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가중표준편차를 이용한 비대칭 모집단에 대한 다변량 공정능력지수

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Multivariate Process Capability Indices for Skewed Populations with Weighted Standard Deviations

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This paper proposes multivariate process capability indices (PCIs) for skewed populations using T^2 and modified process region approaches. The proposed methods are based on the multivariate version of a weighted standard deviation method which adjusts the variance-covariance matrix of quality characteristics and approximates the probability density function using several multivariate normal distributions with the adjusted variance-covariance matrix. Performance of the proposed PCIs is investigated using Monte Carlo simulation, and finite sample properties of the estimators are studied by means of relative bias and mean square error.

Keywords: process capability index, skewed population, weighted standard deviation, T^2 approach, modified process region approach

1. Introduction

Process capability analysis is an important and integral part of the statistical process control activities for the continuous improvement of quality and productivity. The capability of a process is frequently measured by a process capability index (PCI) which is designed to provide a common and easily understood language for quantifying its performance with a single-number summary, and is a dimensionless function of process parameters and specifications. C_p and C_{pk} are widely used PCIs based on univariate quality measurements. However, capability analyses involving more than one quality characteristics are sometimes of interest, and multivariate statistical techniques can be used to analyze several quality characteristics simultaneously.

A difficulty of defining multivariate PCIs is that there is no consensus on the methodology for assessing capability, which arises since the multivariate relationship among the quality characteristics may or may not be reflected in the engineering specifications. In general, the upper specification limit (USL) and the lower specification limit (LSL) may be given for each quality characteristic and these specification ranges, taken together, form a rectangle or hypercube. A multivariate PCI should compare the shapes, locations, sizes, and orientations arising from the statistical distribution with those from the engineering specifications, which can lead to very different definitions of capability in the multivariate domain.

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Several multivariate PCIs have been proposed. Hubele *et al*. (1991) proposed a multivariate capability vector which consists of three components, a ratio of area or volumes, a relative location of the process and specification centers, and a relative location of maximum and minimum of the probability contour and the specification limits. Taam *et al*. (1993) proposed MC_{pm} which is the ratio of specification region to a scaled 99.73 percent process region. Karl *et al*. (1994) considered geometric dimensioning and tolerancing which is a set of standards to describe the physical features and their specified tolerances. Wang and Chen (1998-99) suggested to use principal components analysis. For more detailed reviews, see Kotz and Lovelace (1998) and Wang *et al*. (2000).

These methods, however, are confined to the case of multivariate normal populations. In practice, the normality assumption is usually difficult to justify and is often not appropriate. For example, the measurements from chemical processes, filling processes, and semiconductor processes are often skewed. Recently, Polansky (2001) proposed a nonparametric approach based on a kernel estimate of an integral of a multivariate density. This procedure can do without the normality assumption, but may be somewhat complicated for practitioners and may need a large amount of data to perform well. Therefore, it is necessary to develop a multivariate PCI for skewed populations which is simple and performs reasonably well even with a small data set.

This paper proposes methods of constructing simple multivariate PCIs for skewed populations based on the multivariate version of a 'weighted standard deviation (WSD) method of Chang and Bai (2002). This method adjusts the variance-covariance matrix in accordance to the degree of skewness of the distribution by using different factors in computing the deviations above and below the process mean and approximates the probability density function (PDF) using several multivariate normal distributions with adjusted variance-covariance matrix.

This paper is organized as follows: Section 2 reviews the WSD method. Section 3 proposes two multivariate WSD PCIs using T^2 and modified process region (Hubele *et al*. (1991)) approaches. The T^2 approach uses Hotelling's T^2 statistic to reduce the dimension of quality characteristics, and the process region approach defines the PCI as the ratio of the volume of engineering tolerance region to the volume of modified process region. The performance of the proposed PCIs is investigated in Section 4, and the finite sample properties are studied in Section 5.

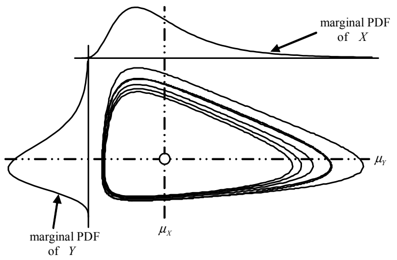
2. Weighted Standard Deviation(WSD) Method

Chang and Bai (2001) and Chang *et al*. (2002) proposed a WSD method to construct univariate control charts and PCIs for skewed populations, respectively. The WSD method is based on the idea that standard deviation σ_X can be divided into upper and lower deviations, σ_U^W and σ_L^W, which represent the degree of the dispersions of the upper and lower sides from mean μ_X, respectively. WSDs σ_U^W and σ_L^W are obtained as $\sigma_U^W = P_X \sigma_X$ and $\sigma_L^W = (1-P_X)\sigma_X$, where $P_X = \Pr\{X \le \mu_X\}$. An asymmetric PDF can be approximated with two normal PDFs with the same mean μ_X but different standard deviations $2\sigma_U^W$ and $2\sigma_L^W$. Details can be found in Chang and Bai (2001).

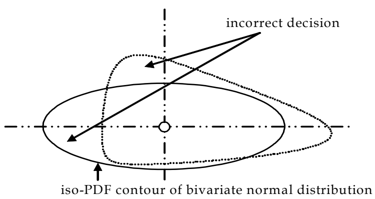
Chang and Bai (2002) suggested a multivariate WSD method and constructed a WSD T^2 control chart for skewed populations. The multivariate WSD method adjusts the variance-covariance matrix with WSDs of each quality characteristic. However, we should not tamper with the correlation matrix since it represents the dependent structure of quality characteristics and a multivariate control chart must reflect this dependency. If the variance-covariance matrix is adjusted and the correlation matrix is maintained, the scale of the conventional control region is adjusted in accordance with skewness but the direction is maintained.

<Figure 1> depicts the iso-PDF contours of the original distribution, bivariate normal distribution, and approximated distribution by the WSD method with two quality characteristics, X and Y. For simplicity, the correlation of X and Y is set to zero. <Figure 1(a)> describes the iso-PDF contours of the original bivariate distribution, which show that the distribution of (X, Y) is skewed. <Figure 1(b)> shows that the iso-PDF contour of the bivariate normal distribution is very different from that of original distribution and the rate of incorrect decisions will be large if the standard method based on the normality assumption is used. <Figure 1(c)> represents the concept of a multivariate version of the WSD method. Similarly to the univariate case, the original bivariate PDF can be approximated with four segments each from four bivariate normal PDFs which are obtained with the combination of normal PDFs derived from marginal PDFs of X and Y. <Figure 1(c)> shows that the iso-PDF contour of the original skewed distribution is approximated with four parts each from the iso-PDF contour of the derived bivariate normal distributions,

