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On the Negative Quadrant Dependence in Three Dimensions

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Abstract

In this note we perform an extreme point analysis on two natural definitions of negative quadrant dependence of three random variables and examine how different these two notions of dependence. We also characterize some special distributions which are both negatively lower orthant dependent and negatively upper orthant dependent.

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1. Introduction

Lehmann (1966) introduced the following notion of negative quadrant dependence (NQD): Let X_1 and X_2 be two random variables with some joint probability distribution function F. X_1 and X_2 (or F) are said to be negatively quadrant dependent(NQD) if

$$P(X_1 \leq x_1, X_2 \leq x_2) \leq P(X_1 \leq x_1)P(X_2 \leq x_2) \tag{1.1}$$

for all real numbers x_1 and x_2. The condition (1.1) is equivalent to

$$P(X_1 > x_1, X_2 > x_2) \leq P(X_1 > x_1)P(X_2 > x_2) \tag{1.2}$$

for all real numbers x_1 and x_2. One faces problems if one wishes to extend the notion of negative quadrant dependence to more than two random variables. If X_1, X_2, and X_3 are three random variables, one could say that X_1, X_2, and X_3 are NQD by adapting either of the conditions (1.1) or (1.2) in a natural way. To be more precise, we say that X_1, X_2, and X_3 are negatively lower orthant dependent(NLOD) if

$$P(X_1 \leq x_1, X_2 \leq x_2, X_3 \leq x_3) \leq P(X_1 \leq x_1)P(X_2 \leq x_2)P(X_3 \leq x_3) \tag{1.3}$$

for all real numbers x_1, x_2, and x_3, and we say that X_1, X_2, and X_3 are negatively upper orthant dependent(NUOD) if

$$P(X_1 > x_1, X_2 > x_2, X_3 > x_3) \leq P(X_1 > x_1)P(X_2 > x_2)P(X_3 > x_3) \tag{1.4}$$

for all real numbers x_1, x_2, and x_3. These two concepts have been examined by Ebrahimi and Ghosh(1981) and by several authors cited in that paper. Some other concepts of negative dependence were introduced by Block, Savits and Shaked(1982) and Joag-Dev and Proschan(1983). In general, these two notions of NLOD and NUOD are not equivalent. Ebrahimi and Ghosh(1981) gave an example of a trivariate distribution which is NUOD, but not NLOD: Let X_1, X_2, and X_3 be three random variables assuming the values (0,1,1), (1,0,1), (1,1,0) and (0,0,0) each with probability 1/4. Then,

$$P(X_1 > 0, X_2 > 0, X_3 > 0) = 0 < \frac{1}{8} = P(X_1 > 0)P(X_2 > 0)P(X_3 > 0);$$
$$P(X_1 \leq 0, X_2 \leq 0, X_3 \leq 0) = \frac{1}{4} > \frac{1}{8} = P(X_1 \leq 0)P(X_2 \leq 0)P(X_3 \leq 0).$$

The main goal of this note is to examine how different are these two notions of dependence. More precisely, we want to perform extreme point analysis on these two notions of dependence. In some special cases, extreme point analysis helps us to characterize all trivariate distributions which are both NLOD and NUOD.

