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ENERGY DECAY FOR THE NONLINEAR WAVE EQUATION IN THE WHOLE SPACE WITH SOME DISSIPATION

IL HYO JUNG

ABSTRACT. We study decay estimates of the energy for the nonlinear wave equation in the whole space. We note that the method of proof is based on the multiplier technique and on the unique continuation, and no geometrical condition is imposed on the boundary.

1. Introduction

In this paper we consider the Cauchy problem for the nonlinear wave equation with a half-linear dissipation;

$$u_{tt} - \triangle u + \rho(x, u_t) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty) \tag{1.1}$$

$$u(x, 0) = u_0, \quad u_t(x, 0) = u_1 \quad \text{in } \mathbb{R}^N \tag{1.2}$$

where $\rho(x, v)$ is some nonlinear function specified later. For the sequel, we need some notations. We set $B_r = \{x \in \mathbb{R}^N | |x| < r\}$ and $\Omega_r := \mathbb{R}^N \setminus B_r$ for $r > 0$.

Let $R > 0$ be arbitrary fixed positive number and $a(x)$ be a nonnegative bounded function on $\mathbb{R}^N$ such that

$$a(x) \geq \epsilon_0 > 0 \ a.e. \text{ for } x \in \Omega_R. \tag{1.3}$$

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We now make the following hypotheses on the dissipative term $\rho(x, v)$.

Hyp. $\rho(x, v)$ is differentiable *a.e.* and nondecreasing function in v such that

$$\rho(x, v) = \tilde{\rho}(v)\chi(B_R) + a(x)v\chi(\Omega_R), \tag{1.4}$$

where $\tilde{\rho}(v)$ satisfies

$$k_0|v|^{r+2} \leq \tilde{\rho}(v)v \leq k_1\{|v|^{r+2} + |v|^2\} \text{ for } (x,t) \in B_R \times \mathbb{R}^+ \tag{1.5}$$

with $k_0, k_1 > 0$, $0 \leq r \leq 2/(N-2)$ and $\chi(A)$ denotes the characteristic function of A

For example, $\tilde{\rho}(v)$ is a function like $\tilde{\rho}(v) = |v|^r v$.

Condition (1 4) means that the dissipative term $\rho(x, u_t)$ has two character; linear and nonlinear More precisely, the dissipative term is the linear function $a(x)u_t$ on Ω_R, which is effective at infinity. On the other hand, it is the nonlinear function $\tilde{\rho}(u_t)$ on B_R satisfying (1.5). By reason of such two character, we may call the dissipation the half-linear dissipation, temporarily.

The main purpose of this paper is to investigate precise decay estimates of the energy for the problem (1 1)-(1 2).

The problem of proving decay estimates of the solutions to the wave equation with some dissipation has attracted a lot of attention in recent years. To our knowledge, these are the only results for the whole space, though the Klein-Gordon type wave equation with nonlinear dissipations like $|u_t|^r u_t$ have been treated by Nakao [4] and [5], Nakao and Jung [6], Nakao and Ono [7], Ono [8], and Mochizuki and Motai [3]. Recently, G Todorova [10] have analyzed the global existence and nonexistence conditions in details for the Cauchy problem.

2. Preliminaries and Statement of the Main Result

Throughout this paper we shall use the following notations .

$$||u||_p \equiv ||u||_{L^p(\Omega)}, \ 1 \leq p < \infty;$$

$H^m(\Omega)(m \geq 0)$ denotes the usual Sobolev space with the norm

$$||f||_{H^m(\Omega)} = \Big(\sum_{|\alpha| \leq m} ||D_x^\alpha f(x)||_2^2 dx \Big)^{\frac{1}{2}} < \infty,$$

where α is the multi-indices For simplicity, we will write $||u||$ for $||u||_2$.

Before stating our main result, let us recall the following well-posedness result, which is given by Lions and Strauss [2] and Nakao [4]

THEOREM 2.1. *Let $(u_0, u_1) \in H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Then, under Hyp., the problem (1.1)-(1 2) admits a unique solution*

$$u(t) \in W^{2,\infty}([0,T); L^2(\mathbb{R}^N)) \cap W^{1,\infty}([0,T), H^1(\mathbb{R}^N)) \cap L^\infty([0,T), H^2(\mathbb{R}^N))$$

for any $T > 0$.