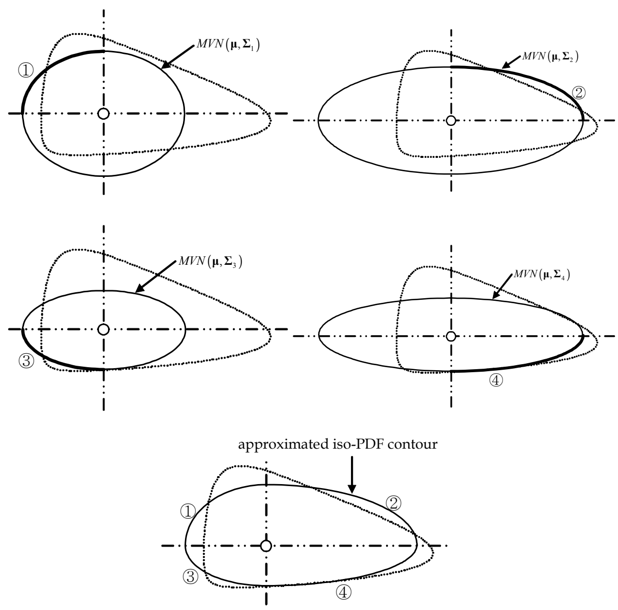
①, ②, ③, and ④. It is similar to the iso-PDF contour of the original PDF, and we can see that the WSD method can effectively approximate the PDF of a bivariate skewed distribution.



(a) iso-PDF contours of skewed distribution



(b) iso-PDF contour of normal distribution



(c) iso-PDF contour approximated with WSD method

MVN(μ, Σ): multivariate normal distribution with mean vector μ and variance-covariance matrix Σ

Figure 1. Iso-PDF contours and concept of multivariate WSD Method.

Assume that ν-variate random vector $X=(X_1,\cdots,X_\nu)^T$ is distributed with mean vector $\mu=(\mu_1,\cdots,\mu_\nu)^T$ and variance-covariance matrix

$$\Sigma=\begin{bmatrix}\sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1\nu}\sigma_1\sigma_\nu \\ & \sigma_2^2 & \cdots & \rho_{2\nu}\sigma_2\sigma_\nu \\ & & \ddots & \vdots \\ & & & \sigma_\nu^2\end{bmatrix}, \quad (1)$$

where 'T' denotes the transpose of a vector or matrix, σ_j is the standard deviation of X_j, and ρ_{ij} is the correlation coefficient of X_i and X_j. For the approximation such as <Figure 1(c)>, the variance-covariance matrix should be adjusted as follows:

$$\Sigma^W= W\Sigma W=\begin{bmatrix}(\sigma_1^W)^2 & \rho_{12}\sigma_1^W\sigma_2^W & \cdots & \rho_{1\nu}\sigma_1^W\sigma_\nu^W \\ & (\sigma_2^W)^2 & \cdots & \rho_{2\nu}\sigma_2^W\sigma_\nu^W \\ & & \ddots & \vdots \\ & & & (\sigma_\nu^W)^2\end{bmatrix}, \quad (2)$$

where

$$W= diag\{W_1,\cdots,W_\nu\},\qquad W_j=\begin{cases}2P_j, & \text{if } X_j>\mu_j,\\ 2(1-P_j), & \text{otherwise},\end{cases} \quad (3)$$

$\sigma_j^W= W_j\,\sigma_j$, and $P_j=\Pr\{X_j\le\mu_j\}$. If X_j is greater than μ_j, the PDF related to X_j is modified by adjusting the jth row and column of the variance-covariance matrix using upper deviation $2P_j\sigma_j$ in place of σ_j. Otherwise, the jth row and column of the variance-covariance matrix is adjusted using lower deviation $2(1-P_j)\sigma_j$. Note that correlation matrix $\rho=\{\rho_{ij}\}$ does not change after the variance-covariance matrix is adjusted. This WSD method approximates the original PDF with segments from 2^ν multivariate normal distributions. <Figure 1(c)> is obtained with four bivariate normal distributions with the same mean vector $\mu=(\mu_X,\mu_Y)^T$ but different variance-covariance matrices as follows:

$$\Sigma_1=\begin{bmatrix}\{2(1-P_X)\sigma_X\}^2 & \rho_{XY}\cdot 2(1-P_X)\sigma_X\cdot 2P_Y\sigma_Y \\ \rho_{XY}\cdot 2(1-P_X)\sigma_X\cdot 2P_Y\sigma_Y & \{2P_Y\sigma_Y\}^2\end{bmatrix},$$

$$\Sigma_2=\begin{bmatrix}\{2P_X\sigma_X\}^2 & \rho_{XY}\cdot 2P_X\sigma_X\cdot 2P_Y\sigma_Y \\ \rho_{XY}\cdot 2P_X\sigma_X\cdot 2P_Y\sigma_Y & \{2P_Y\sigma_Y\}^2\end{bmatrix},$$

$$\Sigma_3=\begin{bmatrix}\{2(1-P_X)\sigma_X\}^2 & \rho_{XY}\cdot 2(1-P_X)\sigma_X\cdot 2(1-P_Y)\sigma_Y \\ \rho_{XY}\cdot 2(1-P_X)\sigma_X\cdot 2(1-P_Y)\sigma_Y & \{2(1-P_Y)\sigma_Y\}^2\end{bmatrix},$$

$$\Sigma_4=\begin{bmatrix}\{2P_X\sigma_X\}^2 & \rho_{XY}\cdot 2P_X\sigma_X\cdot 2(1-P_Y)\sigma_Y \\ \rho_{XY}\cdot 2P_X\sigma_X\cdot 2(1-P_Y)\sigma_Y & \{2(1-P_Y)\sigma_Y\}^2\end{bmatrix}.$$

Since the WSD method approximates the PDF of the original distribution using normal distributions, the standard statistical methods based on the normality assumption can be used with the approximated distribution.

3. Multivariate PCIs Based on the WSD Method

3.1 T^2 Approach

We now propose multivariate T^2 WSD PCIs. The T^2 statistic is often used in multivariate control charts. The univariate C_{pk} is defined as the ratio of the distance of the specification limits from the process mean to the actual process spread $3\sigma_X$:

$$C_{pk} = \min\left\{\frac{USL-\mu_X}{3\sigma_X}, \frac{LSL-\mu_X}{3\sigma_X}\right\} = \min\left\{\frac{USL_Z}{3}, \frac{LSL_Z}{3}\right\}, \quad (4)$$

where USL_Z and LSL_Z are the standardized distances from the mean to the upper and lower specification limits, respectively, and the number 3 means the 99.865th percentile of the standard normal distribution which is the same as $\sqrt{\chi^2_{0.0027}(1)}$. Therefore, formula (4) can be extended to the multivariate case as follows:

$$C_{pk,T^2} = \min\left\{\sqrt{\frac{L_1^T\rho^{-1}L_1}{\chi^2_{0.0027}(\nu)}}, \cdots, \sqrt{\frac{L_{2^\nu}^T\rho^{-1}L_{2^\nu}}{\chi^2_{0.0027}(\nu)}}\right\} = \min\left\{\sqrt{\frac{L_{T^2,1}}{\chi^2_{0.0027}(\nu)}}, \cdots, \sqrt{\frac{L_{T^2,2^\nu}}{\chi^2_{0.0027}(\nu)}}\right\}, \quad (5)$$