2. Preliminaries

Let $\mathcal{B}$ be the Borel σ-field on the line, R and $\mathcal{B}^3$ the product σ-field on R^3. Let μ be a probability measure on $\mathcal{B}^3$ and μ_1, μ_2 and μ_3 the corresponding marginal probability measures on $\mathcal{B}$, i.e., $\mu_1(B) = \mu(B \times R \times R)$ $\mu_2(B) = \mu(R \times B \times R)$ and $\mu_3(B) = \mu(R \times R \times B)$ for every B in $\mathcal{B}$. μ is said to be negatively lower orthant dependent (NLOD) if $\mu\{(-\infty, b] \times (-\infty, c] \times (-\infty, d]\}$ $\leq \mu_1\{(-\infty, b]\}\mu_2\{(-\infty, c]\}\mu_3\{(-\infty, d]\}$ for every b, c, and d in R and μ is said to be negatively upper orthant dependent (NUOD) if $\mu\{(b, \infty) \times (c, \infty) \times (d, \infty)\} \leq \mu_1\{(b, \infty)\}\mu_2\{(c, \infty)\}\mu_3\{(d, \infty)]\}$.
The above notions can be rephrased as in (1.3) and (1.4) respectively.
Let $M_{NLOD}(\lambda, \nu, \eta)$ be a set of NLOD probability measures μ with $\mu_1 = \lambda,\ \mu_2 = \nu,\ \mu_3 = \eta$ and $M_{NUOD}(\lambda, \nu, \eta)$ be a set of NUOD probability measures μ with $\mu_1 = \lambda,\ \mu_2 = \nu,\ \mu_3 = \eta$. We will show that for any three probability measures λ, ν and η, $M_{NLOD}(\lambda, \nu, \eta)$ and $M_{NUOD}(\lambda, \nu, \eta)$ are compact and convex. We need the following definitions and results in this connection. Let (X, d) be a polish space, i.e., a complete separable metric space. Let B_X be Borel σ-field on X and M_X the space of all probability measures on B_X. M_X is equipped with weak topology.

Definition 2.1.(Maki, Thompson, 1973) Let $\underline{x}, \underline{y} \in S$, where S is a linear space. Then the line segment between $\underline{x}$ and $\underline{y}$ is the set defined by $[\underline{x}, \underline{y}] = \{\underline{z}_\alpha\ :\ \underline{z}_\alpha = \alpha\underline{x} + (1-\alpha)\underline{y},\ \ 0 \leq \alpha \leq 1\}$.

Definition 2.2.(Maki, Thompson, 1973) A set K in S, where S is a linear space, is said to be convex if $\underline{x}, \underline{y} \in K \Rightarrow [\underline{x}, \underline{y}] \subset K$.
Thus a set is convex if for every two points in the set the line segment be-

tween them is also contained in the set. In any convex set there are certain special points which correspond to the corner points. These are called extreme points.

Definition 2.3.(Maki, Thompson, 1973) A point $\underline{e}$ in a convex set K is said to be an extreme point of K if there do not exist points $\underline{x}, \underline{y} \in K, \underline{x} \neq \underline{y}$, such that $\underline{e} \in [\underline{x}, \underline{y}], \underline{x} \neq \underline{e}, \underline{y} \neq \underline{e}$

Definition 2.4.(Subramanyam, Rao, 1986) A subset of M_X is uniformly tight if for every $\epsilon > 0$ there exists a compact subset C of X such that $\mu(C) > 1 - \epsilon \;\; for \; every \; \mu \; in \; S.$

The following is known as Prohorov's theorem

Proposition 2.5.(Subramanyam, Rao, 1986) A subset of M_X is relatively compact if and only if S is uniformly tight. S is compact if and only if S is closed and uniformly tight.

Proof. See Billingsley [Theorems 6.1 and 6.2, p37].

Theorem 2.6. Let $M(\lambda, \nu, \eta)$ be the collection of all probability measure μ on $\mathcal{B}^3$ such that $\mu_1 = \lambda$, $\mu_2 = \nu$ and $\mu_3 = \eta$ where μ_1, μ_2 and μ_3 are the corresponding marginal probability measures of μ. Then $M(\lambda, \nu, \eta)$ is compact.

Proof. It is obvious that $M(\lambda, \nu, \eta)$ is a closed subset of M, the space of all probability measures on $\mathcal{B}^3$. We show that $M(\lambda, \nu, \eta)$ is uniformly tight. Let $\epsilon > 0$. There exist compact subsets C_1, C_2, and C_3 of R such that $\lambda(C_1^c) < \epsilon/3$, $\nu(C_2^c) < \epsilon/3$, and $\eta(C_3^c) < \epsilon/3$ $C_1 \times C_2 \times C_3$ is a compact subset of R^3. Let $\mu \in M(\lambda, \nu, \eta)$. Then

$$\begin{aligned}
\mu[(C_1 \times C_2 \times C_3)^c] &\leq \mu(C_1^c \times R \times R \bigcup R \times C_2^c \times R \bigcup R \times R \times C_3^c) \\
&\leq \mu(C_1^c \times R \times R) + \mu(R \times C_2^c \times R) + \mu(R \times R \times C_3^c) \\
&= \mu_1(C_1^c) + \mu_2(C_2^c) + \mu_3(C_3^c) \\
&= \lambda(C_1^c) + \nu(C_2^c) + \eta(C_3^c) < \epsilon.
\end{aligned}$$

This completes the proof in view of Propositons 2.5.

