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ON SLIGHTLY α-CONTINUOUS FUNCTIONS

G.I. CHAE, T. NOIRI AND J.S. KIM

ABSTRACT. In [11] the feeble continuity is introduced and then the weak and strong forms of feeble (or, equivalently α-continuity) continuity are studied. In this note, we introduce a type of function called a slightly α-continuous function and study several properties of it

1. Introduction

Since the concept of feeble continuity is introduced in [9], the weak and strong forms of it are defined and studied here and there. For example, after a year Mashhour, Hasanein and El-Deeb have defined α-continuity in [13] and the notion of almost feeble continuity is, in [12], defined and studied its properties and relations. Among them feeble continuity and α-continuity are equivalent because it is proved in [6] that feebly open sets coincide with α-open sets.

We denote topological spaces by X, Y and Z on which no separation axioms are assumed, and the closure and the interior of a subset S of X by $\mathrm{Cl}_X(S)$ and $\mathrm{Int}_X(S)$ (simply, $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$), respectively. S is said to be *semi-open* [7] if there exists an open set O such that $O \subset S \subset Cl(O)$ and its complement is called *semi-closed*. The intersection of all semi-closed sets containing S is called the *semi-closure* of S and denoted by $sCl(S)$. $S \subset X$ is said to be *α-open*

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if $S \subset \text{Int}(\text{Cl}(\text{Int}(S)))$ and its complement is called *α-closed.* The intersection of all α-closed sets containing S is called the *α-closure* of S and denoted by $\alpha Cl(S)$. It is known in [9] that a *feebly open* set, which coincides with an α-open set, is defined as a set if there is an open set U such that $U \subset S \subset sCl(U)$.

Throughout this paper, we also denote the family of all α-open (resp. semi-open, open and clopen) sets of X by $\alpha O(X)$ (resp. $SO(X)$, $\tau(X)$ and $CO(X)$), and denote the family of α-open (resp. semi-open, open and clopen) sets of X containing x by $\alpha O(X,x)$ (resp. $SO(X,x)$, $\tau(X,x)$ and $CO(X,x)$).

DEFINITION 1.1. A function $f : X \to Y$ is called semi-continuous ($s.C.$) [8] (resp. almost semi-continuous ($a.s.C.$) [1], semi θ-continuous ($s.\theta.C.$) [1] and weakly semi-continuous ($w.s.C.$) [1]) if for each $x \in X$ and each $V \in \tau(Y, f(x))$, there exists $U \in SO(X,x)$ such that $f(U) \subset V$ (resp. $f(U) \subset \text{Int}(\text{Cl}(V))$, $f(s\text{Cl}(U)) \subset \text{Cl}(V)$ and $f(U) \subset \text{Cl}(V)$)

DEFINITION 1.2. A function $f : X \to Y$ is called *slightly semi-continuous* ($sl.s.C.$) [15] (resp. *slightly continuous* ($sl.C.$) [4]) if for each $x \in X$ and each $V \in CO(Y, f(x))$, there exists $U \in SO(X,x)$ (resp. $U \in \tau(X,x)$) such that $f(U) \subset V$.

DEFINITION 1.3 A function $f : X \to Y$ is called *almost continuous* ($a.C.$) [17] (resp. *θ-continuous* ($\theta.C$) [3] and *weakly continuous* ($w.C.$) [7]) if for each $x \in X$ and each $V \in \tau(Y, f(x))$, there is $U \in \tau(X,x)$ such that $f(U) \subset \text{Int}(\text{Cl}(V))$ (resp. $f(\text{Cl}(U)) \subset \text{Cl}(V)$ and $f(U) \subset \text{Cl}(V)$).

2. Slightly α-continuous functions

DEFINITION 2 1. A function $f : X \to Y$ is called *slightly α-continuous* ($sl.\alpha.C.$) if for each $x \in X$ and each $V \in CO(Y, f(x))$, there exists $U \in \alpha O(X,x)$ such that $f(U) \subset V$.

THEOREM 2.1. *For a function $f : X \to Y$, the following are equivalent :*

(a) f is sl.α.C.,
(b) $f^{-1}(V) \in \alpha O(X)$ for each $V \in CO(Y)$,
(c) $[X - f^{-1}(V)] \in \alpha O(X)$ for each $V \in CO(Y)$.

