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UNIFORM DECAY OF SOLUTIONS FOR VISCOELASTIC PROBLEMS

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ABSTRACT. In this paper we prove the existence of solution and uniform decay rates of the energy to viscoelastic problems with nonlinear boundary damping term. To obtain the existence of solutions, we use Faedo-Galerkin's approximation, and also to show the uniform stabilization we use the perturbed energy method

1. Introduction

In this paper, we consider the uniform decay of solutions for viscoelastic problems with nonlinear boundary damping of the following form:

$$
\begin{aligned}
& Ku'' - (1+\|\nabla u\|^2)\Delta u + \int_0^t h(t-\tau)\Delta u(\tau)d\tau = 0 \\
& \qquad \text{on} \quad Q = \Omega \times (0,\infty), \\
& u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{on} \quad x \in \Omega, \\
& u = 0 \quad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0,\infty), \\
& (1+\|\nabla u\|^2)\frac{\partial u}{\partial \nu} - \int_0^t h(t-\tau)\frac{\partial u}{\partial \nu}(\tau)d\tau + g(u') = f(u) \\
& \qquad \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,\infty),
\end{aligned} \tag{1.1}
$$

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where Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary $\Gamma := \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ and Γ_0, Γ_1 have positive measures, and ν denotes the unit outer normal vector pointing towards Γ. When $\Gamma_0 = \emptyset$, problem (1.1) with $h = 0$ results from the mathematical description of small amplitude vibrations of an elastic string (see [7]). In fact, a mathematical model for the deflection of an elastic string of length $L > 0$ is given by the mixed problem for the nonlinear wave equation

$$\rho h \frac{\partial^2 u}{\partial t^2} = \{p_0 + \frac{Eh}{2L}\int_0^L (\frac{\partial u}{\partial x})^2 dx\} \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad 0 < x < L, \quad t \geq 0,$$

where u is the lateral deflection, x the space coordinate, t the time, E the Young modulus, ρ the mass density, h the cross section area and p_0 the initial axial tension

There exists many literature about viscoelastic problems with the memory term acting in the domain. Among the numerous works in this direction, we can cite Jiang and Rivera[4]. When $K = I$, Georgiev and Todorova[3] investigated blow-up properties of the solutions of wave equation with nonlinear damping and source term acting in the domain. For the existence results for Kirchhoff type wave equation with $\partial\Omega = \Gamma_1$ and $K = I$, see Brito[1], Matsuyama[5], Ikehata[6] and Yamada[9]. When $h = 0$ and $K = I$, Bae[8] has studied the uniform decay of solution for the Kirchhoff type wave equations with nonlinear boundary damping $g(t)|u'|^\rho u'$ and boundary source term $\int_0^t g(t-r)|u(r)|^\gamma u(r)dr$.

On the other hand, Cavalcanti et al [2] have studied the global existence and uniform decay of strong solutions of linear wave equation;

$$\begin{aligned}
&K_1 u'' + K_2 u' - \Delta u = 0 \quad \text{on} \quad Q = \Omega \times (0,\infty), \\
&u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{on} \quad x \in \Omega, \\
&u = 0 \quad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0,\infty), \\
&\frac{\partial u}{\partial \nu} + u' + \alpha(t)(|u'|^\rho u' - |u|^\gamma u) = 0 \quad \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,\infty).
\end{aligned}$$

In this paper, we will study the existence of solutions of the Kirchoff type viscoelastic problem (1.1) with nonlinear boundary damping, nonlinear boundary source term. Moreover, we will consider the uniform decay of the energy of the problem (1.1).

2. Assumptions and main result

Throughout this paper we define

$$V = \{u \in H^1(\Omega);\ u = 0 \quad \text{on} \quad \Gamma_1\}, \quad (u,v) = \int_\Omega u(x)v(x)dx,$$

$$(u,v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)d\Gamma, \quad \|u\|^p_{p,\Gamma_0} = \int_{\Gamma_0} |u(x)|^p dx$$
$$\text{and} \quad \|u\|_\infty = \|u\|_{L^\infty(\Omega)}.$$

For simplicity we denote $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{2,\Gamma_0}$ by $\|\cdot\|$ and $\|\cdot\|_{\Gamma_0}$, respectively For our result, we need the following assumptions.

(A_1) Let us consider u_0, $u_1 \in V \cap H^2(\Omega)$ verifying the compatibility conditions

$$u_0 = \Delta u_0 = u_1 = 0 \quad \text{on} \quad \Gamma_1,$$
$$(1 + \|\nabla u_0\|^2)\frac{\partial u_0}{\partial \nu} + g(u_1) = f(u_0) \quad \text{on} \quad \Gamma_0.$$

(A_2) Let K be a function in $W^{1,\infty}(0,\infty) \cap L^\infty(0,\infty)$, $K \geq 0$ such that

$$-K'(t) \geq \delta > 0, \quad \forall\ t \geq 0.$$

(A_3) $f : R \to R$ is a C^1 function such that for some positive constant C_0,

$$|f(s)| \leq C_0|s|^{\gamma+1}, \quad |f'(s)| \leq C_0|s|^\gamma,$$

where $0 < \gamma \leq \frac{1}{n-2}$ if $n \geq 3$ or $\gamma > 0$ if $n = 1, 2$.