where L_i is the coordinates of edge i among 2^ν edges of the hypercube made by standardized specification limits $USL_{Z_k}=(USL_k-\mu_k)/\sigma_k$ and $LSL_{Z_k}=(LSL_k-\mu_k)/\sigma_k$, $k=1,2,\cdots,\nu$. The order of edges is meaningless, but we assume that $L_1=(USL_{Z_1},\cdots,USL_{Z_\nu})^T$ and $L_{2^\nu}=(LSL_{Z_1},\cdots,LSL_{Z_\nu})^T$. For example, if $\nu=2$, $L_1=(USL_{Z_1}, USL_{Z_2})^T$ and $L_4=(LSL_{Z_1}, LSL_{Z_2})^T$, and $L_2=(USL_{Z_1}, LSL_{Z_2})^T$ and $L_3=(LSL_{Z_1}, USL_{Z_2})^T$ or $L_2=(LSL_{Z_1}, USL_{Z_2})^T$ and $L_3=(USL_{Z_1}, LSL_{Z_2})^T$. If all the correlation coefficients are positive, only L_1 and L_{2^ν} can be considered since they have the shortest distances from the mean. <Figure 2> describes the T^2 approach for a bivariate distribution with a positive correlation, and shows that C_{pk,T^2} is the ratio the standardized specification lengths $L_{T^2,1}$ and $L_{T^2,4}$ to the process dispersion $\chi^2_{0.0027}(2)$.

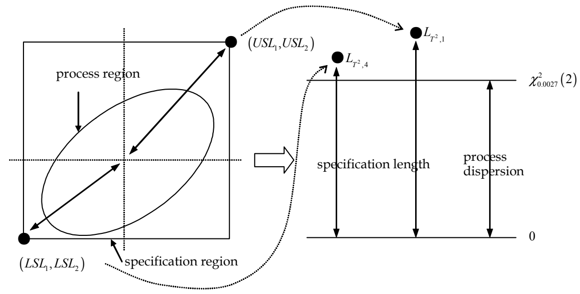


Figure 2. Concept of T^2 approach.

When the distribution is skewed, the WSD method can be applied, so that formula (5) can be extended to

$$C^{WSD}_{pk,T^2} = \min\left\{\sqrt{\frac{L_1^{W^T}\rho^{-1}L_1^W}{\chi^2_{0.0027}(\nu)}}, \cdots, \sqrt{\frac{L_{2^\nu}^{W^T}\rho^{-1}L_{2^\nu}^W}{\chi^2_{0.0027}(\nu)}}\right\} = \min\left\{\sqrt{\frac{L^W_{T^2,1}}{\chi^2_{0.0027}(\nu)}}, \cdots, \sqrt{\frac{L^W_{T^2,2^\nu}}{\chi^2_{0.0027}(\nu)}}\right\}, \quad (6)$$

where the kth component of L_i^W is $USL_{Z_k^W}=(USL_k-\mu_k)/\sigma_k^W$ or $LSL_{Z_k^W}=(LSL_k-\mu_k)/\sigma_k^W$, $k=1,2,\cdots,\nu$.

3.2 Modified Process Region Approach

Hubele *et al*. (1991) proposed a 'modified process region approach' which defines a multivariate PCI as the ratio of the volume of engineering tolerance region to the volume of modified process region. <Figure 3> illustrates the concept of the method. The modified process region can be obtained by drawing the smallest rectangle around the elliptical probability contour. The edges of the rectangle are defined as the upper and lower process limits, UPL_i and LPL_i, $i=1,\cdots,\nu$, determined by solving the system of equations of first derivatives, with respect to X_i, of the quadratic form

$$(X-\mu)^T\Sigma^{-1}(X-\mu) = \chi^2_{0.0027}(\nu).$$

The solutions to the equation provide the upper and lower limits as follows:

$$UPL_i = \mu_i + \sqrt{\frac{\chi^2_{0.0027}(\nu)\cdot det(\Sigma_i^{-1})}{det(\Sigma^{-1})}}, \quad LPL_i = \mu_i - \sqrt{\frac{\chi^2_{0.0027}(\nu)\cdot det(\Sigma_i^{-1})}{det(\Sigma^{-1})}}, \quad (7)$$

where $det(\cdot)$ is the determinant of a matrix and Σ_i is the matrix obtained from Σ by deleting row i and column i. The approach defines the multivariate PCI $C_{p,M}$ as

$$C_{p,M}=\left[\prod_{i=1}^{\nu}\left(\frac{USL_i-LSL_i}{UPL_i-LPL_i}\right)\right]^{1/\nu}. \qquad (8)$$

If the process mean and specification midpoint are incorporated in the PCI and the specification limits are standardized, $C_{pk,M}$ can be defined as

$$C_{pk,M}=\left[\prod_{i=1}^{\nu}\min\left\{\frac{USL_{Z_i}}{UPL_{Z_i}},\ \frac{LSL_{Z_i}}{LPL_{Z_i}}\right\}\right]^{1/\nu}, \qquad (9)$$

where

$$UPL_{Z_i}=\sqrt{\frac{\chi^2_{0.0027}(\nu)\cdot det(\rho_i^{-1})}{det(\rho^{-1})}}, \qquad (10)$$

$$LPL_{Z_i}=-\sqrt{\frac{\chi^2_{0.0027}(\nu)\cdot det(\rho_i^{-1})}{det(\rho^{-1})}}=-UPL_{Z_i}.$$

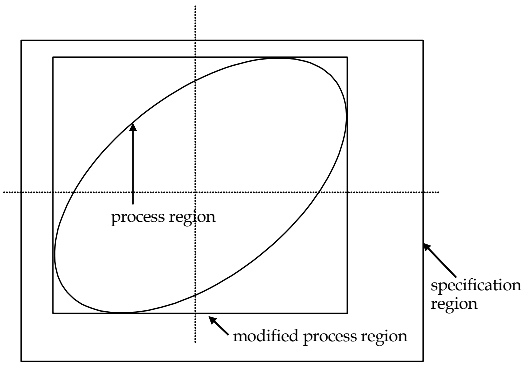


Figure 3. concept of modified process region approach.

When the distribution is skewed, the WSD method can be applied. $USL^W_{Z_i}$ and $LSL^W_{Z_i}$ replace USL_{Z_i} and LSL_{Z_i} in formula (10), respectively, and hence $C^{WSD}_{pk,M}$ is defined as

$$C^{WSD}_{pk,M}=\left[\prod_{i=1}^{\nu}\min\left\{\frac{USL^W_{Z_i}}{UPL_{Z_i}},\ \frac{LSL^W_{Z_i}}{LPL_{Z_i}}\right\}\right]^{1/\nu}. \qquad (11)$$

3.3 Estimation of Parameters

To use PCIs in practice, the parameters must be estimated. If a random sample of size n is obtained, μ and Σ can be estimated by the sample mean

$$\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$$

and the sample variance-covariance matrix

$$S=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})(X_i-\overline{X})^T,$$

respectively. The correlation coefficient of jth and kth random variables ρ_{jk} can be estimated by

$$\hat{\rho}_{jk}=\frac{S_{ij}}{\sqrt{S_{jj}}\sqrt{S_{kk}}},$$

where S_{jk} is the (j,k)th element of S. P_j can be estimated by the number of observations less than or equal to the sample mean of jth quality characteristic $\overline{X}_j$ as

$$\hat{P}_j=\frac{1}{n}\sum_{i=1}^{n}I(\overline{X}_j-X_{ij}),$$

where $I(x)=1$ if $x\geq 0$ or $I(x)=0$ otherwise.