Proof (a) $\Rightarrow$ (b) : Let $V \in CO(Y)$ and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and there is $U_x \in \alpha O(X, x)$ such that $f(U_x) \subset V$ since f is *sl.α.C.* Thus we have $f^{-1}(V) = \bigcup\{U_x : x \in f^{-1}(V)\}$ and so $f^{-1}(V)$ is the union of α-open sets. Hence $f^{-1}(V) \in \alpha O(X)$ because $\alpha O(X)$ is a topology on X. The remainders of proof are easy and are thus omitted. □

The following are obtained easily since $\alpha O(X) \subset SO(X)$ in any space X.

THEOREM 2.2. *Slight continuity implies slight α-continuity.*

THEOREM 2.3. *Slight α-continuity implies slight semi-continuity.*

EXAMPLE 2.1. Let $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, X\}$. Define a function $f : (X, \tau) \to (X, \sigma)$ by $f(a) = f(c) = a$ and $f(b) = b$. Then it is easy to prove f is *sl.s.C.* However f is not *sl.α.C.* because $f^{-1}(\{a\}) = \{a, c\}$ is not α-open in (X, τ).

THEOREM 2.4. *If $f : X \to Y$ is sl.α.C. and $A \in \alpha O(X)$, then the restriction $f|A$ is sl.α.C.*

Proof Let $V \in CO(Y)$. Then $(f|A)^{-1}(V) = A \cap f^{-1}(V) \in \alpha O(X)$ since $\alpha O(X)$ is a topology on X. Therefore, $f|A$ is *sl.α.C.*□

DEFINITION 2.2. A function $f : X \to Y$ is said to be *α-irresolute* [10] if for each $V \in \alpha O(Y)$, $f^{-1}(V) \in \alpha O(X)$, and to be *pre-feebly-open* [6] if for each $U \in \alpha O(X)$, $f(U) \in \alpha O(Y)$.

THEOREM 2.5. *If $f : X \to Y$ is α-irreolute and $g : Y \to Z$ is sl.α.C., then $g \circ f$ is sl.α.C.*

Proof Let $V \in CO(Z)$. Then $g^{-1}(V) \in \alpha O(Y)$. Since f is α-irresolute, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \in \alpha O(X)$. Thus $g \circ f$ is *sl.α.C.* □

THEOREM 2.6. *Let $f : X \to Y$ be α-irresolute and pre-feebly-open surjection, and let $g : Y \to Z$ be a function. Then $g \circ f$ is sl.α.C. if and only if g is sl.α.C.*

Proof. Let $g \circ f$ be sl.α.C. and $V \in CO(Z)$. Then $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \in \alpha O(X)$. Since f is pre-feebly-open, $f(f^{-1}(g^{-1}(V))) \in \alpha O(Y)$. Hence $g^{-1}(V) \in \alpha O(Y)$. Thus g is *sl.α.C.* We have its opposite from Theorem 2.5. □

The following diagram is obtained from the above and the references:

$$\begin{array}{ccccccccccc}
C. & \Rightarrow & a.C. & \Rightarrow & \theta.C. & \Rightarrow & w.C. & \Rightarrow & sl.C. & \Rightarrow & sl.\alpha.C. \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & \swarrow & \\
s.C. & \Rightarrow & a.s.C. & \Rightarrow & \theta.s.C. & \Rightarrow & w.s.C. & \Rightarrow & sl.s.C. & &
\end{array}$$

3. More Characterizations and Comparisions

It is well known that a filterbase $\mathcal{B}$ in X is said to be *residually* in $U \subset X$ if there is $B \in \mathcal{B}$ such that $B \subset U$, and a net $\{s_\lambda : \lambda \in D\}$ in X is said to be *resıdully* in $U \subset X$ if there is a $\lambda_0 \in D$ such that $\lambda_0 \leq \lambda$ implies $\{s_\lambda\} \in U$. We say that a filterbase $\mathcal{B}$ in X converges to $x \in X$ ıf $\mathcal{B}$ ıs residully in every $U \in \tau(X, x)$ and a net $\{s_\lambda\}_{\lambda \in D}$ in X converges to $x \in X$ if $\{s_\lambda\}_{\lambda \in D}$ ıs residully in every $U \in \tau(X, x)$.

In [11] a filterbase having a concept, which is weaker than one of convergent filterbase, is defined to study more properties of α-irresolute functions. We defined the following to obtain more characterizations of *sl.α.C.* function.

DEFINITION 3.1. A filterbase $\mathcal{B}$ in X is said to be *α-converge* (resp. *c-converge*) to $x \in X$ [11] if $\mathcal{B}$ is residully in U for each $U \in \alpha O(X, x)$ (resp. $U \in CO(X, x)$).

