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CHARACTERIZATION OF BEST APPROXIMANTS FROM LEVEL SETS OF CONVEX FUNCTIONS IN NORMED LINEAR SPACES

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ABSTRACT Some new characterization of best approximants from level sets of convex functions in normed linear spaces in terms of norm derivatives are given

1. Introduction

Let $(X, \|\cdot\|)$ be a real normed space and consider the norm derivatives

$$(x, y)_{i(s)} = \lim_{t \to -(+)0} \frac{\left(\|y + tx\|^2 - \|y\|^2\right)}{2t}$$

Note that these mappings are well defined on $X \times X$ and the following properties are valid (see also [1], [3]):

(i) $(x, y)_i = -(-x, y)_s$ if x, y are in X;

(ii) $(x, x)_p = \|x\|^2$ for all x in X;

(iii) $(\alpha x, \beta y)_p = \alpha\beta\,(x, y)_p$ for all x, y in X and $\alpha\beta \geq 0$,

(iv) $(\alpha x + y, x)_p = \alpha\|x\|^2 + (y, x)_p$ for all x, y in X and α a real number,

(v) $(x + y, z)_p \leq \|x\| \cdot \|z\| + (y, z)_p$ for all x, y, z in X,

(vi) The element x in X is Birkhoff orthogonal over y in X (we denote $x \perp y(B)$), i e , $\|x + ty\| \geq \|x\|$ for all t a real number iff $(y, x)_i \leq 0 \leq (y, x)_s$,

(vii) The space X is smooth iff $(y, x)_i = (y, x)_s$ for all x, y in X iff $(\cdot, \cdot)_p$ is linear in the first variable,

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where $p = s$ or $p = i$.

Now, let $(X, \|\cdot\|)$ be a normed linear space and G a nondense subset in X. Suppose $x_0 \in X \setminus Cl(G)$ and $g_0 \in G$.

DEFINITION 1.1. The element g_0 will be called *the best approximation element* of x_0 in G if

$$\|x_0 - g_0\| = \inf_{g \in G} \|x_0 - g\| \tag{1.1}$$

and we shall denote by $\mathcal{P}_G(x_0)$ the set of all elements which satisfy (1.1)

The main aim of this paper is to prove some characterization of best approximants from the level sets of continuous convex mappings in normed linear spaces.

For the classical results in domain, see the monograph [4] due to Ivan Singer

2. The Results

Now, let us denote by

$$F^{\leq}(r) := \{x \in X : F(x) \leq r\}, r \in \mathbb{R}$$

the $r-$*level set* of F and assume that r is such that $F^{\leq}(r)$ is nonempty

The following theorem characterizes best approximants by elements of the level set $F^{\leq}(r)$. This result can also be viewed as an estimation theorem for the continuous convex mappings defined on a normed space in terms of semi-inner product $(\cdot,\cdot)_i$.

THEOREM 2.1. Let $(X, \|\cdot\|)$ be a normed linear space, $F : X \to \mathbb{R}$ a continuous convex mapping on X, $r \in \mathbb{R}$ such that $F^{\leq}(r) \neq \emptyset, x_0 \in X \backslash F^{\leq}(r)$ and $g_0 \in F^{\leq}(r)$. The following statements are equivalent.

(i) $g_0 \in \mathcal{P}_{F^{\leq}(r)}(x_0)$,

(ii) We have the estimation

$$F(x) \geq r + \frac{F(x_0) - r}{\|x_0 - g_0\|^2}(x - g_0, x_0 - g_0)_i \text{ for all } x \in F^{\leq}(r), \tag{2.1}$$

or, equivalently, the estimation

(2.2)
$$F(x) \geq F(x_0) + \frac{F(x_0) - r}{\|x_0 - g_0\|^2}(x - x_0, x_0 - g_0)_i \text{ for all } x \in F^{\leq}(r).$$

Proof "$(i) \Rightarrow (ii)$". Let us observe first that as $x_0 \in X \backslash F^{\leq}(r)$, we have that $F(x_0) > r$.
Now, let $x \in F^{\leq}(r)$. Then $F(x) \leq r$ and if we choose

$$\alpha = F(x_0) - r, \beta := r - F(x),$$

then obviously $\alpha > 0, \beta \geq 0$ and $0 < \alpha + \beta = F(x_0) - F(x)$
Let us consider the element

$$u := \frac{\alpha x + \beta x_0}{\alpha + \beta}$$

Then, by the convexity of F we have

$$F(u) \leq \frac{\alpha F(x) + \beta F(x_0)}{\alpha + \beta} = \frac{(F(x_0) - r) F(x) + (r - F(x))F(x_0)}{F(x_0) - F(x)} = r,$$

which shows that $u \in F^{\leq}(r)$.
As $g_0 \in \mathcal{P}_{F^{\leq}(r)}(x_0)$ and $F^{\leq}(r)$ is a convex set, we get that

$$\|x_0 - g_0\|^2 \leq \|x_0 - ((1-t)g_0 + tg)\|^2$$

for each $g \in F^{\leq}(r)$ and $t \in [0,1]$.
Denoting $w_0 := x_0 - g_0$ and $u_0 := g_0 - g$ we get $\|w_0\|^2 \leq \|w_0 + tu_0\|^2$ for all $t \in [0,1]$, which implies that

$$\frac{(\|w_0 + tu_0\|^2 - \|w_0\|^2)}{2t} \geq 0 \text{ for all } t \in (0,1]$$