Moreover, for the solution $u(t)$ to the problem (1 1)-(1 2), there exists a finite constant $K > 0$ such that for any $T > 0$,

$$||\nabla u_t|| + ||u_t|| \leq K \text{ for } t \in [0, T). \tag{2.1}$$

The main result of this paper is as follows

THEOREM 2 2. *Let $(u_0, u_1) \in H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and $N \geq 3$. Assume that Hyp is satisfied and $N \geq 3$ Then the energy $E(t)$ for the problem (1 1)-(1 2) satisfy the following decay properties*

(i) *If $0 < r < 2/3(N-2)$, then*

$$E(t) \leq C_1(1+t)^{-\gamma_1}$$

with

$$\gamma_1 = \min\left\{ \frac{(2-N)r^2 + 4(2-N)r + 8}{2(r+2)(4-(N-2)r)}, \frac{3(2-N)r^2 + 2(8-3N)r + 8}{2(r+2)} \right\}.$$

(ii) *If $r = 2/3(N-2)$ and $0 < r < -2 + 2\sqrt{N(N-2)}/(N-2)$, then*

$$E(t) \leq C_1(\log(2+t))^{-\gamma_2}$$

with

$$\gamma_2 = \frac{2 - (N-2)r}{2}$$

The proof of Theorem 2.2 relies on the the following lemmas.

First, we need the following well-known lemma without proof here.

LEMMA 2.1. *(Gagliardo-Nirenberg) Let* $1 \le r < p \le \infty$, $1 \le q \le p$ *and* $0 \le k \le m$. *Then we have the inequality*

$$||v||_{W^{k,p}} \le c||v||^{\theta}_{W^{m,q}}||v||^{1-\theta}_{L^r} \text{ for } v \in W^{m,p} \cap L^r$$

with some $c > 0$ *and*

$$\theta = \Big(\frac{k}{N} + \frac{1}{r} - \frac{1}{p}\Big)\Big(\frac{m}{N} + \frac{1}{r} - \frac{1}{q}\Big)^{-1}$$

provided that $0 < \theta \le 1$ *(*$0 < \theta < 1$ *if* $p = \infty$ *and* $mq = N$*).*

LEMMA 2.2. ([5]) *Let* $\phi(t)$ *be a nonnegative function on* $[0, \infty)$ *satisfying the inequality*

$$\sup_{t \le s \le t+T} \phi(s) \le C \sum_{i=1}^{2} (1+t)^{\theta_i} (\phi(t) - \phi(t+1))^{\epsilon_i},\ t \ge 0$$

with some $T > 0$, $C > 0$, $0 < \epsilon_i \le 1$ *and* $\theta_i \le \epsilon_i, i = 1, 2$. *Then* $\phi(t)$ *has the following decay property:*

(1) *If* $0 < \epsilon_i \le 1$ *and* $\theta_i < \epsilon_i, i = 1, 2$, *then*

$$\phi(t) \le C_0 (1+t)^{-\alpha}$$

with $\alpha = \min_{i=1,2}\{(\epsilon_i - \theta_i)/(1-\epsilon_i)\}$

(2) *If* $\theta_1 = \epsilon_1 < 1$ *and* $\theta_2 < \epsilon_2 \le 1$, *then*

$$\phi(t) \le C_0 (\log(2+t))^{-\frac{\epsilon_1}{1-\epsilon_1}}.$$

3. Some Useful Inequalities

Throughout the remainder of this paper, C denotes different positive generic constants, independent of the initial data, in various occurrences.