(A_4) $g : R \to R$ is a nondecreasing C^1-function such that for some positive constants C_1, C_2,

$$C_1|s|^{\rho+2} \le g(s)s \le C_2|s|^{\rho+2}, \ s \in R,$$

where $0 < \rho \le \frac{1}{n-2}$ if $n \ge 3$ or $\rho > 0$ if $n = 1, 2$.

(A_5) Let the function $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonnegative and bounded C^2-function such that $l = 1 - \int_0^\infty h(r)dr > 0$ and for some ξ_i, $i = 1, 2, 3$,

$$-\xi_1 h(t) \le h'(t) \le -\xi_2 h(t) \quad \text{and} \quad 0 \le h''(t) \le \xi_3 h(t), \quad \forall\, t \ge 0.$$

Now we state our main result.

THEOREM 2.1. *Under assumptions (A_1)-(A_5) and $\rho \ge \gamma$, problem (1.1) has a unique strong solution $u : \Omega \to \mathbb{R}$ such that $u \in L^\infty(0, \infty; V)$, $u' \in L^\infty(0, \infty; V)$, $\sqrt{K}u'' \in L^\infty(0, \infty; L^2(\Omega))$, $u'' \in L^2(0, \infty; L^2(\Omega))$. Moreover, if $\rho = \gamma$, then there exist positive constants C_1 and C_2 such that*

$$E(t) \le C_1 E(0) exp(-C_2 \epsilon t), \quad t \ge 0.$$

3. Proof of Theorem 2.1

In this section we are going to show the existence of solution for problem (1.1). Now, we represent by $\{w_j\}_{j \in N}$ a basis in $V \cap H^2(\Omega)$ which is orthonormal in $L^2(\Omega)$, by V_m the finite dimensional subspace of V generated by the first m vectors. Next we define for each $\epsilon > 0$, $K_\epsilon(t) = K(t) + \epsilon$ and $u_{\epsilon m}(t) = \Sigma_{j=1}^m \gamma_{\epsilon j m}(t) w_j$, where $u_{\epsilon m}(t)$ is the solution of the following problem

$$\begin{aligned} &(K_\epsilon(t)u''_{\epsilon m}(t), w) + (1 + \|\nabla u_{\epsilon m}(t)\|^2)(\nabla u_{\epsilon m}(t), \nabla w) + \\ &\qquad (g(u'_{\epsilon m}(t)), w)_{\Gamma_0} \\ &= (f(u_{\epsilon m}(t)), w)_{\Gamma_0} + \int_0^t h(t-r)(\nabla u_{\epsilon m}(r), \nabla w)dr, \\ &u_{\epsilon m}(0) = u'_{\epsilon m}(0) = 0 \quad \text{for all} \quad w \in V_m \end{aligned} \tag{3.1}$$

with the initial conditions,

$$
\begin{aligned}
& u_{\epsilon m}(0) = u_{0m} = \sum_{j=1}^{m}(u_0, w_j)w_j \to u_0 \\
& \qquad \text{strongly in} \quad H_0^1(\Omega) \cap H^2(\Omega), \\
& u'_{\epsilon m}(0) = u_{1m} = \sum_{j=1}^{m}(u_1, w_j)w_j \to u_1 \\
& \qquad \text{strongly in} \quad H_0^1(\Omega).
\end{aligned} \tag{3.2}
$$

Note that we can solve the system (3.1)-(3.2) by Picard's iteration method. In fact, the problem (3.1)-(3.2) has a unique solution on some interval $[0, T_m)$. The extension of these solutions to the whole interval $[0, T]$ is a consequence of the estimates which we are going to prove below.

A Priori Estimate I.

Multiplying (3.1) by $\gamma'_j(t)$, summing over j, we obtain

$$
\begin{aligned}
& \frac{d}{dt}\Big[\frac{1}{2}\|\sqrt{K_\epsilon(t)}u'_{\epsilon m}(t)\|^2 + \frac{1}{2}\|\nabla u_{\epsilon m}(t)\|^2 + \frac{1}{4}\|\nabla u_{\epsilon m}(t)\|^4 \\
& \quad + \frac{1}{\gamma+2}\|u_{\epsilon m}(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}\Big] + (g(u'_{\epsilon m}(t)), u'_{\epsilon m}(t))_{\Gamma_0} \\
& = \frac{1}{2}(K'(t), |u'_{\epsilon m}(t)|^2) + (f(u_{\epsilon m}(t)), u'_{\epsilon m}(t))_{\Gamma_0} \\
& \quad + \frac{d}{dt}\int_0^t h(t-r)(\nabla u_{\epsilon m}(r), \nabla u_{\epsilon m}(t))dr \\
& \quad - h(0)\|\nabla u_{\epsilon m}(t)\|^2 - \int_0^t h'(t-r)(\nabla u_{\epsilon m}(r), \nabla u_{\epsilon m}(t))dr \\
& \quad + (|u_{\epsilon m}(t)|^\gamma u_{\epsilon m}(t), u'_{\epsilon m}(t))_{\Gamma_0}.
\end{aligned} \tag{3.3}
$$