Letting $t \to 0+$ we deduce $(u_0, w_0)_s \geq 0$, which is equivalent to $(g - g_0, x_0 - x_0)_i \leq 0$ for all $g \in F^{\leq}(r)$
Choose $g = u$,where u is defined as above. Then

$$\left(\frac{(F(x_0) - r)x + (r - F(x))\, x_0}{F(x_0) - F(x)} - g_0, x_0 - g_0\right)_i \leq 0 \tag{2.3}$$

for all $x \in F^{\leq}(r)$. However,

$$\begin{aligned}
&\left(\frac{(F(x_0)-r)x+(r-F(x))\,x_0}{F(x_0)-F(x)}-g_0,x_0-g_0\right)_i \\
= &\ \frac{1}{F(x_0)-F(x)}((r-F(x))\,(x_0-g_0)+(F(x_0)-r)(x-g_0),x_0-g_0)_i \\
= &\ \frac{1}{F(x_0)-F(x)}\left(r-F(x)\right)\|x_0-g_0\|^2+(F(x_0)-r)\,(x-g_0,x_0-g_0)_i)
\end{aligned}$$

and then, by (2.3), we obtain

$$(r-F(x))\|x_0-g_0\|^2+(F(x_0)-r)\,(x-g_0,x_0-g_0)_i\geq 0$$

which is equivalent with the desired estimation (2 1)
Now, let us observe that

$$\begin{aligned}
&r+\frac{F(x_0)-r}{\|x_0-g_0\|^2}\,(x-g_0,x_0-g_0)_i \\
= &\ r+\frac{F(x_0)-r}{\|x_0-g_0\|^2}(x-x_0+x_0-g_0,x_0-g_0)_i \\
= &\ r+\frac{F(x_0)-r}{\|x_0-g_0\|^2}\left[(x-x_0,x_0-g_0)_i+\|x_0-g_0\|^2\right] \\
= &\ r+F(x_0)-r+\frac{F(x_0)-r}{\|x_0-g_0\|^2}\,(x-x_0,x_0-g_0)_i \\
= &\ F(x_0)+\frac{F(x_0)-r}{\|x_0-g_0\|^2}\,(x-x_0,x_0-g_0)_i\,,
\end{aligned}$$

which shows that (2.1) and (2.2) are equivalent.
"$(ii)\Rightarrow(i)$". As $x\in F^{\leq}(r)$, then $0\geq F(x)-r$
On the other hand, by (2.1) we have

$$F(x)-r\geq+\frac{F(x_0)\ -\ r}{\|x_0-g_0\|^2}\,(x-g_0,x_0-g_0)_i$$

for all $x\in F^{\leq}(r)$. Consequently,

$$0\geq\frac{F(x_0)\ -\ r}{\|x_0-g_0\|^2}\,(x-g_0,x_0-g_0)_i\ \text{ for all } x\in F^{\leq}(r)\,.$$

As $F(x_0) - r > 0$, we get

$$0 \geq (x - g_0, x_0 - g_0)_s \text{ for all } x \in F^{\leq}(r);$$

which is clearly equivalent to

$$(g_0 - x, x_0 - g_0)_s \geq 0 \text{ for all } x \in F^{\leq}(r) \tag{2.4}$$

A simple calculation shows that

$$\begin{aligned}(g_0 - x, x_0 - g_0)_s &= (x_0 - x - (x_0 - g_0), x_0 - g_0)_s \\ &= (x_0 - x, x_0 - g_0)_s - \|x_0 - g_0\|^2\end{aligned}$$

and then, by the above inequality, we deduce

$$(x_0 - x, x_0 - g_0)_s \geq \|g_0 - x_0\|^2 \; for \; all \; x \in F^{\leq}(r) \tag{2 5}$$

On the other hand, by Schwarz's inequality we have

$$\|x_0 - x\| \, \|x_0 - g_0\| \geq (x_0 - x, x_0 - g_0)_s \tag{2 6}$$

and then (2 5) and (2.6) yield that $\|x_0 - x\| \geq \|g_0 - x_0\|$ for all $x \in F^{\leq}(r)$, and the theorem is proved. □

REMARK 2 2. If $g_0 \in \mathcal{P}_{F^{\leq}(r)}(x_0)$, then $F(g_0) = r$. Indeed, as $g_0 \in F^{\leq}(r)$, then $F(g_0) \leq r$. On the other hand, choosing $x = g_0$ in (2 1) we get $F(g_0) \geq r$, and then the required equality holds

For other recent results concerning the estimation of linear functionals or sublinear functionals in terms of semi-inner products, see the papers [1]-[3]

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