In this section, we will derive some useful inequalities to prove Theorem

LEMMA 3.1. *Let $q(x) = (q_1(x), q_2(x), \ldots, q_N(x)) \in (W^{1,\infty}(\mathbb{R}^N))^N$ be a vector field on $\mathbb{R}^N$ and $\varphi(x)$ a proper function on $\mathbb{R}^N$. Then for a solution $u(t)$ to the problem (1.1)-(1.2), we have the following identities :*

$$0 = \frac{d}{dt}E(t) + \int_{\mathbb{R}^N} \rho(x, u_t)u_t dx \tag{3.1}$$

$$\begin{aligned} 0 &= \frac{d}{dt}\left\{\int_{\mathbb{R}^N} \varphi(x)u_t u dx + \frac{1}{2}\int_{\Omega_R} \varphi(x)a(x)|u|^2 dx\right\} \\ &\quad - \int_{\mathbb{R}^N} \varphi(x)|u_t|^2 dx + \int_{\mathbb{R}^N} \nabla u \cdot \nabla(\varphi u) dx \\ &\quad + \int_{B_R} \varphi(x)\tilde{\rho}(u_t) u dx \end{aligned} \tag{3.2}$$

$$\begin{aligned} 0 &= \frac{d}{dt}\left\{\int_{\mathbb{R}^N} u_t q(x) \cdot \nabla u dx\right\} + \frac{1}{2}\int_{\mathbb{R}^N} \nabla \cdot q(x)|u_t|^2 dx \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^N} \nabla \cdot q(x)|\nabla u|^2 dx + \int_{\mathbb{R}^N} Q_{N\times N}\nabla u \cdot \nabla u dx \\ &\quad + \int_{\mathbb{R}^N} \rho(x, u_t) q(x) \cdot \nabla u dx, \end{aligned} \tag{3.3}$$

where $Q_{N\times N} = (a_{ij})$ is the $N \times N$ matrix with $a_{ij} = \partial q_i/\partial x_j$, $i, j = 1, 2, \ldots, N$ as its components

The proof of Lemma 3.1 is based on standard multiplier technique, using u_t, $\varphi(x)u$ and $q(x) \cdot \nabla u$ as multipliers, and the interested reader should refer to Komornik[1] or Nakao[5]

In order to obtain some estimate, we prepare the following Proposition:

PROPOSITION 3.1. *There exists $T_0 > 0$, independent of u, such that if $T > T_0$, then the inequality*

$$\int_0^T \int_{B_R} |u|^2 dx dt \le C(T) \int_0^T \int_{\mathbb{R}^N} \rho(x, u_t)u_t dx dt + \epsilon \int_0^T E(t) dt \tag{3.4}$$

holds for any $\epsilon > 0$.

Proof. We shall use a contradiction method(cf. Nakao[5]). Assume that (3.4) does not hold. Then, there exists a sequence of solutions $\{u_n\}$ of the problem (1.1)-(1 3) such that

$$\int_0^T \int_{B_R} |u_n|^2 dxdt \geq n \int_0^T \int_{\mathbb{R}^N} \rho(x, u_{nt}) u_{nt} dxdt + \epsilon \int_0^T E_{u_n}(t)dt, \tag{3.5}$$

where $E_{u_n}(t)$ is defined by $E(t)$ with u_n instead of u.

Setting

$$\lambda_n^2 \equiv \int_0^T \int_{B_R} |u_n|^2 dxdt \text{ and } v_n(t) \equiv \frac{u_n(t)}{\lambda_n},$$

we get by (3.5),

$$n \int_0^T \int_{\mathbb{R}^N} \frac{\rho(x, u_{nt})}{\lambda_n} v_{nt} dxdt + \epsilon \int_0^T E_{v_n}(t)dt \leq 1,$$

where $E_{v_n}(t)$ is defined by $E(t)$ with u replaced by v_n.

Thus we obtain

$$\int_0^T \int_{B_R} |v_n|^2 dxdt = 1 \text{ for all } n \geq 1, \tag{3.6}$$

$$\int_0^T \int_{\mathbb{R}^N} \frac{\rho(x, u_{nt})}{\lambda_n} v_{nt} dxdt \to 0 \text{ as } n \to \infty \tag{3.7}$$

and

$$\int_0^T E_{v_n}(t)dt = \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} (|v_{nt}|^2 + |\nabla v_n|^2) dxdt \leq \frac{2}{\epsilon} < \infty. \tag{3.8}$$