Note that the assumption (A_3), Holder's inequality, Young's inequality and imbedding $L^{\rho+2}(\Gamma_0) \hookrightarrow L^{\gamma+2}(\Gamma_0)$ give us

$$
\begin{aligned}
&(f(u_{\epsilon m}(t)), u'_{\epsilon m}(t))_{\Gamma_0} + (|u_{\epsilon m}(t)|^\gamma u_{\epsilon m}(t), u'_{\epsilon m}(t))_{\Gamma_0} \\
&\leq (C_0+1)\int_{\Gamma_0} |u_{\epsilon m}(t)|^{\gamma+1}|u'_{\epsilon m}(t)|d\Gamma \\
&\leq C_1(\eta)\|u_{\epsilon m}(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \eta\|u'_{\epsilon m}(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} \\
&\leq C_1(\eta)\|u_{\epsilon m}(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + C_2(\eta) + \eta\|u'_{\epsilon m}(t)\|^{\rho+2}_{\rho+2,\Gamma_0}.
\end{aligned} \tag{3.4}
$$

Considering Schwarz's inequality and taking the assumption (A_5) into account, we deduce

$$
\begin{aligned}
&\int_0^t h'(t-r)(\nabla u_{\epsilon m}(r), \nabla u_{\epsilon m}(t))dr \leq \frac{\xi_1^2}{2}\|\nabla u_{\epsilon m}(t)\|^2 \\
&+ \frac{1}{2}\|h\|_{L^1(0,\infty)}\int_0^t h(t-r)\|\nabla u_{\epsilon m}(r)\|^2 dr.
\end{aligned} \tag{3.5}
$$

Combining the above inequalities, and integrating it over $(0,t)$, assumptions (A_2) and (A_4) imply

$$
\begin{aligned}
&E_{\epsilon m}(t) + \frac{1}{\gamma+2}\|u_{\epsilon m}(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \frac{\delta}{2}\int_0^t \|u'_{\epsilon m}(s)\|^2 ds \\
&\quad + \int_0^t (C_1-\eta)\|u'_{\epsilon m}(s)\|^{\rho+2}_{\rho+2,\Gamma_0} ds \\
&\leq E_{\epsilon m}(0) + \frac{1}{\gamma+2}\|u_{0\epsilon m}\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \int_0^t C_2(\eta) \\
&\quad + C_1(\eta)\|u_{\epsilon m}(s)\|^{\gamma+2}_{\gamma+2,\Gamma_0} ds \\
&\quad + (\frac{\xi_1^2}{2} - h(0))\int_0^t \|\nabla u_{\epsilon m}(s)\|^2 ds \\
&\quad + \int_0^t h(t-r)(\nabla u_{\epsilon m}(r), \nabla u_{\epsilon m}(t))dr \\
&\quad + \frac{1}{2}\|h\|_{L^1(0,\infty)}\int_0^t\int_0^s h(s-r)\|\nabla u_{\epsilon m}(r)\|^2 drds
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
&\leq E_{\epsilon m}(0) + \frac{1}{\gamma+2}\|u_{0\epsilon m}\|_{\gamma+2,\Gamma_0}^{\gamma+2} \\
&+ \int_0^t C_2(\eta) + C_1(\eta)\|u_{\epsilon m}(s)\|_{\gamma+2,\Gamma_0}^{\gamma+2} ds \\
&+ (\frac{\xi_1^2}{2} - h(0) + \frac{1}{2}\|h\|_{L^1(0,\infty)}^2 \\
&+ \|h\|_{L^1(0,\infty)}\|h\|_{L^\infty(0,\infty)}) \int_0^t \|\nabla u_{\epsilon m}(s)\|^2 ds,
\end{aligned} \tag{3.6}
$$

where $E_{\epsilon m}(t) = \frac{1}{2}\|\sqrt{K_\epsilon(t)}u'_{\epsilon m}(t)\|^2 + \frac{1}{2}\|\nabla u_{\epsilon m}(t)\|^2 + \frac{1}{4}\|\nabla u_{\epsilon m}(t)\|^4$.

Employing Gronwall's lemma we obtain the first estimate:

$$
\begin{aligned}
&\|\sqrt{K_\epsilon(t)}u'_{\epsilon m}(t)\|^2 + \|\nabla u_{\epsilon m}(t)\|^2 + \int_0^t \|u'_{\epsilon m}(s)\|^2 ds \\
&+ \int_0^t \|u'_{\epsilon m}(s)\|_{\rho+2,\Gamma_0}^{\rho+2} \leq L_1,
\end{aligned} \tag{3.7}
$$

where $L_1 > 0$ is a positive constant independent of m and $t > 0$.

