 \end{bmatrix}.$$

Thus we have

$$z + \frac{x(ad - bc)}{1 - xy} \geq 0 \text{ for all } x \geq 0.$$

It follows that $\det(D) = ad - bc > 0$. We have

$$\begin{bmatrix} (D^{-1})^t & O \\ O & D \end{bmatrix} \in H_\circ. \tag{7}$$

By (4), (5), (6), and (7), we have $g \in \Gamma^+ H_\circ \Gamma^-$. $\square$

THEOREM 4 4. We have $\Gamma_\Omega = \{g \in G : g \cdot \Omega \subset \Omega\}$.

Proof. Let $S = \{g \in G : g \cdot \Omega \subset \Omega\}$ and let $g \in \Gamma_\Omega$. Since $g \in \Gamma_\Omega = \Gamma^+ H_\circ \Gamma^-$, the element g is of the form

$$\begin{bmatrix} I & A \\ O & I \end{bmatrix} \begin{bmatrix} (D^{-1})^t & O \\ O & D \end{bmatrix} \begin{bmatrix} I & O \\ B & I \end{bmatrix}.$$

By Theorem 3.3, we have

$$g\begin{bmatrix} I & X \\ O & I \end{bmatrix} \in \begin{bmatrix} I & A+(D^t)^{-1}X(I+BX)^{-1}D^{-1} \\ O & I \end{bmatrix} P$$

for all

$$X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \in \Omega$$

Let

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

for some $a \geq 0, b \leq 0$. Then $g \cdot X = A + (D^t)^{-1}X(I + BX)^{-1}D^{-1}$ is of the form

$$\begin{bmatrix} 0 & a + \dfrac{x\det(D)}{1-xb} \\ -(a + \dfrac{x\det(D)}{1-xb}) & 0 \end{bmatrix}.$$

Since $\det(D) > 0$ and since $x > 0$,

$$a + \frac{x\det(D)}{1-xb} > 0$$

Therefore $g \cdot X \in \Omega$ and hence $\Gamma_\Omega \subset S$.

Conversely, suppose that $g \in S$ Set for each $n \in \mathbb{N}$,

$$h_n = \begin{bmatrix} I & X_n \\ O & I \end{bmatrix}, \text{ where } X_n = \begin{bmatrix} 0 & \frac{1}{n} \\ -\frac{1}{n} & 0 \end{bmatrix}$$

Then $h_n \in \Gamma_\Omega$ and $h_n \cdot \overline{\Omega} \subset \Omega$ It follows that

$$gh_n \ \overline{\Omega} \subset g \cdot \Omega \subset \Omega \subset \overline{\Omega}$$

Thus we have $gh_n \in \Gamma_\Omega$ for all $n \in \mathbb{N}$. Since h_n converges to the identity of G and since Γ_Ω is closed, we have

$$g = \lim_{n\to\infty} gh_n \in \Gamma_\Omega$$

This completes the proof. □

Let

$$\mathrm{SL}(2,\mathbb{R})^+ := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{R}) \cdot a, b, c, d \geq 0 \right\}.$$

Then $\mathrm{SL}(2,\mathbb{R})^+$ is a closed subsemigroup of $\mathrm{SL}(2,\mathbb{R})$ with the following factorization.

LEMMA 4.5. We have $\mathrm{SL}(2,\mathbb{R})^+ = UDL$, where

$$U = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \geq 0 \right\}, L = \left\{ \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} : x \geq 0 \right\}$$

and

$$D = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a > 0 \right\}$$

Proof. Proposition 3.2 in [6]. □

Now we let

$$H' = \left\{ \begin{bmatrix} \lambda I & O \\ O & \frac{1}{\lambda} I \end{bmatrix} \ \lambda > 0 \right\}$$
$$H'' = \left\{ \begin{bmatrix} D & O \\ O & (D^{-1})^t \end{bmatrix} : D \in \mathrm{SL}(2,\mathbb{R}) \right\}.$$

Then H' and H'' are subgroups of $H_\circ$ with $H_\circ = H''H'$.

THEOREM 4.6. We have

(1) $H' \exp \mathbf{C} = \Gamma^+ H' \Gamma^-$ and is a subsemigroup of $H_\circ \exp \mathbf{C}$.
(2) $\Gamma^+ H' \Gamma^-$ is isomorphic to $\mathrm{SL}(2,\mathbb{R})^+$.

Proof. (1) Let $g, h \in \Gamma^+ H' \Gamma^-$ Then

$$g = \begin{bmatrix} I & X \\ O & I \end{bmatrix} \begin{bmatrix} \lambda I & O \\ O & \frac{1}{\lambda} I \end{bmatrix} \begin{bmatrix} I & O \\ Y & I \end{bmatrix}$$
$$h = \begin{bmatrix} I & X' \\ O & I \end{bmatrix} \begin{bmatrix} \lambda' I & O \\ O & \frac{1}{\lambda'} I \end{bmatrix} \begin{bmatrix} I & O \\ Y' & I \end{bmatrix} \tag{8}$$

for some $\lambda, \lambda' > 0$ $x, x' \geq 0$ and $y, y' \leq 0$ with

$$X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}, \quad X' = \begin{bmatrix} 0 & x' \\ -x' & 0 \end{bmatrix},$$
$$Y = \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix}, \quad Y' = \begin{bmatrix} 0 & y' \\ -y' & 0 \end{bmatrix}.$$

