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ON MAXIMAL PRERADICAL RATIONAL EXTENSIONS

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ABSTRACT The concepts of t-rational extensions and t-essential extensions of modules, where t a preradical for R-Mod, are introduced The structures of such extensions are determined Relations between maximal t-rational extensions and other concepts of modules are studied.

1. Introduction

The notion of rational extensions of modules was introduced by Utumi [7] and Findlay-Lambek [2]. For a preradical t, as torsion theoritically, we will define t-rational extensions of modules This is a dual concept of t-corational extensions [5].

In general, rational extensions do not preserve direct sums. However, t-rational extensions preserve direct sums In this paper, first, we determine the form of t-rational extension of t-torsionfree module

Next, we show that every t-torsion free module has the maximal t-rational extention uniquely up to isomorphism. Moreover, we characterize t-rationally complete module

Throughout this paper, R denotes a ring with identity and all modules are unitary left R-modules. We denote the category of all modules by R-Mod and the injective hull of a module A by $E(A)$.

For a preradical t of R-Mod, a module A is said to be *t-torsion* (resp. *t-torsionfree*) if $t(A) = A$ (resp. $t(A) = 0$) The t-torsion class (resp t-torsionfree class) of modules or the class of t-torsion (resp

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t-torsionfree) modules will be denoted by $T(t)$ (resp $F(t)$). Also, for two preradicals t and s of R-Mod, we shall say that *s is less than t* (or *t is larger than s*) if $s(A) \subseteq t(A)$ for every module A.

We denote the left linear topology corresponding to a left exact preradical t by $L(t)$, that is,

$$L(t) = \{_R I \leq R \mid R/I \in T(t)\}$$

Also, for each module M, $Z(M)$ denotes the singular submodule of M, that is,

$$Z(M) = \{x \in M \mid Ix = 0, \ for\ some\ essential\ left\ ideal\ I\ of\ R\}$$

We refer the reader to Stenstròm [6] for additional terminology and properties of preradicals and torsion theories.

2. t-rational extensions

Let Q be a module We define a preradical k_Q by

$$k_Q(M) = \cap\{Kerf \mid f \in Hom_R(M, Q)\}$$

for all modules M. As is well known, k_Q is the largest preradical t such that $t(Q) = 0$ In fact, k_Q is a radical All preradicals in this paper are over R-Mod for a fixed ring R Let t be a preradical We call an exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B$ is a *t-rational extension* of A if B is in $F(t)$ and any submodule of $Coker(\alpha)$ is in $T(t)$ If α is an inclusion map, then we say that A is a *t-rational submodule* of B. It is immediate that an exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B$ is a rational extension of A if and only if it is a k_B-rational extension of A Also, if an exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B$ is a t-rational extension of A, then it is a rational extension of A

The converse of the above remark is not true in general, as follows:

EXAMPLE 2 1. Let R be the ring $\mathbf{Z}$ of integers We put $A = 12\mathbf{Z}$, $\mathbf{B} = \mathbf{Z}$ and $t = Soc$ Since $Z(B) = 0$ and A is an essential submodule of B, $0 \longrightarrow A \xrightarrow{\imath} B$ is a rational extension of A, where $\imath$ an inclusion map. However, $Soc(B/A) = 2\mathbf{Z}/12\mathbf{Z} \neq B/A$. Thus $0 \longrightarrow A \xrightarrow{\imath} B$ is not a t-rational extension of A

On the other hand, for any preradical t, an exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B$ is called a *t-essential extension* of A if it is an essential extension of A and every submodule of $Coker(\alpha)$ is in $T(t)$. If α is an inclusion map, we say that A is a *t-essential submodule* of B

Obviously, we see that if $0 \longrightarrow A \xrightarrow{\alpha} B$ is a t-rational extension of A, then it is a t-essential extension of A.

The converse of this remark is not true in general, as follows:

EXAMPLE 2 2. Let R be the ring $\mathbf{Z}$ of rational integers. We put $A = 4\mathbf{Z}/8\mathbf{Z}$, $B = \mathbf{Z}/8\mathbf{Z}$ and $t = Z$ be the singular torsion functor. Then $Z(B) = B$, and so $Z(B/A) = B/A$ Also since A is an essential submodule of B, $0 \longrightarrow A \xrightarrow{i} B$ is a t-essential extension of A, where i is an inclusion map However, $0 \longrightarrow A \xrightarrow{i} B$ is not a t-rational extension of A

We note that for any left exact preradical t, an exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B$ is a t-rational extension of A if and only if B is in $F(t)$ and $Coker(\alpha)$ is in $T(t)$.