From the assumptions (A_4) on g and (3.7), we get

$$
\int_0^t \|g(u'_{\epsilon m}(s))\|_{\Gamma_0}^2 ds \leq L_2, \tag{3.8}
$$

where $L_2 > 0$ is a positive constant independent of m and $t > 0$.

A Priori Estimate II.

Now differentiating (3.1), multiplying the result by $\gamma_j''(t)$ and summing over j, we get

$$
\begin{aligned}
(3.9)\qquad & \frac{1}{2}\frac{d}{dt}[\|\sqrt{K_\epsilon(t)}u''_{\epsilon m}(t)\|^2 + (1+\|\nabla u_{\epsilon m}(t)\|^2)\|\nabla u'_{\epsilon m}(t)\|^2 \\
&\quad + 2(\nabla u_{\epsilon m}(t), \nabla u'_{\epsilon m}(t))^2] + \frac{1}{2}(K'_\epsilon(t), |u''_{\epsilon m}(t)|^2) \\
&\quad \pm \int_{\Gamma_0} g'(u'_{\epsilon m}(t))(u''_{\epsilon m}(t))^2 d\Gamma \\
&= (f'(u_{\epsilon m}(t))u'_{\epsilon m}(t), u''_{\epsilon m}(t))_{\Gamma_0} \\
&\quad + 3(\nabla u_{\epsilon m}(t), \nabla u'_{\epsilon m}(t))\|\nabla u'_{\epsilon m}(t)\|^2 - h(0)\|\nabla u'_{\epsilon m}(t)\|^2 \\
&\quad + h(0)\frac{d}{dt}(\nabla u_{\epsilon m}(t), \nabla u'_{\epsilon m}(t)) \\
&\quad + \frac{d}{dt}\int_0^t h'(t-r)(\nabla u_{\epsilon m}(r), \nabla u'_{\epsilon m}(t))dr \\
&\quad - h'(0)(\nabla u_{\epsilon m}(t), \nabla u'_{\epsilon m}(t)) \\
&\quad - \int_0^t h''(t-r)(\nabla u_{\epsilon m}(r), \nabla u'_{\epsilon m}(t))dr.
\end{aligned}
$$

Making use of the Schwarz inequality, Young Inequality and taking assumption (A_3) into account, we have

$$
\begin{aligned}
(3.10)\qquad & (f'(u_{\epsilon m}(t))u'_{\epsilon m}(t), u''_{\epsilon m}(t))_{\Gamma_0} \\
&\le C_0\|u_{\epsilon m}(t)\|^\gamma_{2(\gamma+1),\Gamma_0}\|u'_{\epsilon m}(t)\|_{2(\gamma+1),\Gamma_0}\|u''_{\epsilon m}(t)\|_{\Gamma_0} \\
&\le C\|\nabla u_{\epsilon m}(t)\|^\gamma\|\nabla u'_{\epsilon m}(t)\|\|u''_{\epsilon m}(t)\|_{\Gamma_0} \\
&\le C(\eta)L_1^{2\gamma}\|\nabla u'_{\epsilon m}(t)\|^2 + \eta\|u''_{\epsilon m}(t)\|^2_{\Gamma_0}.
\end{aligned}
$$

Integrating (3.9) over $(0,t)$, we have

$$
\begin{aligned}
&E_{\epsilon 1m}(t) + \frac{\delta}{2}\int_0^t \|u''_{\epsilon m}(s)\|^2 ds \\
&\quad + \int_0^t \int_{\Gamma_0} (g'(u'_{\epsilon m}(s)) - \eta)(u''_{\epsilon m}(s))^2 d\Gamma ds \\
&\le E_{\epsilon 1m}(0) + 3\int_0^t \|\nabla u_{\epsilon m}(s)\|\|\nabla u'_{\epsilon m}(s)\|^3 ds \\
&\quad + h(0)[\|\nabla u_{\epsilon m}(t)\|\|\nabla u'_{\epsilon m}(t)\| + \|\nabla u_{0m}\|\|\nabla u_{1m}\|] \\
&\quad + \frac{1}{2}[\|h\|^2_{L^1(0,\infty)} + \|h\|_{L^1(0,\infty)}\|h\|_{L^\infty(0,\infty)} \\
&\quad + |h'(0)|]\int_0^t \|\nabla u_{\epsilon m}(r)\|^2 dr \\
&\quad + \frac{1}{2}(\xi_1^2 + \xi_3^2 - 2h(0) + |h'(0)| \\
&\quad + C(\eta)L_1^{2\gamma})\int_0^t \|\nabla u'_{\epsilon m}(s)\|^2 ds,
\end{aligned}
\tag{3.11}
$$

where $E_{\epsilon 1m}(t) = \frac{1}{2}[\|\sqrt{K_\epsilon(t)}u''_{\epsilon m}(t)\|^2 + (1+\|\nabla u_{\epsilon m}(t)\|^2)\|\nabla u'_{\epsilon m}(t)\|^2 + 2(\nabla u_{\epsilon m}(t), \nabla u'_{\epsilon m}(t))^2]$. Considering the first estimate and employing Gronwall's inequality, for sufficiently small η,

$$
\|\sqrt{K_\epsilon(t)}u''_{\epsilon m}(t)\|^2 + \|\nabla u'_{\epsilon m}(t)\|^2 + \int_0^t \|u''_{\epsilon m}(s)\|^2 ds \le L_4, \tag{3.12}
$$

where L_4 is a positive constant independent of $m \in N$ and $t \in [0,T]$.