The following result is the main result of this section.

Theorem 2.7. Let $M(\lambda, \nu, \eta)$ be the collection of all probability measure μ on $\mathcal{B}^3$ such that $\mu_1 = \lambda$, $\mu_2 = \nu$ and $\mu_3 = \eta$ where μ_1, μ_2 and μ_3 are the corresponding marginal probability measures of μ. For any given probability measures λ, ν, and η on $\mathcal{B}$, $M_{NLOD}(\lambda, \nu, \eta)$ is compact and convex.

Proof. $M_{NLOD}(\lambda, \nu, \eta)$ is a closed subset $M(\lambda, \nu, \eta)$ from the following observation. Let μ_n, $n \geq 1$, be a sequence in $M_{NLOD}(\lambda, \nu, \eta)$ converging weakly to a μ in $M_{NLOD}(\lambda, \nu, \eta)$. Note that ($\mu_n \Rightarrow \mu$) implies $\mu(\partial A) = 0$ for any μ-continuity set A, where ∂A is boundary of A. Then for any b, c, and d in R (see Billingsley [1, Theorem 2.1, p11]),

$$\begin{aligned}
&\mu\{(-\infty, b] \times (-\infty, c] \times (-\infty, d]\} \\
&\quad = \mu\{(-\infty, b) \times (-\infty, c) \times (-\infty, d)\} \\
&\quad \leq \lim\ \inf \mu_n\{(-\infty, b) \times (-\infty, c) \times (-\infty, d)\} \\
&\quad \leq \lambda\{(-\infty, b)\}\nu\{(-\infty, c)\}\eta\{(-\infty, d)\} \\
&\quad \leq \lambda\{(-\infty, b]\}\nu\{(-\infty, c]\}\eta\{(-\infty, d]\}.
\end{aligned}$$

Hence, $\mu \in M_{NLOD}(\lambda, \nu, \eta)$. This implies that $M_{NLOD}(\lambda, \nu, \eta)$ is compact. We now show that $M_{NLOD}(\lambda, \nu, \eta)$ is convex. Let $\mu, \xi \in M_{NLOD}(\lambda, \nu, \eta)$ and $0 \leq \alpha \leq 1$. Then for any b, c, d, in R,

$$\begin{aligned}
& (\alpha\mu + (1-\alpha)\xi)\{(-\infty, b] \times (-\infty, c] \times (-\infty, d]\} \\
& -\lambda\{(-\infty, b]\}\nu\{(-\infty, c]\}\eta\{(-\infty, d]\} \\
= \ & \alpha\mu\{(-\infty, b] \times (-\infty, c] \times (-\infty, d]\} \\
& +(1-\alpha)\xi\{(-\infty, b] \times (-\infty, c] \times (-\infty, d]\} \\
& -\lambda\{(-\infty, b]\}\nu\{(-\infty, c]\}\eta\{(-\infty, d]\} \\
\leq \ & \alpha\lambda\{(-\infty, b]\}\nu\{(-\infty, c]\}\eta\{(-\infty, d]\} \\
& +(1-\alpha)\lambda\{(-\infty, b]\} \times \nu\{(-\infty, c]\} \times \eta\{(-\infty, d]\} \\
& -\lambda\{(-\infty, b]\}\nu\{(-\infty, c]\}\eta\{(-\infty, d]\} = 0.
\end{aligned}$$

Consequently, $\alpha\mu + (1-\alpha)\xi \in M_{NLOD}(\lambda, \nu, \eta)$. This completes the proof.

Remark Note that for any given probability measures λ, ν, and η on $\mathcal{B}$ $M_{NUOD}(\lambda, \nu, \eta)$ is compact and convex.