Let t be a preradical We say that a left R-module A is *t-injective* if the functor $Hom_R(-, A)$ preserves exactness for all exact sequences

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

with $C'' \in T(t)$.

PROPOSITION 2 3. Let t be a left exact radical, and $0 \longrightarrow A \xrightarrow{\alpha} B$ and $0 \longrightarrow B \xrightarrow{\beta} C$ be two exact sequences Then $0 \longrightarrow A \xrightarrow{\alpha} B$ is a t-rational extension of A and $0 \longrightarrow B \xrightarrow{\beta} C$ is a t-rational extension of B if and only if $0 \longrightarrow A \xrightarrow{\beta\alpha} C$ is a t-rational extension of A.

Proof Assume that both $0 \longrightarrow A \xrightarrow{\alpha} B$ and $0 \longrightarrow B \xrightarrow{\beta} C$ are t-rational extensions Then since

$$0 \longrightarrow \beta(B)/\beta\alpha(A) \longrightarrow C/\beta\alpha(A) \longrightarrow C/\beta(B) \longrightarrow 0$$

is exact,

$$(C/\beta\alpha(A))/(\beta(B)/\beta\alpha(A)) \cong C/\beta(B)$$

On the other hand, since $\beta(B)/\beta\alpha(A) \cong B/\alpha(A)$, $\beta(B)/\beta\alpha(A)$ and $C/\beta(B)$ are in $T(t)$, we see that $C/\beta\alpha(A)$ is in $T(t)$. Thus $0 \longrightarrow A\xrightarrow{\beta\alpha}C$ is a t-rational extension of A.

Conversely, suppose that $0 \longrightarrow A\xrightarrow{\beta\alpha}C$ is a t-rational extension of A We must show that $B/\alpha(A)$ and $c/\beta(B)$ are in $T(t)$. Since

$$B/\alpha(A) \cong \beta(B)/\beta\alpha(A) \subseteq C/\beta\alpha(A),$$

$B/\alpha(A)$ is in $T(t)$. On the other hand, since

$$C/\beta(B) \cong (C/\beta\alpha(A))/(\beta(B)/\beta\alpha(A))$$

which is contained in $T(t)$, it follows that $C/\beta(B)$ is in $T(t)$. This completes the proof. □

PROPOSITION 2.4. Let t be a left exact preradical and $\{0 \longrightarrow A_\lambda\xrightarrow{\alpha_\lambda}B_\lambda\}_{\lambda\in\Lambda}$ be a family of t-rational extensions. Then

$$0 \longrightarrow \oplus_{\lambda\in\Lambda}A_\lambda \overset{\oplus_{\lambda\in\Lambda}\alpha_\lambda}{\longrightarrow} \oplus_{\lambda\in\Lambda} B_\lambda$$

is a t-rational extension

Proof. Since B_λ is in $F(t)$ for all $\lambda \in \Lambda$, $\oplus_{\lambda\in\Lambda}B_\lambda$ is in $F(t)$. Also,

$$f \ : \ \oplus_{\lambda\in\Lambda}B_\lambda/ \oplus_{\lambda\in\Lambda} \alpha_\lambda(A_\lambda) \longrightarrow \oplus_{\lambda\in\Lambda}(B_\lambda/\alpha_\lambda(A_\lambda)),$$

which is defined by

$$f((b_\lambda)_{\lambda\in\Lambda} + \oplus_{\lambda\in\Lambda}\alpha_\lambda(A_\lambda)) = (b_\lambda) + \alpha_\lambda(A_\lambda))_{\lambda\in\Lambda}$$

is a monomorphism Since $\oplus_{\lambda\in\Lambda}(B_\lambda/\alpha_\lambda(A_\lambda))$ is in $T(t)$,

$$\oplus_{\lambda\in\Lambda}B_\lambda/ \oplus_{\lambda\in\Lambda} \alpha_\lambda(A_\lambda)$$

is in $T(t)$ Thus $0 \longrightarrow \oplus_{\lambda\in\Lambda}A_\lambda \overset{\oplus_{\lambda\in\Lambda}\alpha_\lambda}{\longrightarrow} \oplus_{\lambda\in\Lambda} B_\lambda$ is a t-rational extension □

Let t be a preradical. We call a module A *t-uniform (resp strongly t-uniform)* if every nonzero submodule of A is t-essential (resp t-rational) submodule Clearly, every strongly t-uniform module is t-uniform. Also, every nonzero submodule of a strongly t-uniform module is strongly t-uniform If $_RR$ is t-uniform (resp. strongly t-uniform), we say that the ring R is *left t-uniform (resp. left strongly t-uniform)*

If A is in $F(t)$ and t-uniform, then it is strongly t-uniform.