By the estimates we can extract subsequence $(u_{\epsilon\mu})$ of $(u_{\epsilon m})$ such that

(3.13)
$$u_{\epsilon\mu} \to u_\epsilon \quad \text{weak star} \quad L^\infty(0,T;V),$$
(3.14)
$$u'_{\epsilon\mu} \to u'_\epsilon \quad \text{weak star} \quad L^\infty(0,T;V),$$
(3.15)
$$\sqrt{K_\epsilon}u''_{\epsilon\mu} \to \sqrt{K_\epsilon}u''_\epsilon \quad \text{weak star} \quad L^\infty(0,T;L^\infty(\Omega)),$$
(3.16)
$$u_{\epsilon\mu} \to u_\epsilon \quad \text{weak star} \quad L^\infty(0,T;L^{\gamma+2}(\Gamma_0)),$$
(3.17)
$$u'_{\epsilon\mu} \to u'_\epsilon \quad \text{weak star} \quad L^\infty(0,T;L^{\rho+2}(\Gamma_0)),$$
(3.18)
$$u'_{\epsilon\mu} \to u'_\epsilon \quad \text{weak} \quad L^2(0,T;L^2(\Gamma_0)),$$
(3.19)
$$u''_{\epsilon\mu} \to u''_\epsilon \quad \text{weak} \quad L^2(0,T;L^2(\Gamma_0)).$$

The convergence (3.13)-(3.15) and (3.19) are sufficient to pass to the limit in the linear terms of problem (3 1). Next we are going to consider the nonlinear ones.

Analysis of the nonlinear terms.

Taking (3.7) into account, we note that

$$\int_0^T \int_{\Gamma_0} |f(u_{\epsilon\mu})|^{\frac{\gamma+2}{\gamma+1}} d\Gamma dt \le C_1 \int_0^T \int_{\Gamma_0} |u_{\epsilon\mu}|^{\gamma+2} d\Gamma dt \le L,$$

$$\int_0^T \int_{\Gamma_0} |g(u'_{\epsilon\mu})|^{\frac{\rho+2}{\rho+1}} d\Gamma dt \le C_1 \int_0^T \int_{\Gamma_0} |u'_{\epsilon\mu}|^{\rho+2} d\Gamma dt \le L,$$

so we have ϕ, $\xi \in L^2(0,T;L^2(\Gamma_0))$ such that

$$f(u_{\epsilon\mu}) \to \phi \quad \text{weakly in} \quad L^{\frac{\gamma+2}{\gamma+1}}(\Gamma_0 \times (0,T)),$$
$$g(u'_{\epsilon\mu}) \to \xi \quad \text{weakly in} \quad L^{\frac{\rho+2}{\rho+1}}(\Gamma_0 \times (0,T)).$$

From the first and second estimates we deduce

$$\begin{aligned}
&(u_{\epsilon\mu}) \quad \text{is bounded in} \quad L^2(0,T;H^{\frac{1}{2}}(\Gamma_0)),\\
&(u'_{\epsilon\mu}) \quad \text{is bounded in} \quad L^2(0,T;H^{\frac{1}{2}}(\Gamma_0)),\\
&(u''_{\epsilon\mu}) \quad \text{is bounded in} \quad L^2(0,T;L^2(\Gamma_0)).
\end{aligned}$$

Taking into consideration that the imbedding $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Gamma)$ is continuous and compact and using Aubin compactness theorem, we can extract a subsequence, still represented by $(u_{\epsilon\mu})$, such that

$$\begin{aligned}
(3.20) \qquad & u_{\epsilon\mu} \to u_\epsilon \quad \text{strongly in} \quad L^2(0,T;L^2(\Gamma_0)),\\
& u'_{\epsilon\mu} \to u'_\epsilon \quad \text{strongly in} \quad L^2(0,T;L^2(\Gamma_0)),
\end{aligned}$$

which imply that

$$(3.21) \qquad u_{\epsilon\mu} \to u_\epsilon \quad \text{a.e. on} \quad \Sigma_0 \quad \text{and} \quad u'_{\epsilon\mu} \to u'_\epsilon \quad \text{a.e. on} \quad \Sigma_0$$

and therefore

$$\begin{aligned}
& f(u_{\epsilon\mu}) \to f(u_\epsilon) \quad \text{weakly in} \quad L^{\frac{\gamma+2}{\gamma+1}}(\Gamma_0 \times (0,T)),\\
& g(u'_{\epsilon\mu}) \to g(u'_\epsilon) \quad \text{weakly in} \quad L^{\frac{\rho+2}{\rho+1}}(\Gamma_0 \times (0,T)).
\end{aligned}$$