DEFINITION 3.2. Let $(D, \leqslant)$ be a directed set. A net $\{s_\lambda : \lambda \in D\}$ in X is said to *α-converge* (resp. *c-converge*) to $x \in X$ if $\{s_\lambda\}_{\lambda \in D}$ is residually in U for each $U \in \alpha O(X, x)$ (resp. $U \in CO(X, x)$).

THEOREM 3.1. *For a function $f : X \to Y$, the following are equivalent:*

(a) f is sl.α.C. at x.

(b) If a filterbase $\mathcal{B}$ in X is residually in each $U \in \alpha O(X,x)$, then $f(\mathcal{B})$ in Y is residually in every $V \in CO(Y, f(x))$.

(c) If a net $\{s_\lambda\}_{\lambda\in D}$ in X is residually in each $U \in \alpha O(X,x)$, then $\{f(s_\lambda)\}_{\lambda \in D}$ is residually in every $V \in CO(Y, f(x))$.

Proof. (a) $\Rightarrow$ (b) Let (a) be true and $V \in CO(Y, f(x))$ and $U \in \alpha O(X,x)$ such that $f(U) \subset V$. Assume a filterbase $\mathcal{B}$ in X is residully in each $U \in \alpha O(X,x)$. Then there is $E \in \mathcal{B}$ such that $E \subset U$ So we have $f(E) \subset f(U) \subset V$, which proves (b).

(b) $\Rightarrow$ (c)· Let (b) be true and $V \in CO(Y, f(x))$. Assume a net $\{s_\lambda\}_{\lambda\in D}$ is residually in each $U \in \alpha O(X,x)$. Thus there is $\lambda_0 \in D$ such that $\lambda_0 \leq \lambda$ implies $s_\lambda \in U$. To show (c) let $E_k = \{s_\lambda : k \leq \lambda\}$ and $\mathcal{B} = \{E_k\}$. Then $\mathcal{B}$ is also residually in the U since it is a filterbase in X which is generated by $\{s_\lambda\}_{\lambda\in D}$ Thus from (b), $f(\mathcal{B}) = \{f(E_k)\}$ is residually in $V \in CO(Y, f(x))$, that is, there is an $f(E_{k_0}) \in f(\mathcal{B})$ such that $f(E_{k_0}) \subset V$ and there is thus a $k_0 \in D$ such that $f(s_{k_0}) \in V$ and $k_0 \leq \lambda$ implies $f(s_\lambda) \in V$ because $E_{k_0} = \{s_\lambda : k_0 \leq \lambda\}$ Hence $\{f(s_\lambda)\}_{\lambda\in D}$ is is residually in V. So (c) holds.

(c) $\Rightarrow$ (a)· Suppose that f is not $sl.\alpha.C$ at $x \in X$. Then there exists a $V \in CO(Y, f(x))$ such that $f(U) \not\subset V$ for each $U \in \alpha O(X,x)$. Thus $U \not\subset f^{-1}(V)$. For each $U \in \alpha O(X,p)$, we have $U \subset Y - f^{-1}(V) = f^{-1}(Y - V)$ So $U \cap f^{-1}(Y - V) \neq \emptyset$. In order to find a net not α-converging to $f(x)$, we may partially order $\alpha O(X,x)$ by set-inclusion and also direct it by $\leq$ as defined by $A \leq B$ iff $B \subset A$ for each $A, B \in \alpha O(X,x)$. Let $s : \alpha O(X,x) \to X$ be a selection function defined by $s(U) \equiv s_U \in U \cap f^{-1}(Y - V)$ for each $U \in \alpha O(X,x)$. Then $\{s_U\}_{U\in\alpha O(X,x)}$ is a net in X α-converging to x. Since $s_U \in U \cap f^{-1}(Y - V)$ and so $f(s_U) \in f[U \cap f^{-1}(Y - V)] \subset f(U) - V$, we have $f(s_U) \notin V$ for each $U \in \alpha O(X,x)$. Thus $\{f(s_U)\}_{U\in\alpha O(X,x)}$ is not residually in $V \in CO(Y, f(x))$. It contradicts. Thus f is $sl.\alpha.C.$ □

COROLLARY 3.1. *For a function $f : X \to Y$, the following are equivalent:*

(a) f is sl.α.C.

(b) For each $x \in X$ and each filterbase $\mathcal{B}$ in X α-converging to x, $f(\mathcal{B})$ c-converges to $f(x)$.

(c) For each $x \in X$ and each net $\{s_\lambda\}_{\lambda \in D}$ in X α-converging to x, $\{f(s_\lambda)\}_{\lambda \in D}$ c-converges $f(x)$.