PROPOSITION 2 5. Let t be a left exact radical and M be a submodule of a module N If M is both a strongly t-uniform and a t-rational submodule of N, then N is strongly t-uniform.

Proof Let N' be a nonzero submodule of N Then $0 \neq N' \cap M$ is a t-rational submodule of M. Also, since $N' \cap M \subseteq M \subsetneq N$ and M is a t-rational submodule of N, $N' \cap M$ is a t-rational submodule of N, by Proposition 2 3 Thus N' is a t-rational submodule of N Consequently, N is strongly t-uniform □

A ring R is called a *domain* if R has no nonzero divisors of zero and a *left Ore domain* if, in addition, $Ra \cap Rb \neq 0$, for all $a \neq 0$ and $b \neq 0$ in R.

LEMMA 2 6. A ring R is left Ore domain if and only if $Z({}_RR) = 0$ and R is left uniform

Proof. Suppose that R is left Ore domain Then clearly, R is left uniform and so that $L(Z)$, the family of essential left ideals of R, actually consists of all nonzero left ideals. If $a \in Z({}_RR)$, $Ia = 0$, for some $I \in L(Z)$. Thus $a = 0$.

Conversely, assume that $ab = 0$ and $a \neq 0$ in R. Then $Rab = 0$ and $Ra \in L(Z)$, that is, $b \in Z(R) = 0$. Consequently, R is a domain □

COROLLARY 2 7. Let t be a preradical. If R is left strongly t-uniform, then R is a left Ore domain The converse is not true in general

Proof Assume that R is strongly t-uniform Then clearly, R is left uniform and $Z = k_{E(R)}$ which is indicated previously, that is, $Z({}_RR) = 0$ Therefore, by Lemma 2 6, R a is left Ore domain

Next, we put $R = \mathbf{Z}$, $t = Soc$ and $I = 8\mathbf{Z}$ Then R is left Ore domain But

$$Soc(R/I) = 4\mathbf{Z}/8\mathbf{Z} \neq R/I$$

This means I is not even a t-rational submodule of R □

THEOREM 2 8. Let t be a preradical and $0 \longrightarrow A \xrightarrow{\alpha} B$ is a t-rational extension of A and $0 \longrightarrow A \xrightarrow{\alpha'} B'$ an exact sequence If there exist homomorphisms $f : B \longrightarrow B'$ and $g : B' \longrightarrow B$ such that $f\alpha = \alpha'$ and $g\alpha' = \alpha$, then $gf = 1_B$

Proof. By the assumption, $gf\alpha = \alpha$ and so $(gf - 1_B)\alpha = 0$. Thus we have

$$\alpha(A) \subseteq Ker(gf - 1_B)$$

and

$$B/Ker(gf - 1_B) \cong Im(gf - 1_B) \subseteq B$$

Since $B/Ker(gf - 1_B)$ is t-torsionfree and $B/\alpha(A)$ is t-torsion, we see that $B/Ker(gf - 1_B) = 0$, that is, $Ker(gf - 1_B) = B$. Hence $gf - 1_B = 0$, that is, $gf = 1_B$. □

COROLLARY 2.9. Let t be a preradical, and both $0 \longrightarrow A \xrightarrow{\alpha} B$ and $0 \longrightarrow A \xrightarrow{\alpha'} B'$ t-rational extensions of A. If there exist homomorphisms $f : B \longrightarrow B'$ and $g : B' \longrightarrow B$ such that $f\alpha = \alpha'$ and $g\alpha' = \alpha$, then $B \cong B'$.

3. Maximal t-rational extensions

Let t be a preradical. We call an exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B$ a *maximal t-rational extension* of A if

(i) $0 \longrightarrow A \xrightarrow{\alpha} B$ is a t-rational extension of A,

(ii) For any t-rational extension $0 \longrightarrow A \xrightarrow{\alpha'} B'$ of A , there exists a homomorphism $f : B' \longrightarrow B$ such that $f\alpha' = \alpha$.

By Corollary 2.9, we obtain the following important statement

THEOREM 3 1. Let t be a left exact radical and A a t-torsion free module Then there exists a maximal t-rational extension of A, uniquely up to isomorphism.