These imply, applying Rellich compactness theorem and replacing the sequence v_n with a subsequence if needed, that

(3.9) $v_n \to v$ weak-star in $L^\infty([0,T]; H_0^1(\mathbb{R}^N)) \cap W^{1,\infty}([0,T]; L^2(\mathbb{R}^N))$

(3 10) $v_n \to v$ strongly in $L^2([0,T] \times B_R$

Therefore (3.6), (3.7), (3.9) and (3.10) lead to the following limit problem :

$$v_{tt} - \triangle v = 0 \text{ in } \mathbb{R}^N \times [0, T] \tag{3.11}$$

with

$$\int_0^T \int_{B_R} |v|^2 dx dt = 1 \tag{3.12}$$

and

$$v_t(x,t) = 0 \text{ on } supp\ a(\cdot) \times [0, T].$$

Since $\Omega_R \subset$ supp $a(\cdot)$, by a general result of unique continuation(cf. Tataru[9]), there exists $T_0 > 0$ such that if $T > T_0$,

$$v_t(x,t) = 0 \text{ on } \mathbb{R}^N \times [0, T]. \tag{3.13}$$

Noting that (3.13) means that $v(x,t) = v(x)$, independent of t and using (3.11), we have

$$-\triangle v(x) = 0 \text{ in } \mathbb{R}^N.$$

Since $v \in H^1(\mathbb{R}^N)$ and $N \geq 3$, $v(x)$ must be identically zero in $\mathbb{R}^N$, which is a contradiction to (3.12). This completes the proof of Proposition 3.1. □

From now on we set

$$\begin{aligned} X(t) = & \int_{\mathbb{R}^N} (u_t \phi(r) x \cdot \nabla u dx + \alpha u_t u) dx \\ & + \frac{\alpha}{2} \int_{\Omega_R} a(x)|u|^2 dx + kE(t) \end{aligned} \tag{3.14}$$

Here α and $k > 0$ are some constants, and $\phi(r)$, $r = |x|$ is a Lipschitz continuous function on $[0, \infty)$ as follows

$$\phi(r) = \begin{cases} \epsilon_0 & \text{if } r \leq R \\ \frac{\epsilon_0 R}{r} & \text{if } r \geq R, \end{cases}$$

where ϵ_0 and R are positive constants given in (1.3).

Then we obtain the following(for proof, see [6]) :

PROPOSITION 3.2. *For $T > T_0$ and a large $k > 0$, there exists some constants $\overline{\epsilon}_1 > 0$ and $\overline{C} > 0$ such that the solution u of the problem (1.1)-(1.2) satisfies for any $t \geq 0$,*

$$X(t+T) + k\int_t^{t+T}\int_{\mathbb{R}^N} \rho(x,u_t)u_t dxds + \overline{\epsilon}_1 \int_t^{t+T} E(s)ds \leq X(t) + \overline{C}\int_t^{t+T}\int_{B_R} \tilde{\rho}(u_t)^2 dxds. \tag{3.15}$$

We observe that $X(t)$ is equivalent to $E(t) + ||u(t)||^2$ if k is sufficiently large Indeed, we have :

LEMMA 3.2. *For a large $k > 0$, there exist constants $C_1 > 0$ and $C_2 > 0$ such that for any $t \geq 0$,*

$$C_1(E(t) + ||u(t)||^2) \leq X(t) \leq C_2(E(t) + ||u(t)||^2) \tag{3.16}$$

Proof. Since the second inequality of (3 16) holds trivially, it's sufficient to show the first inequality of (3 16). Simple calculations using the Young inequality show that for some constant $C > 0$,

$$-CE(t) \leq \int_{\mathbb{R}^N} u_t \phi(r) x \cdot \nabla u dx \tag{3.17}$$

and for any $\epsilon > 0$ (may be small)

$$-\epsilon\int_{\mathbb{R}^N} |u|^2 dx - C(\epsilon)\int_{\mathbb{R}^N} |u_t|^2 dx \leq \int_{\mathbb{R}^N} \alpha u_t u dx. \tag{3.18}$$