Thus

$$\begin{aligned}
& f(u_{\epsilon\mu}) \to f(u_\epsilon) \quad \text{weakly in} \quad L^2(0,T;L^2(\Gamma_0)),\\
& g(u'_{\epsilon\mu}) \to g(u'_\epsilon) \quad \text{weakly in} \quad L^2(0,T;L^2(\Gamma_0)).
\end{aligned}$$

Using standard arguments, we can show from the above estimates that

(3.22)

$$K(t)u''(t) - \Delta[u(t) + \|\nabla u(t)\|^2 u(t) - \int_0^t h(t-r)u(r)dr] = 0$$

$$\text{in} \quad L^2_{loc}(0,T;L^2(\Omega))$$

Making use of the generalized Green's formula we deduce that

$$(3.23) \quad \frac{\partial}{\partial\nu}[(1 + \|\nabla u(t)\|^2)u(t) - \int_0^t h(t-r)u(r)dr] + g(u') = f(u)$$

$$\text{in} \quad L^2_{loc}(0,T;L^2(\Gamma_0))$$

This completes the proof of the existence of solutions of (1.1). The uniqueness is obtained in a usual way, so we omit the proof here. □

4. Uniform decay

In this section we derive the decay estimates for the energy of (1.1). We define the energy $E(t)$ of the problem (1.1) by

$$E(t) = \frac{1}{2}\|\sqrt{K(t)}u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{4}\|\nabla u(t)\|^4. \tag{4.1}$$

Then the derivative of energy is given by

$$\begin{aligned} E'(t) &= \frac{1}{2}(K'(t), |u'(t)|^2) + \int_0^t h(t-r)(\nabla u(r), \nabla u'(t))dr \\ &\quad - (g(u'(t)), u'(t))_{\Gamma_0} + (f(u(t)), u'(t))_{\Gamma_0}. \end{aligned} \tag{4.2}$$

Define $(h \,\Box\, u)$ by

$$(h \,\Box\, u)(t) := \int_0^t h(t-r)\|u(t) - u(r)\|^2 dr. \tag{4.3}$$

Next consider the modified energy $e(t)$:

$$\begin{aligned} e(t) &= \frac{1}{2}\|\sqrt{K(t)}u'(t)\|^2 + \frac{1}{4}\|\nabla u(t)\|^4 + \frac{1}{2}(h \,\Box\, \nabla u)(t) \\ &\quad + + \frac{1}{2}(1 - \int_0^t h(r)dr)\|\nabla u(t)\|^2 \\ &\quad + \frac{1}{\gamma+2}\alpha(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \int_{\Gamma_0}\int_0^u f(\eta)d\eta d\Gamma, \end{aligned} \tag{4.4}$$

where $\alpha \in W^{1,\infty}(0,\infty) \cap L^1(0,\infty)$ with $-m_0\alpha(t) \le \alpha'(t) \le -m_1\alpha(t)$, $\alpha(t) \ge m_2$ for all $t \ge 0$, $m_i \ge 0$, $i = 0, 1, 2$, $m_1 > 2(\gamma + 2)$ and $\|\alpha\|_\infty \le C_1$, C_1 is a constant in assumption (A_4). Then (4.2) and (4.4) imply

$$\begin{aligned} e'(t) &= \frac{1}{2}(K'(t), |u'(t)|^2) - (g(u'(t)), u'(t))_{\Gamma_0} \\ &\quad - \frac{1}{2}h(t)\|\nabla u(t)\|^2 + \frac{1}{2}(h' \,\Box\, \nabla u)(t) \\ &\quad + \alpha(t)(|u(t)|^\gamma u(t), u'(t))_{\Gamma_0} + \frac{1}{\gamma+2}\alpha'(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}. \end{aligned} \tag{4.5}$$

Considering Young's inequality, we get

$$\alpha(t)(|u(t)|^\gamma u(t), u'(t))_{\Gamma_0} \tag{4.6}$$
$$\leq \eta\alpha(t)\|u'(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \eta^{-\frac{1}{\gamma+1}}\alpha(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}.$$

Choosing $\eta = 2^{-(\gamma+1)}$ then $C_1 - \eta\|\alpha\|_\infty > 0$. Thus for $\gamma = \rho$ and sufficiently small $\eta > 0$, the assumptions (A_2) and (A_5) imply

$$\begin{aligned} e'(t) \leq & -\frac{\delta}{2}\|u'(t)\|^2 - \beta\|u'(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} - \frac{1}{2}h(t)\|\nabla u(t)\|^2 \\ & - \frac{\xi_2}{2}(h \,\Box\, \nabla u)(t) - \beta_1\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}, \end{aligned} \tag{4.7}$$

where $\beta = C_1 - \eta\|\alpha\|_\infty > 0$ and $\beta_1 = m_2(\frac{m_1}{\gamma+2} - \eta^{-\frac{1}{\gamma+1}}) > 0$. On the other hand we note that from assumption (A_5)

$$\begin{aligned} E(t) \leq & \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2l}(1 - \int_0^t h(r)dr)\|\nabla u(t)\|^2 \\ & + \frac{1}{4}\|\nabla u(t)\|^4 \leq l^{-1}e(t) \end{aligned} \tag{4.8}$$

and therefore it is enough to obtain the desired exponential decay for the modified energy $e(t)$ which will be done below.