Proof We put that $t(E(A)/A) = B/A$, where $A \subseteq B \subseteq E(A)$ Then as is easily seen, $0 \longrightarrow A \xrightarrow{i} B$ is a t-rational extension of A where i is an inclusion map. Let $0 \longrightarrow A \xrightarrow{\alpha'} B'$ be any t-rational extension of A. Then we have the following commutative diagram.

$$\begin{array}{ccccc} 0 \longrightarrow & A & \xrightarrow{\alpha'} & B' \\ & {\scriptstyle i'}\downarrow & \swarrow & \\ & E(A) & & \end{array}$$

where i' is an inclusion map. Thus $f\alpha' = i'$ and f is monomorphism, we show that $f(B') \subseteq B$, since $f\alpha'(A) = A$ and $\alpha(A) \subseteq B'$, $A \subsetneq Imf$. Also since

$$(Imf + B)/B \cong Imf/(Imf \cap B) \cong (Imf/A)/((Imf \cap B)/A)$$

and

$$Imf/A \;\cong\; B'/\alpha'(A)$$

$(Imf + B)/B$ is in $T(t)$. Since

$$0 \longrightarrow B/A \longrightarrow (Imf + B)/A \longrightarrow (Imf + B)/B \longrightarrow 0$$

is exact, B/A and $(Imf + B)/B$ are in $T(t)$. Thus $(Imf + B)/A$ is in $T(t)$.

Moreover,

$$B/A = t(E(A)/A) \supseteq t((Imf + B)/A) = (Imf + B)/A$$

and

$$t(B/A) \;=\; B/A.$$

Thus $B = Imf + B$, that is, $Imf \subseteq B$. Hence $0 \longrightarrow A \xrightarrow{i} B$ is a maximal t-rational extension of A.

Finally, let two exact sequences $0 \longrightarrow A \xrightarrow{\alpha_1} B_1$ and $0 \longrightarrow A \xrightarrow{\alpha_2} B_2$ be maximal t-rational extensions of A. Then by Corollary 2.9, we have $B_1 \cong B_2$ □

LEMMA 3 2 ([3], Proposition 3.3). Let t be a preradical and $0 \longrightarrow A \xrightarrow{\alpha} B$ an exact sequence If B is t-injective and $Coker(\alpha)$ is in $F(t)$, then A is t-injective.

Proof. Since $\alpha(A) \cong A$, we will show that $\alpha(A)$ is t-injective. Let

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Y/X \longrightarrow 0$$

be an exact sequence with $Y/X \in T(t)$ Then there exists a homomorphism $g: Y \longrightarrow B$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & Y & \longrightarrow & Y/X & \longrightarrow & 0 \\ & & & & {\scriptstyle f}\downarrow \swarrow & & & & \\ & & & & \alpha(A) & & & & \\ & & & & {\scriptstyle j}\downarrow & & & & \\ & & & & B & & & & \end{array}$$

Thus g induces a homomorphism $g' \cdot Y/X \longrightarrow B/\alpha(A)$, since Y/X is in $T(t)$ and $B/\alpha(A)$ is in $F(t)$, $g' = 0$. Hence $g(Y) \subseteq \alpha(A)$, that is, $\alpha(A)$ is t-injective. □

THEOREM 3.3. Let t be a left exact radical If $0 \longrightarrow A \xrightarrow{\alpha} B$ is the maximal t-rational extension of A, then B is t-injective.

Proof From Lemma 3 2 and the method of construction of maximal t-rational extensions of A in Theorem 3.1, B is t-injective. □

Let t be a preradical. We call a module A *t-rationally complete* if $0 \longrightarrow A \xrightarrow{\alpha} B$ a t-rational extension of A implies A is isomorphic to B.

THEOREM 3.4. Let t be a left exact radical and A a t-torsion free module. Then A is t-rationally complete if and only if A is t-injective.

Proof Assume that A is t-rationally complete By Theorem 3 1, there exists a maximal t-rational extension $0 \longrightarrow A \xrightarrow{\alpha} B$ of A. From Theorem 3.3, B is t-injective

Conversely, suppose that A is t-injective. If $0 \longrightarrow A \xrightarrow{\alpha} B$ is a t-rational extension of A, then the fact that $B/\alpha(A)$ is in $T(t)$ implies $\alpha(A)$ is a direct summand of B. Also, since $\alpha(A)$ is an essential submodule of B, $B = \alpha(A)$, that is, $A \cong B$ Thus A is t-rationally complete. □

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