3. Extreme Points

We consider the case where each of X_1, X_2, and X_3 assumes two values 1 and 2, say. Let $P_{ijk} = P(X_1 = i, X_2 = j, X_3 = k), i = 1, 2;\ j = 1, 2;\ k = 1, 2.$

The joint probability law of $X_1,\ X_2\ and\ X_3$ is written, for convenience,

$$P = \begin{pmatrix} P_{111} & P_{112} & P_{121} & P_{122} \\ P_{211} & P_{212} & P_{221} & P_{222} \end{pmatrix}$$

and let $p_1 = P(X_1 = 1);\ q_1 = P(X_2 = 1);\ r_1 = P(X_3 = 1); p_2 = 1 - p_1;\ q_2 = 1 - q_1;\ \ and\ \ r_2 = 1 - r$. Then we have

$$P_{111} + P_{112} + P_{121} + P_{122} = p_1, \tag{3.1}$$

$$P_{211} + P_{212} + P_{221} + P_{222} = p_2, \tag{3.2}$$

$$P_{111} + P_{211} + P_{112} + P_{212} = q_1, \tag{3.3}$$

$$P_{121} + P_{221} + P_{122} + P_{222} = q_2, \tag{3.4}$$

$$P_{111} + P_{121} + P_{211} + P_{221} = r_1, \tag{3.5}$$

$$P_{112} + P_{122} + P_{212} + P_{222} = r_2. \tag{3.6}$$

We note that, P is NLOD iff

$$P_{111} \leq p_1 q_1 r_1, \tag{3.7}$$

$$P_{11\cdot} \leq p_1 q_1, \tag{3.8}$$

$$P_{1\cdot 1} \leq p_1 r_1, \tag{3.9}$$

$$P_{\cdot 11} \leq q_1 r_1. \tag{3.10}$$

P is NUOD iff

$$P_{222} \leq p_2 q_2 r_2, \tag{3.11}$$

$$P_{22\cdot} \leq p_2 q_2, \tag{3.12}$$

$$P_{2\cdot 2} \leq p_2 r_2, \tag{3.13}$$

$$P_{\cdot 22} \leq q_2 r_2, \tag{3.14}$$

where $P_{ii.} = P(X_1 = i, X_2, i), P_{i.i}(X_1 = i, X_3 = i)$ and $P_{.ii}(X_2 = i, X_3 = i)$. Let $0 < p_1 < 1$, $0 < q_1 < 1$, and $0 < r_1 < 1$ be three fixed numbers. Let $M_{NLOD}(p_1,\ q_1,\ r_1)$ be the collection of all trivariate distributions $P = (P_{ijk})$ with support contained in $\{(i,\ j,\ k)\ ;\ i = 1,2,\ j = 1,2,$ and $k = 1,2\}$ such that P is NLOD, and the marginal distributions of $X_1,\ X_2,$ and X_3 under P are $p_1,\ 1-p_1\ ;\ q_1,\ 1-q_1\ ;$ and $r_1,\ 1-r_1$, respectively. The set $M_{NUOD}(p_1,\ q_1,\ r_1)$ is defined analogously.
For the positive quadrant dependence Nguyen and Sampson(1985) have looked into properties of sets of the above type for bivariate distributions with fixed marginals. Subramanyan and Bhaskara Rao(1986) have developed an algebraic method for identifying the extreme points of bivariate distributions.

Now we apply this method to trivariate distributions. Being simplexes, the sets $M_{NLOD}(p_1,\ q_1,\ r_1)$ and $M_{NUOD}(p_1,\ q_1,\ r_1)$ have each finite number of extreme points. Once we identify the extreme points of the set $M_{NLOD}(p_1,\ q_1,\ r_1)$ $(or\ M_{NUOD}(p_1,\ q_1,\ r_1))$ say, we can express every member of $M_{NLOD}(p_1,\ q_1,\ r_1)$ $(or\ M_{NUOD}(p_1,\ q_1,\ r_1))$as a convex combiantion of its extreme points. We describe now a method of identifying the extreme points of $M_{NUOD}(p_1, q_1,\ r_1)$ as well as $M_{NLOD}(p_1, q_1,\ r_1)$. First, we take up the case of $M_{NLOD}(p_1,\ q_1,\ r_1)$. Any $P = (P_{ijk}) \in M_{NLOD}(p_1, q_1, r_1)$ must satisfy the inequalities (3.7), (3.8), (3.9) and (3.10). Also, due to marginality restrictions, that is, from (3.1), (3.3) and (3.5) we should have