Reporting (3.17) and (3 18) in (3.14) and noting that $\int_{B_R} |u|^2 dx \leq C\{\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\Omega_R} |u|^2 dx\}$ for some constant $C > 0$ and noting that

$a(x) \geq \epsilon_0 > 0$ in Ω_R, we get for any $\epsilon > 0$,

$$\begin{aligned} X(t) &\geq -\epsilon \int_{\mathbb{R}^N} |u|^2 dx - C(\epsilon) \int_{\mathbb{R}^N} |u_t|^2 dx \\ &\quad + \frac{\alpha\epsilon_0}{2} \int_{\Omega_R} |u|^2 dx + (k - C)E(t) \\ &\geq \left(\frac{\alpha\epsilon_0}{4C} - \epsilon\right) \int_{B_R} |u|^2 dx + \frac{\alpha\epsilon_0}{4} \int_{\Omega_R} |u|^2 dx - \frac{\alpha\epsilon_0}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &\quad - C(\epsilon) \int_{\mathbb{R}^N} |u_t|^2 dx + (k - C)E(t) \\ &\geq \min\left\{ \left(\frac{\alpha\epsilon_0}{4C} - \epsilon\right), \frac{\alpha\epsilon_0}{4} \right\} \int_{\mathbb{R}^N} |u|^2 dx \\ &\quad + \left(k - C - 2\max\left\{\frac{\alpha\epsilon_0}{2}, C(\epsilon)\right\}\right) E(t). \end{aligned}$$

Therefore we can always choose a proper constant $C_1 > 0$ if $\epsilon > 0$ is sufficiently small and k sufficiently large, which competes the proof of the Lemma. □

4. Proof of Theorem 2.2

We recall that the method used to prove the Theorem essentially relies on the multiplier technique and on some difference inequalities due to Nakao

First, let us derive the estimate of L^2-norm $||u(t)||$ of $u(t)$ To this end, we will use Proposition 3 2.

Indeed we have .

PROPOSITION 4.1. *For a solution $u(t)$ to the problem (1.1)-(1 2), we have the estimate of L^2-norm of $u(t)$,*

$$||u(t)|| \leq C(k_1, K, E(0))(1+t)^{\frac{r(N-2)}{2(4-(N-2)r)}}, \tag{4.1}$$

where $0 \leq r \leq 2/(N-2)$, $N \geq 3$.

Proof. First we note from (3 1) that for any $t > 0$

$$\int_0^t \int_{\mathbb{R}^N} \rho(x, u_t) u_t \, dx dt \leq E(0), \tag{4.2}$$

that is,

$$\int_0^\infty ||u_t||_{r+2}^{r+2} dt \le CE(0) < \infty$$

Using the Gagliardo-Nirenberg inequality (see Lemma 2.1), the Hölder inequality, (1.5) and (2.1), we obtain, for any $t > 0$,

$$
\begin{aligned}
(4.3) \qquad & \int_0^t \int_{B_R} \tilde{\rho}(u_t)^2 dx ds \\
\le\ & k_1^2 \int_0^t \int_{B_R} |u_t|^{2(r+1)} dx ds \\
\le\ & Ck_1^2 \int_0^t ||u_t||_{r+2}^{2(r+1)(1-\theta)} ||u_t||_{H^1(B_R)}^{2(r+1)\theta} ds \\
\le\ & Ck_1^2 K^{2(r+1)\theta} \int_0^t ||u_t||_{r+2}^{2(r+1)(1-\theta)} ds \\
\le\ & Ck_1^2 K^{2(r+1)\theta} \left(\int_0^t ||u_t||_{r+2}^{r+2} ds \right)^{\frac{2(r+1)(1-\theta)}{r+2}} \left(\int_0^t ds \right)^{\frac{2\theta(r+1)-r}{r+2}} \\
\le\ & Ck_1^2 K^{2(r+1)\theta} E(0)^{\frac{2(r+1)(1-\theta)}{r+2}} (t+1)^{\frac{2\theta(r+1)-r}{r+2}} \\
\equiv\ & C(k_1, K, E(0))(1+t)^{\frac{r(N-2)}{4-(N-2)r}}
\end{aligned}
$$

with $\theta = Nr/(r+1)(4-(N-2)r)$, where we have used $0 \le r \le 2/(N-2)$ $(\le 4/(N-2))$