3.4 An Illustrative Example

We illustrate the use of the proposed WSD PCIs with the bivariate process data in Sultan (1986) dealing with 25 observations of the Brinell hardness (X_1) and the tensile strength (X_2) of a process. These data were also used by Chan *et al*. (1988), Chen (1994), and Wang and Chen (1998-99). <Table 1> presents the data, and <Figure 4> depicts the histograms of X_1 and X_2 and shows that the marginal distributions of X_1 and X_2 are slightly skewed to the left. The specification limits for X_1 and X_2 were set at (112.7, 2451.3) and (32.7, 73.3), respectively.

Table 1. Data for the example

i	1	2	3	4	5	6	7	8	9	10	11	12	13
X_{i1}	143	200	160	181	148	178	162	215	161	141	175	187	187
X_{i2}	34.3	57.0	47.5	53.4	47.8	51.5	45.9	59.1	48.4	47.3	57.3	58.5	58.2
i	14	15	16	17	18	19	20	21	22	23	24	25	
X_{i1}	186	172	182	177	204	178	196	160	183	179	194	181	
X_{i2}	57.0	49.4	57.2	50.6	55.1	50.9	57.9	45.5	53.9	51.2	57.5	55.6	

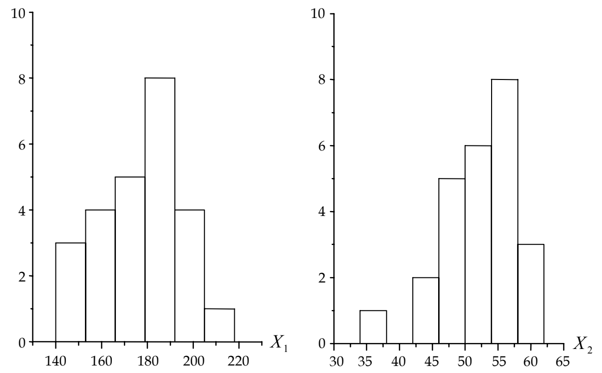


Figure 4. Histograms of data in <table 1>.

From the data, $\overline{X}_1=177.2$, $\overline{X}_2=52.32$, $\widehat{\sigma}_1=18.38$, $\widehat{\sigma}_2=5.80$, $\widehat{\rho}_{12}=0.80$, $\widehat{P}_1=0.40$, and $\widehat{P}_2=0.48$. We can obtain $LSL_{Z_1}=(LSL_1-\overline{X}_1)/\widehat{\sigma_1}=(112.7-177.2)/18.38=-3.51$, $USL_{Z_1}=3.49$, $LSL_{Z_2}=-3.38$, $USL_{Z_2}=3.62$, $LSL^W_{Z_1}=LSL_{Z_1}/2(1-\widehat{P}_1)=-2.93$, $USL^W_{Z_1}=4.36$, $LSL^W_{Z_2}=-3.25$, $USL^W_{Z_2}=3.77$. Also,

$$L_{T^2,1}=\boldsymbol{L}_1^T\boldsymbol{\rho}^{-1}\boldsymbol{L}_1=[-3.51\ \ -3.38]\begin{bmatrix}1 & 0.8\\ 0.8 & 1\end{bmatrix}\begin{bmatrix}-3.51\\ -3.38\end{bmatrix}=13.57,$$

$$L_{T^2,4}=\boldsymbol{L}_4^T\boldsymbol{\rho}^{-1}\boldsymbol{L}_4=[3.49\ \ 3.62]\begin{bmatrix}1 & 0.8\\ 0.8 & 1\end{bmatrix}\begin{bmatrix}3.49\\ 3.62\end{bmatrix}=14.08,$$

$$L^W_{T^2,1}=\boldsymbol{L}_1^{W^T}\boldsymbol{\rho}^{-1}\boldsymbol{L}_1=[-2.93\ \ -3.25]\begin{bmatrix}1 & 0.8\\ 0.8 & 1\end{bmatrix}\begin{bmatrix}-2.93\\ -3.25\end{bmatrix}=10.87,$$

$$L^W_{T^2,4}=\boldsymbol{L}_4^{W^T}\boldsymbol{\rho}^{-1}\boldsymbol{L}_4=[4.36\ \ 3.77]\begin{bmatrix}1 & 0.8\\ 0.8 & 1\end{bmatrix}\begin{bmatrix}4.36\\ 3.77\end{bmatrix}=19.23,$$

$$UPL_{Z_i}=\sqrt{\frac{\chi^2_{0.0027}(2)\cdot det(\boldsymbol{\rho}_i^{-1})}{det(\boldsymbol{\rho}^{-1})}}=\sqrt{\frac{11.83\cdot 2.78}{2.78}}=3.44$$

and $LPL_{Z_i}=-UPL_{Z_i}=-3.44$ for $i=1,2$. Then the estimated PCIs are:

The T^2 approach —

$$\widehat{C}_{pk,T^2}=\min\left\{\sqrt{\frac{L_{T^2,1}}{\chi^2_{0.0027}(2)}},\ \sqrt{\frac{L_{T^2,4}}{\chi^2_{0.0027}(2)}}\right\}=\min\left\{\sqrt{\frac{13.57}{11.83}},\ \sqrt{\frac{14.08}{11.83}}\right\}=\min\{1.07,\ 1.09\}=1.07$$

$$\widehat{C}^{WSD}_{pk,T^2}=\min\left\{\sqrt{\frac{L^W_{T^2,1}}{\chi^2_{0.0027}(2)}},\ \sqrt{\frac{L^W_{T^2,4}}{\chi^2_{0.0027}(2)}}\right\}=\min\left\{\sqrt{\frac{10.87}{11.83}},\ \sqrt{\frac{19.23}{11.83}}\right\}=\min\{0.96,\ 1.27\}=0.96$$

The modified process region approach —

$$\widehat{C}_{pk,M}=\left[\prod_{i=1}^{2}\min\left\{\frac{USL_{Z_i}}{UPL_{Z_i}},\frac{LSL_{Z_i}}{LPL_{Z_i}}\right\}\right]^{1/2}=\left[\min\left\{\frac{3.49}{3.44},\frac{-3.51}{-3.44}\right\}\cdot\min\left\{\frac{3.62}{3.44},\frac{-3.38}{-3.44}\right\}\right]^{1/2}=[1.01\cdot 0.98]^{1/2}=0.99$$

$$\widehat{C}^{WSD}_{pk,M}=\left[\prod_{i=1}^{2}\min\left\{\frac{USL^W_{Z_i}}{UPL_{Z_i}},\frac{LSL^W_{Z_i}}{LPL_{Z_i}}\right\}\right]^{1/2}=\left[\min\left\{\frac{4.36}{3.44},\frac{-2.93}{-3.44}\right\}\cdot\min\left\{\frac{3.77}{3.44},\frac{-3.25}{-3.44}\right\}\right]^{1/2}=[0.85\cdot 0.94]^{1/2}=0.89$$

$\widehat{C}^{WSD}_{pk,T^2}$ and $\widehat{C}^{WSD}_{pk,M}$ are smaller than $\widehat{C}_{pk,T^2}$ and $\widehat{C}_{pk,M}$, respectively, and this shows that the process is less satisfactory than $\widehat{C}_{pk,T^2}$ and $\widehat{C}_{pk,M}$ indicate.