For this purpose let λ be the positive number such that $\|v\|^2 \leq \lambda\|\nabla v\|^2, \quad \forall v \in V$ and for every $\epsilon > 0$ let us define the perturbed modified energy by

$$e_\epsilon(t) = e(t) + \epsilon\psi(t), \quad \text{where} \quad \psi(t) = (K(t)u'(t), u(t)).$$

Applying Cauchy Schwarz's inequality, we easily obtain the following proposition.

PROPOSITION 4.1. *We have the inequality for any* $\epsilon > 0$

$$| e_\epsilon(t) - e(t)| \leq \epsilon\lambda^{\frac{1}{2}}\|K\|^{\frac{1}{2}}_\infty e(t), \quad \forall\, t \geq 0.$$

The following proposition is the useful instrument for the energy decay of problem (1.1).

PROPOSITION 4.2. *There exist $C > 0$ and ϵ_1 such that for $\epsilon \in (0, \epsilon_1]$*

$$e'_\epsilon(t) \leq -\epsilon C e(t).$$

Proof. Using the equation (1.1), we have

$$\begin{aligned}
\psi'(t) &= \|\sqrt{K(t)}u'(t)\|^2 + (K'(t)u'(t), u(t)) \\
&- (1 + \|\nabla u(t)\|^2)\|\nabla u(t)\|^2 - (g(u'(t)), u(t))_{\Gamma_0} \\
&+ (f(u(t)), u(t))_{\Gamma_0} + \int_0^t h(t-r)(\nabla u(r), \nabla u(t))dr \\
&= -e(t) + \frac{3}{2}\|\sqrt{K(t)}u'(t)\|^2 + (K'(t)u'(t), u(t)) \\
&- \frac{1}{2}\|\nabla u(t)\|^2 - \frac{3}{4}\|\nabla u(t)\|^4 + \frac{1}{2}(h \,\square\, \nabla u)(t) \\
&- \frac{1}{2}\int_0^t h(r)dr\|\nabla u(t)\|^2 - (g(u'(t)), u(t))_{\Gamma_0} \\
&+ (f(u(t)), u(t))_{\Gamma_0} + \int_0^t h(t-r)(\nabla u(r), \nabla u(t))dr \\
&+ \frac{1}{\gamma+2}\alpha(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}.
\end{aligned} \tag{4.9}$$

Now applying Schwarz's inequality and (4.3), we get

$$\begin{aligned}
&\int_0^t h(t-r)(\nabla u(r), \nabla u(t))dr \\
&\leq \frac{1}{2}(h \,\square\, \nabla u)(t) + \frac{3}{2}\|\nabla u(t)\|^2 \int_0^t h(r)dr.
\end{aligned} \tag{4 10}$$

From Schwarz's inequality and Young's inequality we get

$$\begin{aligned}
|(g(u'(t)), u(t))_{\Gamma_0}| &\leq C_2\|u'(t)\|^{\rho+1}_{\rho+2,\Gamma_0}\|u(t)\|_{\rho+2,\Gamma_0} \\
&\leq C_4(\eta)\|u'(t)\|^{\rho+2}_{\rho+2,\Gamma_0} + \eta\|u(t)\|^{\rho+2}_{\rho+2,\Gamma_0}
\end{aligned} \tag{4.11}$$

and

$$
\begin{aligned}
(K'(t)u'(t), u(t)) &\leq \lambda^{\frac{1}{2}} \|K'\|_{\infty}^{\frac{1}{2}} \|u'(t)\| \|\nabla u(t)\| \\
&\leq \lambda C_5(\eta) \|K'\|_{\infty} \|u'(t)\|^2 + \eta \|\nabla u(t)\|^2.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\psi'(t) \leq & -e(t) + (\frac{3}{2}\|K\|_{\infty} + \lambda C_5(\eta)\|K'\|_{\infty})\|u'(t)\|^2 \\
& - (\frac{1}{2} - \eta)\|\nabla u(t)\|^2 - \frac{3}{4}\|\nabla u(t)\|^4 \\
& + (h \,\square\, \nabla u)(t) + \int_0^t h(r)dr \|\nabla u(t)\|^2 \\
& + C_4(\eta)\|u'(t)\|_{\rho+2,\Gamma_0}^{\rho+2} + \eta \|u(t)\|_{\rho+2,\Gamma_0}^{\rho+2} \\
& + (C_0 + \frac{1}{\gamma+2}\|\alpha\|_{\infty})\|u(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}
\end{aligned} \tag{4.12}
$$