Thus by Lemma 3.2 and (4.3), we have for $t \ge 0$,

$$X(t) \le X(0) + C(k_1, K, E(0))(t+1)^{\frac{r(N-2)}{4-(N-2)r}},$$

that is,

$$(4\ 4) \qquad ||u(t)|| \le C(k_1, K, E(0))(t+1)^{\frac{r(N-2)}{2(4-(N-2)r)}},$$

where $0 \le r \le 2/(N-2)$ □

We are now in a position to prove Theorem 2.2

Multiplying equation (1.1) by u_t and integrating over $[t, t+T]\times\mathbb{R}^N$, $t>0$, and recalling the definition of $\rho(x,u_t)$, we have

$$
\begin{aligned}
(4.5)\qquad & \int_t^{t+T}\int_{\mathbb{R}^N}\rho(x,u_t)u_t dxds \\
= & \int_t^{t+T}\int_{B_R}\tilde{\rho}(u_t)u_t dxds + \int_t^{t+T}\int_{\Omega_R} a(x)|u_t|^2 dxds \\
= & \; E(t)-E(t+T)\equiv D(t)^{r+2}.
\end{aligned}
$$

Also multiplying equation (1.1) by u and integrating we have

$$
\begin{aligned}
(4.6)\qquad & \int_t^{t+T}\int_{\mathbb{R}^N}(|\nabla u|^2-|u_t|^2)dxds \\
= & \int_{\mathbb{R}^N}(u_t(t)u(t)-u_t(t+T)u(t+T))dx \\
& -\int_t^{t+T}\int_{\mathbb{R}^N}\rho(x,u_t)u dxds.
\end{aligned}
$$

Next we note from(4.5) that

$$
\begin{aligned}
(4.7)\qquad & \int_t^{t+T}\int_{\mathbb{R}^N}|u_t|^2 dxds \\
= & \int_t^{t+T}\int_{B_R}|u_t|^2 dxds + \int_t^{t+T}\int_{\Omega_R}|u_t|^2 dxds \\
\le & \int_t^{t+T}\int_{B_R}|u_t|^2 dxds + \frac{1}{\epsilon_0}D(t)^{r+2}.
\end{aligned}
$$

Combining (4.6) and (4.7) yields

$$\begin{aligned}
(4.8)\ &\int_t^{t+T}\int_{\mathbb{R}^N}(|u_t|^2+|\nabla u|^2)dxds\\
&\le \int_{\mathbb{R}^N}(u_t(t)u(t)-u_t(t+T)u(t+T))dx\\
&-\int_t^{t+T}\int_{\mathbb{R}^N}\rho(x,u_t)udxds+2\int_t^{t+T}\int_{B_R}|u_t|^2dxds+\frac{2}{\epsilon_0}D(t)^{r+2}\\
&\le C\left\{||u_t(t)||||u(t)||+||u_t(t+T)||||u(t+T)||\right.\\
&\left.+\int_t^{t+T}\int_{\mathbb{R}^N}|\rho(x,u_t)||u|dxds+\int_t^{t+T}\int_{B_R}|u_t|^2dxds+D(t)^{r+2}\right\}\\
&\equiv I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t).
\end{aligned}$$

Using the Hölder inequality and (4.5), we get

$$\begin{aligned}
(4.9)\quad I_4(t) &\le \left(\int_t^{t+T}\int_{B_R}dxds\right)^{\frac{r}{r+2}}\left(\int_t^{t+T}\int_{B_R}|u_t|^{r+2}\right)^{\frac{2}{r+2}}\\
&\le CD(t)^2.
\end{aligned}$$

In order to estimate terms $I_1(t)$ and $I_2(t)$, first, we observe that by (4.7) and (4.9),

$$(4.10)\qquad \int_t^{t+T}||u_t||^2ds\le C\left\{D(t)^2+D(t)^{r+2}\right\}$$

From the last inequality, we easily see that

$$(4.11)\qquad ||u_t(s)||\le C\left\{D(s)+D(s)^{\frac{r+2}{2}}\right\}\ \text{for } t\le s\le t+T.$$