4. Performance of the WSD PCIs

The performances of the proposed PCIs are studied when the distribution is multivariate normal, multivariate lognormal, Hougaard's bivariate Weibull, or Cheriyan and Ramabhadran's bivariate gamma. See Kotz *et al.* (2000) for detailed discussions on theses distributions. For all cases, it is assumed that $USL_i=3$ and $LSL_i=-3$, and the distribution is shifted and scaled to produce the same value of $\mu_i=0$ and $\sigma_i=1$, $i=1,\cdots,\nu$.

<Tables 2> gives C_{pk,T^2}, and $C_{pk,M}$ as the sample size n increases when the distribution is bivariate normal and the correlation coefficient ρ is nonnegative. It also gives nonconforming proportion per million (NPM) and MC_p. MC_p is calculated by $-(1/3)\Phi^{-1}((NPM/2)\times 10^{-6})$ since $NPM\times 10^{-6}=2\Phi(-3C_p)$

under the normality assumption, where $\Phi(\cdot)$ is the cumulative standard normal distribution function. If the value of PCI is close to that of MC_p or MC_{pk}, the PCI can be considered to describe the process capability very well in terms of nonconforming proportion. The Table shows that:

i) If the parameters are known ($n=\infty$), MC_p slightly increases as the correlation coefficient ρ increases but C_{pk,T^2} decreases and $C_{pk,M}$ does not change. Also, C_{pk,T^2} overestimates the process capability when the correlation is low and underestimates it when the correlation is high, and $C_{pk,M}$ underestimates it for all ranges of correlation. For highly correlated normal populations, C_{pk,T^2} performs better than $C_{pk,M}$, and vise versa.

ii) If the parameters are unknown ($n=50, 100, 200$), the biases of $\hat{C}_{pk,T^2}$ and $\hat{C}_{pk,M}$ are negative.

<Table 3> and <Table 4> give C_{pk,T^2}, C^{WSD}_{pk,T^2}, $C_{pk,M}$, and $C^{WSD}_{pk,M}$ for bivariate lognormal, Weibull, and gamma distributions with correlation $\rho=0.3$ or

Table 2. NPM, MC_p, C_{pk,T^2} and $C_{pk,M}$ for bivariate normal distributions

ρ	NPM	MC_p	$n=50$		$n=100$		$n=200$		$n=\infty$	
			C_{pk,T^2}	$C_{pk,M}$	C_{pk,T^2}	$C_{pk,M}$	C_{pk,T^2}	$C_{pk,M}$	C_{pk,T^2}	$C_{pk,M}$
0.0	5,392	0.928	1.233	0.850	1.227	0.855	1.226	0.859	1.234	0.872
0.1	5,389	0.928	1.174	0.850	1.168	0.854	1.168	0.859	1.176	0.872
0.2	5,377	0.928	1.123	0.850	1.118	0.855	1.118	0.859	1.126	0.872
0.3	5,352	0.928	1.077	0.851	1.073	0.854	1.073	0.859	1.082	0.872
0.4	5,308	0.929	1.036	0.850	1.032	0.854	1.033	0.859	1.043	0.872
0.5	5,236	0.931	1.000	0.851	0.998	0.855	0.997	0.859	1.007	0.872
0.6	5,120	0.933	0.969	0.852	0.964	0.854	0.965	0.859	0.975	0.872
0.7	4,940	0.937	0.938	0.851	0.935	0.855	0.936	0.859	0.946	0.872
0.8	4,655	0.943	0.911	0.852	0.909	0.856	0.909	0.859	0.919	0.872
0.9	4,179	0.955	0.885	0.851	0.882	0.854	0.885	0.859	0.895	0.872

$n=\infty$: parameters-known case

Table 3. NPM, MC_p, C_{pk,T^2}, C^{WSD}_{pk,T^2}, $C_{pk,M}$ and $C^{WSD}_{pk,M}$ for bivariate distributions (parameters known)

(a) $\rho=0.3$

dist.	(γ_1, γ_2)	NPM	MC_p	C_{pk,T^2}	C^{WSD}_{pk,T^2}	$C_{pk,M}$	$C^{WSD}_{pk,M}$
lognormal	(1,1)	20,295	0.774	1.082	0.962	0.872	0.776
	(1,2)	25,883	0.743	1.082	0.927	0.872	0.746
	(1,3)	27,700	0.734	1.082	0.907	0.872	0.727
	(2,2)	31,332	0.718	1.082	0.889	0.872	0.717
	(2,3)	33,095	0.710	1.082	0.868	0.872	0.699
	(3,3)	34,841	0.703	1.082	0.845	0.872	0.682
Weibull	(1,1)	19,344	0.780	1.082	0.948	0.872	0.764
	(1,2)	27,495	0.735	1.082	0.904	0.872	0.726
	(1,3)	30,221	0.722	1.082	0.880	0.872	0.702
	(2,2)	35,243	0.702	1.082	0.856	0.872	0.690
	(2,3)	37,948	0.692	1.082	0.829	0.872	0.667
	(3,3)	40,513	0.683	1.082	0.801	0.872	0.645
gamma	(1,1)	19,602	0.778	1.082	0.955	0.872	0.770
	(1,2)	27,168	0.736	1.082	0.908	0.872	0.729
	(1,3)	31,228	0.718	1.082	0.876	0.872	0.696
	(2,2)	33,245	0.710	1.082	0.856	0.872	0.690
	(2,3)	37,270	0.694	1.082	0.820	0.872	0.659
	(3,3)	39,731	0.686	1.082	0.781	0.872	0.630

(b) $\rho=0.8$

dist.	(γ_1, γ_2)	NPM	MC_p	C_{pk,T^2}	C_{pk,T^2}^{WSD}	$C_{pk,M}$	$C_{pk,M}^{WSD}$
lognormal	(1,1)	16,843	0.797	0.919	0.817	0.872	0.776
	(1,2)	21,323	0.767	0.919	0.792	0.872	0.746
	(1,3)	22,476	0.761	0.919	0.782	0.872	0.727
	(2,2)	25,531	0.744	0.919	0.755	0.872	0.717
	(2,3)	26,731	0.738	0.919	0.739	0.872	0.699
	(3,3)	28,092	0.732	0.919	0.719	0.872	0.682
Weibull	(1,1)	16,961	0.796	0.919	0.805	0.872	0.764
	(1,2)	23,458	0.755	0.919	0.775	0.872	0.726
	(1,3)	25,271	0.746	0.919	0.766	0.872	0.702
	(2,2)	29,334	0.726	0.919	0.727	0.872	0.690
	(2,3)	30,989	0.719	0.919	0.707	0.872	0.667
	(3,3)	32,840	0.711	0.919	0.680	0.872	0.645
gamma	(1,1)	15,352	0.808	0.919	0.811	0.872	0.770
	(1,2)	*	*	*	*	*	*
	(1,3)	*	*	*	*	*	*
	(2,2)	24,558	0.749	0.919	0.727	0.872	0.690
	(2,3)	*	*	*	*	*	*
	(3,3)	28,625	0.730	0.919	0.664	0.872	0.630