DEFINITION 3.3. A space X is called;

(a) *α-Hausdorff* [11] (resp. *ultra Hausdorff* (written as UT_2) [18]) if every two distinct points of X can be separated by disjoint α-open (resp. clopen) sets,

(b) *ultra normal* [18] if each pair of nonempty disjoint closed sets can be separated by disjoint clopen sets,

(c) *mildly compact* [18] if every clopen cover of X has a finite subcover,

(d) *quasi H-closed* (written as QHC) [16] if every open cover of X has a finite proximate subcover,

(d) *F-closed* (written as FC) [2] if every α-open cover of X has a finite proximate subcover.

DEFINITION 3.4. A space X is called *α-normal* if each pair of nonempty disjoint closed sets can be separated by disjoint α-open sets.

THEOREM 3.2. *If $f : X \to Y$ is an sl.α.C. injection and Y is UT_2, then X is α-Hausdorff.*

Proof. Let x_1, $x_2 \in X$ and $x_1 \neq x_2$. Then there are V_1, $V_2 \in CO(Y)$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \emptyset$ because Y is UT_2. By Theorem 2.1, $x_i \in f^{-1}(V_i) \in \alpha O(X)$ for $i = 1, 2$. Since $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$, X is α-Hausdorff. □

THEOREM 3.3. *If $f : X \to Y$ is an sl.α.C. and closed injection and Y is ultra normal, then X is α-normal.*

Proof. Let F_1 and F_2 be any disjoint closed subsets of X. Since Y is ultra normal, two disjoint closed subsets of Y, $f(F_1)$ and $f(F_2)$, are separated by disjoint clopen sets V_1 and V_2, respectively. So by Theorem 2.1, $F_i \subset f^{-1}(V_i), f^{-1}(V_i) \in \alpha O(X)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_1) = \emptyset$. Thus X is α-normal. □

LEMMA 3.1. *QHC spaces coincide with FC spaces.*

Proof. Let X be FC. Then X is also QHC because $\tau(X) \subset \alpha O(X)$. Conversely, let X be QHC and let $\mathcal{G} = \{U_i \mid U_i \in \alpha O(X), i \in \nabla\}$ such that $X \subset \bigcup_{i\in\nabla} U_i$. Then for each $i \in \nabla$, $U_i \subset$ IntClInt(U_i) because $U_i \in \alpha O(X)$. Thus $X \subset \bigcup_{i\in\nabla}$ IntClInt(U_i). Since X is QHC and $\mathcal{G}^* = \{\text{IntClInt}(U_i) \mid i \in \nabla\}$ is an open cover of X, there exists a finite subset $\nabla_0 = \{i_1, i_2, \ldots, i_m\}$ of ∇ such that $X \subset \bigcup_{k=1}^{k=m}$ Cl(IntClInt(U_{i_k})) Since ClIntClInt$(U_{i_k}) \subset$ Cl(U_{i_k}) for $k = 1, 2, \ldots, m$, we have $X \subset \bigcup_{k=1}^{k=m}Cl(U_{i_k})$. Hence X is FC. □

THEOREM 3.4. *If $f : X \to Y$ is an sl.α.C surjection and X is quasi H-closed, then Y is mildly compact.*

Proof. Let $\{V_\lambda \mid V_\lambda \in CO(Y), \lambda \in \nabla\}$ be a cover of Y. Since f is $sl\ \alpha.C.$, $f^{-1}(V_\lambda) \in \alpha O(X)$ for each $\lambda \in \nabla$. Thus $\{f^{-1}(V_\lambda) \mid \lambda \in \nabla\}$ is an α-open cover of X. Since X is quasi H-closed and is thus FC from Lemma 3.1, there is a finite subclass $\nabla_\circ$ of ∇ such that $X = \bigcup_{\alpha\in\nabla_\circ}Cl(f^{-1}(V_\alpha))$. Since $f^{-1}(V_\alpha) \in \alpha O(X)$, $f^{-1}(V_\alpha) \subset$ IntClInt$(f^{-1}(V_\alpha))$ and so Cl$(f^{-1}(V_\alpha)) \subset$ ClIntClInt$(f^{-1}(V_\alpha)) \subset$ ClIntCl$(f^{-1}(V_\alpha))$. Moreover, by Theorem 2.1 $f^{-1}(V_\alpha)$ is α-closed and ClIntCl$(f^{-1}(V_\alpha)) \subset f^{-1}(V_\alpha)$ Cosequently, we obtain $X = \bigcup_{\alpha\in\nabla_0}Cl(f^{-1}(V_\alpha)) \subset \bigcup_{\alpha\in\nabla_0} f^{-1}(V_\alpha)$. Therefore, $Y = \bigcup_{\alpha\in\nabla_\circ} V_\alpha$. Hence Y is mildly compact. □

EXAMPLE 3.1. Let $(R,\mathcal{I})$ and $(R,\mathcal{U})$ be the indiscrete and the usual space of set of real numbers, respectively Then the identity I : $(R,\mathcal{I}) \to (R,\mathcal{U})$ is $sl.\alpha.C.$, but not $a.C$.