Accordingly, using Proposition 4 1 and (4.11), we find that

$$(4.12)\qquad I_1(t)+I_2(t)\le C(1+t)^{\frac{r(N-2)}{2(4-(N-2)r)}}\left\{D(t)+D(t)^{\frac{r+2}{2}}\right\}.$$

Finally, let us derive the estimate of $I_3(t)$.

$$
\begin{aligned}
(4.13) \qquad I_3(t) &\le k_1 \int_t^{t+T} \int_{B_R} |u_t|^{r+1} |u| dx ds \\
&\quad + \int_t^{t+T} \int_{\Omega_R} a(x) |u_t| |u| dx ds \\
&\equiv J_1(t) + J_2(t)
\end{aligned}
$$

Using the Hölder inequality and the similar way as in the proof of Proposition 4.1 (see (cf. 4 3)), we have

$$
\begin{aligned}
(4.14) \quad & J_1(t) \\
\le\ & k_1 \left(\int_t^{t+T} \int_{B_R} |u_t|^{2(r+1)} dx ds \right)^{\frac{1}{2}} \left(\int_t^{t+T} \int_{\mathbb{R}^N} |u|^2 dx ds \right)^{\frac{1}{2}} \\
\le\ & C(k_1, T, K, E(0))(1+t)^{\frac{r(N-2)}{2(4-(N-2)r)}} \left(\int_t^{t+T} \|u_t\|_{r+2}^{2(r+1)(1-\theta)} ds \right)^{\frac{1}{2}} \\
\le\ & C(k_1, T, K, E(0))(1+t)^{\frac{r(N-2)}{2(4-(N-2)r)}} \left(\int_t^{t+T} \|u_t\|_{r+2}^{r+2} ds \right)^{\frac{(r+1)(1-\theta)}{r+2}} \\
\le\ & C(k_1, T, K, E(0))(1+t)^{\frac{r(N-2)}{2(4-(N-2)r)}} D(t)^{(r+1)(1-\theta)} \\
=\ & C(k_1, T, K, E(0))(1+t)^{\frac{r(N-2)}{2(4-(N-2)r)}} D(t)^{\frac{(r+2)(2-(N-2)r)}{4-(N-2)r}}
\end{aligned}
$$

with $\theta = Nr/(r+1)(4-(N-2)r)$, where we have used $0 < r \le 2/(N-2)$ and

$$
\begin{aligned}
(4.15) \qquad & J_2(t) \\
\le\ & C \left(\int_t^{t+T} \int_{\Omega_R} |u_t|^2 dx ds \right)^{\frac{1}{2}} \left(\int_t^{t+T} \int_{\mathbb{R}^N} |u|^2 dx ds \right)^{\frac{1}{2}} \\
\le\ & C(k_1, T, K, E(0))(1+t)^{\frac{r(N-2)}{2(4-(N-2)r)}} D(t)^{\frac{r+2}{2}}.
\end{aligned}
$$

Reportıng (4.14) and (4.15) ın (4.13) and combınıng (4 8), (4 9), (4 12) and (4.13), we obtaın

$$(4.16)\quad \int_t^{t+T} E(s)ds$$
$$\le C\left\{(1+t)^{\frac{r(N-2)}{2(4-(N-2)r)}}D(t)+D(t)^2+D(t)^{r+2}\right.$$
$$\left.+(1+t)^{\frac{r(N-2)}{2(4-(N-2)r)}}D(t)^{(r+1)(1-\theta)}+(1+t)^{\frac{r(N-2)}{2(4-(N-2)r)}}D(t)^{\frac{r+2}{2}}\right\}$$
$$\le C(1+t)^{\frac{r(N-2)}{2(4-(N-2)r)}}\left\{D(t)+D(t)^{\frac{(r+2)(2-(N-2)r)}{4-(N-2)r}}\right\}.$$

Applyıng Lemma 2.2 to the above inequality we obtaın the estimates in Theorem.

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Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail ilhjung@pusan.ac.kr