*: not applicable

$\rho=0.8$ and various skewnesses (γ_1, γ_2). NPM and MC_p are also given in <Table 3>. The tables show that:

i) When the parameters are known, MC_p decreases as skewness increases and C_{pk,T^2}^{WSD} and $C_{pk,M}^{WSD}$ reflect such a phenomenon, i.e., they decrease as skewness becomes large whereas C_{pk,T^2} and $C_{pk,M}$ remain constant; <Table 3>

Table 4. C_{pk,T^2}, C_{pk,T^2}^{WSD}, $C_{pk,M}$ and $C_{pk,M}^{WSD}$ for bivariate distributions(parameters unknown)

(a) $\rho=0.3$

dist.	(γ_1, γ_2)	$n=50$				$n=100$				$n=200$			
		C_{pk,T^2}	C_{pk,T^2}^{WSD}	$C_{pk,M}$	$C_{pk,M}^{WSD}$	C_{pk,T^2}	C_{pk,T^2}^{WSD}	$C_{pk,M}$	$C_{pk,M}^{WSD}$	C_{pk,T^2}	C_{pk,T^2}^{WSD}	$C_{pk,M}$	$C_{pk,M}^{WSD}$
lognormal	(1,1)	1.095	1.026	0.860	0.794	1.082	0.995	0.860	0.788	1.077	0.980	0.861	0.783
	(1,2)	1.111	1.011	0.869	0.778	1.093	0.973	0.867	0.766	1.083	0.950	0.865	0.756
	(1,3)	1.143	1.019	0.888	0.777	1.111	0.968	0.876	0.756	1.091	0.936	0.868	0.741
	(2,2)	1.313	0.998	0.882	0.765	1.106	0.948	0.874	0.744	1.090	0.920	0.868	0.731
	(2,3)	1.160	1.001	0.898	0.762	1.119	0.939	0.881	0.733	1.097	0.904	0.873	0.716
	(3,3)	1.187	1.007	0.914	0.758	1.138	0.935	0.892	0.725	1.108	0.892	0.878	0.704
Weibull	(1,1)	1.085	1.002	0.856	0.782	1.076	0.975	0.857	0.775	1.074	0.960	0.859	0.769
	(1,2)	1.108	0.982	0.870	0.760	1.089	0.942	0.863	0.743	1.080	0.922	0.863	0.735
	(1,3)	1.141	0.984	0.886	0.751	1.102	0.927	0.871	0.725	1.090	0.905	0.869	0.716
	(2,2)	1.127	0.953	0.880	0.735	1.101	0.906	0.871	0.713	1.086	0.880	0.866	0.701
	(2,3)	1.158	0.953	0.897	0.726	1.118	0.893	0.880	0.698	1.096	0.862	0.872	0.683
	(3,3)	1.193	0.958	0.918	0.722	1.135	0.881	0.891	0.685	1.105	0.841	0.877	0.665
gamma	(1,1)	1.091	1.103	0.858	0.788	1.079	0.984	0.858	0.780	1.076	0.970	0.860	0.776
	(1,2)	1.115	0.992	0.869	0.762	1.092	0.950	0.864	0.747	1.082	0.928	0.863	0.737
	(1,3)	1.152	0.987	0.887	0.744	1.111	0.930	0.874	0.722	1.094	0.905	0.869	0.710
	(2,2)	1.135	0.960	0.881	0.737	1.106	0.909	0.872	0.714	1.089	0.883	0.867	0.702
	(2,3)	1.174	0.954	0.901	0.722	1.119	0.882	0.877	0.687	1.098	0.854	0.872	0.675
	(3,3)	1.211	0.943	0.920	0.705	1.141	0.861	0.890	0.667	1.109	0.822	0.877	0.649

(b) $\rho=0.8$

dist.	(γ_1,γ_2)	$n=50$				$n=100$				$n=200$			
		C_{pk,T^2}	C^{WSD}_{pk,T^2}	$C_{pk,M}$	$C^{WSD}_{pk,M}$	C_{pk,T^2}	C^{WSD}_{pk,T^2}	$C_{pk,M}$	$C^{WSD}_{pk,M}$	C_{pk,T^2}	C^{WSD}_{pk,T^2}	$C_{pk,M}$	$C^{WSD}_{pk,M}$
lognormal	(1,1)	0.928	0.879	0.862	0.797	0.916	0.850	0.860	0.788	0.913	0.834	0.861	0.783
	(1,2)	0.950	0.878	0.874	0.784	0.932	0.840	0.869	0.769	0.921	0.815	0.865	0.757
	(1,3)	0.983	0.888	0.889	0.780	0.948	0.836	0.875	0.756	0.932	0.811	0.871	0.744
	(2,2)	0.968	0.870	0.887	0.772	0.941	0.816	0.875	0.746	0.928	0.788	0.870	0.733
	(2,3)	0.997	0.877	0.902	0.769	0.959	0.814	0.886	0.738	0.938	0.780	0.877	0.721
	(3,3)	1.026	0.888	0.923	0.771	0.973	0.809	0.896	0.731	0.948	0.769	0.883	0.709
Weibull	(1,1)	0.923	0.866	0.858	0.785	0.914	0.836	0.858	0.776	0.912	0.821	0.861	0.770
	(1,2)	0.947	0.857	0.870	0.761	0.928	0.817	0.865	0.746	0.918	0.794	0.863	0.735
	(1,3)	0.995	0.870	0.891	0.757	0.951	0.815	0.876	0.732	0.931	0.791	0.870	0.717
	(2,2)	0.968	0.839	0.883	0.740	0.939	0.783	0.872	0.716	0.925	0.755	0.867	0.702
	(2,3)	1.007	0.848	0.903	0.736	0.956	0.774	0.881	0.700	0.937	0.744	0.875	0.686
	(3,3)	1.032	0.847	0.922	0.728	0.975	0.766	0.894	0.688	0.944	0.724	0.879	0.667
gamma	(1,1)	0.924	0.868	0.861	0.793	0.915	0.841	0.859	0.783	0.913	0.827	0.861	0.778
	(1,2)	*	*	*	*	*	*	*	*	*	*	*	*
	(1,3)	*	*	*	*	*	*	*	*	*	*	*	*
	(2,2)	0.960	0.820	0.886	0.742	0.931	0.771	0.871	0.715	0.921	0.750	0.867	0.702
	(2,3)	*	*	*	*	*	*	*	*	*	*	*	*
	(3,3)	1.009	0.791	0.922	0.709	0.959	0.727	0.891	0.669	0.938	0.698	0.879	0.651

ii) When the parameters are unknown, C_{pk,T^2} and $C_{pk,M}$ increase as skewness becomes large, whereas C^{WSD}_{pk,T^2} and $C^{WSD}_{pk,M}$ decrease in most cases. All PCIs are overestimated especially when the sample size is small; <Table 4>

iii) $C^{WSD}_{pk,M}$ is closer to MC_p in <Table 3(a)>, and C^{WSD}_{pk,T^2} is closer to MC_p in <Table 3(b)> except for $(\gamma_1,\gamma_2)=(1,2)$. Also, <Table 4> and additional extensive study we have conducted indicate that C^{WSD}_{pk,T^2} is closer to MC_p for highly correlated populations as sample size becomes large, and vise versa. This shows that the T^2 approach is superior to the modified process region approach for highly correlated populations in most cases, and vise versa.