From (4.7), (4.12) and the assumption (A_2) and considering $\rho = \gamma$, we get

$$
\begin{aligned}
e'_{\epsilon}(t) &= e'(t) + \epsilon \psi'(t) \\
&\leq -\epsilon e(t) - (\frac{\delta}{2} - \epsilon[\frac{3}{2}\|K\|_{\infty} + \lambda C_5(\eta)\|K'\|_{\infty}])\|u'(t)\|^2 \\
&\quad - \frac{1}{2}h(t)\|\nabla u(t)\|^2 - (\frac{1}{2} - \eta)\epsilon\|\nabla u(t)\|^2 \\
&\quad - \frac{3}{4}\epsilon\|\nabla u(t)\|^4 - (\frac{\xi_2}{2} - \epsilon)(h\square\nabla u)(t) \\
&\quad - (\beta - C_4(\eta)\epsilon)\|u'(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} \\
&\quad - [\beta_1 - (C_0 + \frac{1}{\gamma+2}\|\alpha\|_{\infty})]\|u(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}.
\end{aligned} \tag{4.13}
$$

Defining

$$
\epsilon_1 = \min\{\frac{\xi_2}{2}, \frac{\delta}{2}[\frac{3}{2}\|K\|_{\infty} + \lambda C_5(\eta)\|K'\|_{\infty}]^{-1}, \frac{\beta}{C_4(\eta)}, \frac{\beta_1(\gamma+2)}{C_0(\gamma+2) + \|\alpha\|_{\infty}}\}
$$

Then for each $\epsilon \in (0, \epsilon_1]$, we have

$$e'_\epsilon(t) \leq -\epsilon C e(t) \tag{4.14}$$

if $\|h\|_{L^1(0,\infty)}$ is sufficiently small. □

Continuing the proof of Theorem 2.1

Let $\epsilon_0 = \min\{\frac{1}{2\lambda^{\frac{1}{2}}\|K\|_\infty^{\frac{1}{2}}}, \epsilon_1\}$ and let us consider $\epsilon \in (0, \epsilon_0]$. As we have $\epsilon < \frac{1}{2\lambda^{\frac{1}{2}}\|K\|_\infty^{\frac{1}{2}}}$, we conclude from Proposition 4.1

$$(1 - \epsilon\lambda^{\frac{1}{2}}\|K\|_\infty^{\frac{1}{2}})e(t) < e_\epsilon(t) < (1 + \epsilon\lambda^{\frac{1}{2}}\|K\|_\infty^{\frac{1}{2}})e(t)$$

and so

$$\frac{1}{2}e(t) < e_\epsilon(t) < \frac{3}{2}e(t). \tag{4.15}$$

Thus we have

$$e'_\epsilon(t) \leq -\frac{2}{3}C_1\epsilon e_\epsilon(t)$$

and

$$\frac{d}{dt}[e_\epsilon(t)exp(\frac{2}{3}C_1\epsilon t)] \leq 0. \tag{4.16}$$

Integrating (4.16), inequality (4.15) implies

$$e(t) \leq 3e(0)exp(-\frac{2}{3}C_1\epsilon t). \tag{4.17}$$

Hence from (4.9) and (4.17) we get

$$E(t) \leq l^{-1}e(t) \leq 3e(0)l^{-1}exp(-\frac{2}{3}C_1\epsilon t), \quad t \geq 0.$$

This completes the proof of Theorem 2.1. □

REFERENCES

[1] E H Brito, *Nonlinear initial boundary value problems*, Nonlinear Anal **11** (1987), 125-137

[2] M M Cavalcanti, V N Domingos Cavalcanti, J S Filho and J.A Soriano, *Existence and uniform decay of solutions of a parabolic-hyperbolic equation with nonlinear boundary damping and boundary source term*, Comm. Anal Geo **10** (2002), 451-466

[3] V Georgiev and G Todorova, *Existence of a solution of the wave equations with nonlinear damping and source terms*, J. Diff Eqs **109** (1994), 295-308.

[4] S Jiang and J E Munoz Rivera, *A global existence for the Dirichlet problems in nonlinear n-dimensional viscoelasticity*, Diff Int Eqs **9** (1996), 791-810.

[5] T Matsuyama, *Quasilinear hyperbolic-hyperbolic singular perturbation with nonmonotone nonlinearity*, Nonlinear Analysis, T M A. **35** (1999), 589-607

[6] T Matsuyama and R Ikehata, *On global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping terms*, J. Math Anal Appl **204** (1996), 729-753

[7] K Narasimha, *Nonlinear vibration of an elastic string*, J Sound Vib **8** (1968), 134-146

[8] J Y Park and J J Bae, *uniform decay of solution for wave equation of Kirchhoff type with nonlinear boundary damping and memory term*, Nonlinear Analysis, T M A **50** (2002), 871-884.

[9] Y Yamada, *Some nonlinear degenerate wave equations*, Nonlinear Analysis, T. M. A **11** (1987), 1155-1168

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