THEOREM 3.5. *If $f : X \to Y$ is sl.α.C. and Y is extremally disconnected, then f is a.C.*

Proof. Let $x \in X$ and $V \in \tau(Y, f(x))$. Since Y is extremally disconnected, $Cl(V) \in CO(Y)$ and by Theorem 2.1 $f^{-1}(Cl(V))$ is α-open and α-closed in X. Therefore, we have $x \in f^{-1}(V) \subset f^{-1}(\mathrm{Cl}(V)) \subset \mathrm{IntClInt}(f^{-1}(\mathrm{Cl}(V))) \subset \mathrm{ClIntCl}(f^{-1}(Cl(V))) \subset f^{-1}(\mathrm{Cl}(V))$. Putting $U = \mathrm{IntClInt}(f^{-1}(Cl(V)))$, U is an open set of X, $x \in U$ and $f(U) \subset \mathrm{Cl}(V) = \mathrm{IntCl}(V)$. This shows that f is $a.C.$□

REFERENCES

[1] Arya, S P and Bhamini, M B , *Some weaker forms of semi-continuous functions*, Ganita **33** (1982), 124 – 134

[2] Chae, G.I and Lee, D W , *F-closed spaces*, Kyungpook Math. J. **27(2)** (1987), 127–134.

[3] Fomin, S., *Extensions of topological spaces*, Ann of Math **44** (1943), 471 –480

[4] Jain, R C., *The Role of Regularly Open Sets in General Topology*, Ph.D Thesis, Meerut Univ , Institute of advanced studies, Meerut-India (1980)

[5] Jankovic, D S and Reilly, I L , *On Semi Separation Prperties*, Indian J. Pure Appl Math. **16(9)** (1985), 957–964

[6] Lee, H W., Chae, G I and Lee, D W , *Feebly irresolute functions*, Sunshin Women's University Report **21** (1985), 273–280

[7] Levine, N , *A decomposition of continuity in topological spaces*, Amer. Math Monthly **68** (1961), 44 – 46

[8] Levine, N , *Semi-open sets and semi-continuity in topological spaces*, Amer Math Monthly **70** (1963), 36–41

[9] Maheshwari, S N and Tapi, U.P , *Notes on some applications on feebly open sets*, Madhya Bharti J University of Saugar (1979)

[10] Maheshwari, S N and Thakur, S S , *On α-irresolute mappings*, Tamkang J Math **11** (1979), 209–214

[11] Maheshwari, S N , Chae, G I and Thakur, S.S , *On α-convergent filters*, U I.T Report **12(2)** (1981), 297–300

[12] Maheshwari, S N , Chae, G I and Jain, P.C., *Almost feebly continuous functions*, U.I T Report **13(1)** (1982), 195–197

[13] Mashhour, A.S , Hasanein, I A and El-Deeb, S N , *α-continuous and α-open mappings*, Acta Math. Hung , **41** (1983), 213–218

[14] Noiri, T., *On semi-continuous mappings*, Atti Accad Naz Lincei Rend. Cl. Sci. Fis Mat. Natur **(8)54** (1973), 210–214

[15] Nour, T M., *Slightly semi-continuous functions*, Bull. Calcutta Math Soc **87** (1995), 187–190.

[16] Porter, J and Thomas, J., *On H-closed and minimal Hausdorff spaces*, Trans. Amer Math Soc **138** (1969), 159–170
[17] Singal, M K and Singal, A R., *Almost continuous mappings*, Yokohama Math J **16** (1968), 63–73
[18] Staum, R , *The algebra of bounded continuous functions into a nonarchimedean field*, Pacific J. Math **50** (1974), 169–185

Department of Mathematics
The University of Ulsan
Ulsan 680–749, Korea

Department of Mathematics
Yatsushiro College of Technology
Yatsushiro, Kumamoto 866–8501, Japan

Teacher of ShungShin High School
Ulsan, Korea