<Table 5> and <Table 6> present PCIs for two 4-variate lognormal distributions, where case 1 uses ρ_1 describing low positive correlations ($\rho_{ij}<0.5$) and case 2 uses ρ_2 containing high correlations ($\rho_{ij}\geq 0.5$) as follows:

$$\rho_1=\begin{bmatrix}1 & 0.2 & 0.3 & 0.1\\ & 1 & 0.2 & 0.4\\ & & 1 & 0.3\\ & & & 1\end{bmatrix},$$

$$\rho_2=\begin{bmatrix}1 & 0.8 & 0.6 & 0.7\\ & 1 & 0.8 & 0.5\\ & & 1 & 0.6\\ & & & 1\end{bmatrix}.$$

Table 5. NPM, MC_p, C_{pk,T^2}, C^{WSD}_{pk,T^2}, $C_{pk,M}$ and $C^{WSD}_{pk,M}$ for 4-variate lognormal distributions (parameters known)

case	$(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$	NPM	MC_p	C_{pk,T^2}	C^{WSD}_{pk,T^2}	$C_{pk,M}$	$C^{WSD}_{pk,M}$
1	(1.0,0.5,1.0,0.5)	15,916	0.804	1.128	1.030	0.744	0.680
	(1.5,1.0,1.0,1.5)	24,016	0.752	1.128	0.980	0.744	0.647
	(1.5,1.5,2.5,2.5)	27,417	0.735	1.128	0.933	0.744	0.614
	(0.5,2.5,2.0,1.5)	22,865	0.759	1.128	0.973	0.744	0.633
2	(1.0,0.5,1.0,0.5)	13,352	0.825	0.865	0.810	0.744	0.680
	(1.5,1.0,1.0,1.5)	19,631	0.778	0.865	0.754	0.744	0.647
	(1.5,1.5,2.5,2.5)	22,308	0.762	0.865	0.712	0.744	0.614
	(0.5,2.5,2.0,1.5)	19,230	0.780	0.865	0.755	0.744	0.633

The results are similar to the bivariate cases. Since the WSD PCIs is designed to reflect the skewness, the WSD PCIs decrease as skewness becomes large.

5. Finite Sample Properties

The relative biases and mean square errors (MSEs) of C^{WSD}_{pk,T^2} and $C^{WSD}_{pk,M}$ are investigated for small and moderate sample sizes. 10,000 values of deviations and squared deviations, each based on $n=50, 100, 200$ random variates, are computed and averaged to obtain the relative bias and MSE. For all cases, it is assumed that $USL_i=3$ and $LSL_i=-3$ and the distribution is shifted and scaled to produce the same value of $\mu_i=0$ and $\sigma_i=1$, $i=1,2$.

<Table 7> presents the relative biases and MSEs for bivariate lognormal, Weibull, and gamma distributions with $\rho=0.3$ or $\rho=0.8$, and shows that:

i) For all cases, $\hat{C}^{WSD}_{pk,T^2}$ and $\hat{C}^{WSD}_{pk,M}$ overestimate the true value of C^{WSD}_{pk,T^2} and $C^{WSD}_{pk,M}$.

Table 6. C_{pk,T^2}, C^{WSD}_{pk,T^2}, $C_{pk,M}$ and $C^{WSD}_{pk,M}$ for 4-variate lognormal distributions (parameters unknown)

case	$(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$	$n=50$				$n=100$				$n=200$			
		C_{pk,T^2}	C^{WSD}_{pk,T^2}	$C_{pk,M}$	$C^{WSD}_{pk,M}$	C_{pk,T^2}	C^{WSD}_{pk,T^2}	$C_{pk,M}$	$C^{WSD}_{pk,M}$	C_{pk,T^2}	C^{WSD}_{pk,T^2}	$C_{pk,M}$	$C^{WSD}_{pk,M}$
1	(1.0,0.5,1.0,0.5)	1.169	1.122	0.729	0.682	1.141	1.077	0.731	0.683	1.131	1.054	0.734	0.683
	(1.5,1.0,1.0,1.5)	1.186	1.099	0.735	0.665	1.151	1.038	0.734	0.658	1.135	1.009	0.735	0.653
	(1.5,1.5,2.5,2.5)	1.228	1.097	0.750	0.650	1.173	1.013	0.742	0.632	1.149	0.976	0.740	0.625
	(0.5,2.5,2.0,1.5)	1.207	1.111	0.743	0.659	1.165	1.045	0.740	0.649	1.142	1.007	0.737	0.641
2	(1.0,0.5,1.0,0.5)	0.895	0.902	0.730	0.684	0.874	0.859	0.732	0.684	0.866	0.836	0.734	0.683
	(1.5,1.0,1.0,1.5)	0.915	0.885	0.739	0.671	0.886	0.823	0.736	0.661	0.871	0.787	0.735	0.654
	(1.5,1.5,2.5,2.5)	0.963	0.895	0.754	0.655	0.911	0.805	0.743	0.634	0.888	0.762	0.741	0.626
	(0.5,2.5,2.0,1.5)	0.963	0.906	0.748	0.664	0.913	0.831	0.742	0.651	0.887	0.793	0.740	0.643

Table 7. Relative biases and MSEs of $\hat{C}^{WSD}_{pk,T^2}$ and $\hat{C}^{WSD}_{pk,M}$ for bivariate distributions