We have

$$gh = \begin{bmatrix} I & A \\ O & I \end{bmatrix} \begin{bmatrix} (D^{-1})^t & O \\ O & D \end{bmatrix} \begin{bmatrix} I & O \\ B & O \end{bmatrix}, \tag{9}$$

where

$$A = \begin{bmatrix} 0 & \dfrac{\lambda^2 x' - xyx' + x}{1 - yx'} \\ -\dfrac{\lambda^2 x' - xyx' + x}{1 - yx'} & 0 \end{bmatrix} \in \mathrm{skew}(2, \mathbb{R})^+$$

$$B = \begin{bmatrix} 0 & \dfrac{\lambda'^2 y - yx'y' + y'}{1 - yx'} \\ -\dfrac{\lambda'^2 y - yx'y' + y'}{1 - yx'} & 0 \end{bmatrix} \in \mathrm{skew}(2, \mathbb{R})^-$$

$$D = \frac{1 - yx'}{\lambda\lambda'} I, \ \frac{1 - yx'}{\lambda\lambda'} > 0$$

Thus we have $gh \in \Gamma^+ H' \Gamma^-$ Therefore, $\Gamma^+ H' \Gamma^-$ is a subsemigroup of Γ_Ω. Now since H' is closed, $\Gamma^+ H' \Gamma^-$ is a closed subsemigroup of G By considering the tangent wedge of this semigroup, we conclude that $H' \exp \mathbf{C} \subset \Gamma^+ H' \Gamma^-$ Conversely, suppose that $g \in \Gamma^+ H' \Gamma^-$. Then since $g \in \Gamma_\Omega = H_\circ \exp \mathbf{C}$, $g = h \exp C$ for some $h \in H_\circ, C \in \mathbf{C}$. We show that $h \in H'$ Let

$$C = \begin{bmatrix} O & X \\ Y & O \end{bmatrix} \text{ with } X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -y \\ y & 0 \end{bmatrix}, x, y \geq 0.$$

Then by a direct matrix computation, the $H_\circ$-part of $\exp C$ is

$$h_0 := \begin{bmatrix} \mathrm{sech}(xy)^{\frac{1}{2}} I & 0 \\ 0 & \cosh(xy)^{\frac{1}{2}} I \end{bmatrix}.$$

Therefore $hh_0 \in H'$ which implies that $h \in H'$

(2) Define a mapping $f : \Gamma^+ H' \Gamma^- \to \mathrm{SL}(2, \mathbb{R})^+$ by

$$f(g) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix}$$

for

$$g = \begin{bmatrix} I & X \\ O & I \end{bmatrix} \begin{bmatrix} \lambda I & O \\ O & \frac{1}{\lambda} I \end{bmatrix} \begin{bmatrix} I & O \\ Y & I \end{bmatrix} \in \Gamma^+ H' \Gamma^-,$$

with

$$X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix}$$

Since the factorization of $\Gamma^+ H'\Gamma^-$ is unque and since the factorization of $\mathrm{SL}(2,\mathbb{R})^+$ is unique, the map f is well-defined and is injective. Clearly, f is surjective To complete the proof, it remains that f is a homomorphism from $\Gamma^+ H'\Gamma^-$ to $\mathrm{SL}(2,\mathbb{R})^+$.

Let $g, h \in \Gamma^+ H'\Gamma^-$ be of the form (8) Then by (9) we have

$$f(gh) = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \rho & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -l & 1 \end{bmatrix},$$

where

$$\begin{aligned} u &= \frac{\lambda^2 x' - xyx' + x}{1 - yx'} \geq 0 \\ l &= \frac{\lambda'^2 y - yx'y' + y'}{1 - yx'} \leq 0 \\ \rho &= \frac{\lambda\lambda'}{1 - yx'} > 0. \end{aligned}$$

Since

$$\begin{aligned} f(g) &= \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix} \\ f(h) &= \begin{bmatrix} 1 & x' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda' & 0 \\ 0 & \frac{1}{\lambda'} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -y' & 1 \end{bmatrix}, \end{aligned}$$

we have

$$f(g)f(h) = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where

$$\begin{aligned} b &= \frac{\lambda}{\lambda'} x' - \frac{1}{\lambda\lambda'} xyx' + \frac{1}{\lambda\lambda'} x \geq 0 \\ c &= -\frac{\lambda'}{\lambda} y + \frac{1}{\lambda\lambda'} yx'y - \frac{1}{\lambda\lambda'} y' \geq 0 \\ d &= \frac{1 - yx'}{\lambda\lambda'} > 0. \end{aligned}$$

Since $d \neq 0$,

$$f(g)f(h) = \begin{bmatrix} 1 & bd^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d^{-1} & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d^{-1}c & 1 \end{bmatrix}$$

and $bd^{-1} = u, d^{-1}c = -l$ and $d^{-1} = \rho$ The proof now is complete $\square$

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