(a) $\rho=0.3$

dist.	(γ_1, γ_2)	relative bias						MSE					
		$n=50$		$n=100$		$n=200$		$n=50$		$n=100$		$n=200$	
		C^{WSD}_{pk,T^2}	$C^{WSD}_{pk,M}$	C^{WSD}_{pk,T^2}	$C^{WSD}_{pk,M}$	C^{WSD}_{pk,T^2}	$C^{WSD}_{pk,M}$	C^{WSD}_{pk,T^2}	$C^{WSD}_{pk,M}$	C^{WSD}_{pk,T^2}	$C^{WSD}_{pk,M}$	C^{WSD}_{pk,T^2}	$C^{WSD}_{pk,M}$
lognormal	(1, 1)	0.0604	0.0204	0.0360	0.0163	0.0172	0.0085	0.0306	0.0122	0.0154	0.0068	0.0072	0.0035
	(1, 2)	0.0953	0.0459	0.0470	0.0238	0.0256	0.0146	0.0457	0.0188	0.0201	0.0098	0.0094	0.0050
	(1, 3)	0.1214	0.0650	0.0625	0.0366	0.0334	0.0207	0.0587	0.0255	0.0246	0.0130	0.0114	0.0068
	(2, 2)	0.1295	0.0704	0.0663	0.0381	0.0347	0.0199	0.0618	0.0262	0.0265	0.0132	0.0120	0.0063
	(2, 3)	0.1553	0.0895	0.0801	0.0475	0.0448	0.0268	0.0755	0.0327	0.0311	0.0157	0.0146	0.0080
	(3, 3)	0.1938	0.1162	0.1024	0.0611	0.0548	0.0332	0.0943	0.0410	0.0386	0.0191	0.0176	0.0095
Weibull	(1, 1)	0.0558	0.0227	0.0278	0.0134	0.0136	0.0073	0.0258	0.0106	0.0120	0.0058	0.0055	0.0028
	(1, 2)	0.0844	0.0436	0.0394	0.0214	0.0222	0.0131	0.0375	0.0165	0.0152	0.0079	0.0071	0.0040
	(1, 3)	0.1124	0.0679	0.0577	0.0375	0.0294	0.0198	0.0513	0.0237	0.0203	0.0114	0.0088	0.0055
	(2, 2)	0.1125	0.0628	0.0573	0.0326	0.0286	0.0164	0.0496	0.0218	0.0208	0.0106	0.0091	0.0049
	(2, 3)	0.1563	0.0906	0.0775	0.0464	0.0398	0.0237	0.0710	0.0299	0.0275	0.0138	0.0115	0.0063
	(3, 3)	0.1949	0.1151	0.0994	0.0581	0.0501	0.0296	0.0895	0.0372	0.0346	0.0164	0.0146	0.0078
gamma	(1, 1)	0.0586	0.0216	0.0336	0.0158	0.0166	0.0086	0.0325	0.0124	0.0156	0.0067	0.0074	0.0034
	(1, 2)	0.0943	0.0469	0.0463	0.0251	0.0227	0.0127	0.0478	0.0186	0.0202	0.0091	0.0094	0.0046
	(1, 3)	0.1276	0.0692	0.0600	0.0341	0.0294	0.0170	0.0639	0.0250	0.0245	0.0119	0.0109	0.0058
	(2, 2)	0.1232	0.0673	0.0596	0.0330	0.0305	0.0172	0.0627	0.0250	0.0258	0.0116	0.0119	0.0055
	(2, 3)	0.1583	0.0903	0.0802	0.0460	0.0403	0.0227	0.0825	0.0323	0.0321	0.0143	0.0145	0.0068
	(3, 3)	0.2000	0.1161	0.0997	0.0579	0.0525	0.0315	0.1026	0.0393	0.0400	0.0172	0.0179	0.0081

(b) $\rho = 0.8$

dist.	(γ_1, γ_2)	relative bias						MSE					
		$n=50$		$n=100$		$n=200$		$n=50$		$n=100$		$n=200$	
		C^{WSD}_{pk,T^2}	$C^{WSD}_{pk,M}$	C^{WSD}_{pk,T^2}	$C^{WSD}_{pk,M}$	C^{WSD}_{pk,T^2}	$C^{WSD}_{pk,M}$	C^{WSD}_{pk,T^2}	$C^{WSD}_{pk,M}$	C^{WSD}_{pk,T^2}	$C^{WSD}_{pk,M}$	C^{WSD}_{pk,T^2}	$C^{WSD}_{pk,M}$
lognormal	(1, 1)	0.0748	0.0274	0.0422	0.0188	0.0215	0.0100	0.0272	0.0173	0.0138	0.0097	0.0067	0.0051
	(1, 2)	0.1115	0.0519	0.0569	0.0286	0.0309	0.0166	0.0414	0.0263	0.0183	0.0138	0.0084	0.0071
	(1, 3)	0.1355	0.0706	0.0727	0.0437	0.0372	0.0224	0.0541	0.0349	0.0218	0.0184	0.0094	0.0095
	(2, 2)	0.1536	0.0790	0.0771	0.0390	0.0426	0.0224	0.0592	0.0370	0.0254	0.0180	0.0117	0.0090
	(2, 3)	0.1883	0.1019	0.0991	0.0546	0.0555	0.0320	0.0754	0.0467	0.0316	0.0229	0.0148	0.0118
	(3, 3)	0.2326	0.1293	0.1289	0.0718	0.0691	0.0386	0.0988	0.0606	0.0412	0.0285	0.0184	0.0139
Weibull	(1, 1)	0.0756	0.0287	0.0413	0.0178	0.0197	0.0085	0.0242	0.0151	0.0117	0.0082	0.0052	0.0040
	(1, 2)	0.1068	0.0502	0.0511	0.0254	0.0270	0.0143	0.0375	0.0234	0.0149	0.0114	0.0066	0.0057
	(1, 3)	0.1346	0.0741	0.0638	0.0409	0.0322	0.0220	0.0547	0.0333	0.0183	0.0163	0.0074	0.0080
	(2, 2)	0.1530	0.0748	0.0767	0.0374	0.0407	0.0212	0.0548	0.0325	0.0219	0.0150	0.0094	0.0070
	(2, 3)	0.1941	0.1000	0.1012	0.0552	0.0522	0.0293	0.0768	0.0435	0.0283	0.0197	0.0119	0.0094
	(3, 3)	0.2481	0.1292	0.1270	0.0672	0.0638	0.0329	0.0981	0.0543	0.0376	0.0247	0.0150	0.0110
gamma	(1, 1)	0.0651	0.0266	0.0337	0.0149	0.0187	0.0098	0.0275	0.0180	0.0134	0.0096	0.0064	0.0049
	(1, 2)	*	*	*	*	*	*	*	*	*	*	*	*
	(1, 3)	*	*	*	*	*	*	*	*	*	*	*	*
	(2, 2)	0.1261	0.0758	0.0641	0.0390	0.0294	0.0169	0.0548	0.0367	0.0227	0.0164	0.0104	0.0080
	(2, 3)	*	*	*	*	*	*	*	*	*	*	*	*
	(3, 3)	0.1972	0.1291	0.0992	0.0667	0.0524	0.0344	0.0881	0.0558	0.0346	0.0244	0.0150	0.0112

ii) For a given skewness, both relative bias and MSE decrease as n increases, and for given n, they increase as skewness becomes large.

iii) Both relative bias and MSE of $\hat{C}^{WSD}_{pk,M}$ are smaller than those of $\hat{C}^{WSD}_{pk,T^2}$.

6. Concluding Remarks

This paper proposed two simple methods of constructing multivariate process capability indices for an arbitrary skewed population. These methods use T^2 statistic and modified process region to evaluate the capability of multivariate processes, and the multivariate weighted standard deviation method is used to reflect the skewness. This method adjusts the variance-covariance matrix in accordance to the degree of skewness of the underlying distribution. Numerical analyses indicate that the proposed WSD PCIs are close to the matched PCI for skewed populations, and this shows that the WSD PCIs can describe the process capability of a skewed population adequately.

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