$$P_{111} + P_{112} + P_{121} \leq p_1, \tag{3.15}$$

$$P_{111} + P_{112} + P_{211} \leq q_1, \tag{3.16}$$

$$P_{111} + P_{112} + P_{211} \leq q_1, \tag{3.17}$$

$$P_{111} + P_{121} + P_{211} \leq r_1, \tag{3.18}$$

respectively. The following are the natural nonnegativity conditions:

$$P_{112} \geq 0, \tag{3.19}$$

$$P_{121} \geq 0, \tag{3.20}$$

$$P_{211} \geq 0. \tag{3.21}$$

All these inequalities (3.7) to (3.10) and (3.15) to (3.20) involve $P_{111},\ P_{112},\ P_{121}$, P_{211} only. If some four number $P_{111},\ P_{112},\ P_{121},\ P_{211}$ satisfy the inequalities (3.7) to (3.10) and (3.15) to (3.20), then one could obtain

$$P_{122} = p_1 - (P_{111} + P_{112} + P_{121}), \tag{3.22}$$

$$P_{212} = q_1 - (P_{111} + P_{112} + P_{211}), \tag{3.23}$$

$$P_{221} = r_1 - (P_{111} + P_{121} + P_{211}), \tag{3.24}$$

by (3.1), (3.3) and (3.5). Finally from (3.2), (3.22) and (3.23) we also have

$$P_{222} = 1 - p_1 - q_1 - r_1 + P_{111} + (P_{111} + P_{112} + P_{121} + P_{211}). \tag{3.25}$$

The numbers $P_{122},\ P_{212},\ P_{221}$ will be nonnegative. If $P_{222} \geq 0$ then

$$P = (P_{ijk}) \in M_{NLOD}(p_1, q_1, r_1).$$

A standard method of identifying the extreme points of $M_{NLOD}(p_1,\ q_1,\ r_1)$ is as follows. Select 4 inequalities from (3.7) to (3.10) and (3.15) to (3.20). Replace the inequality signs by equality signs. Solve the resultant system of 4 linear equations in 4 unknowns $P_{111},\ P_{112},\ P_{121}$, and P_{211}. If there is a solution, and this solution satisfies the remaining inequalities, determine $P_{122},\ P_{212},\ P_{221}$, and P_{222} from the equations (3.21), (3.22), (3.23) and (3.24) respectively. If $P_{222} \geq 0$, then $P = (P_{ijk}) \in M_{NLOD}(p_1, q_1, r_1)$. It is easy to check that this P is an extreme point of $M_{NLOD}(p_1,\ q_1,\ r_1)$, and every extreme point of $M_{NLOD}(p_1,\ q_1,\ r_1)$ arises this way. For ideas concerning this approach, one may refer to Subramanyam and Bhaskara Rao(1986).

A computer program is easy to write which will identify the extreme points of $M_{NLOD}(p_1, q_1, r_1)$.

Remark. By the similar method in the above context we can consider the $M_{NUOD}(p_1, q_1, r_1)$ case and examine extreme points of $M_{NUOD}(p_1, q_1, r_1)$. Pursuing the above approach, we have isolated the extreme points of M_{NLOD} $(p_1,\ q_1,\ r_1)$ and $M_{NUOD}(p_1,\ q_1,\ r_1)$ when $p_1 = q_1 = r_1 = \frac{1}{2}$, given in Table 1. The above extreme points of the sets $M_{NLOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $M_{LUOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ reveal the following insights.

Table 1: Extreme Points of $M_{NLOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $M_{NUOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

Serial No.	$M_{NLOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$M_{NUOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
1	$P_1 = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	$P_1 = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
2	$P_2 = \frac{1}{8}\begin{bmatrix} 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 \end{bmatrix}$	$P_2 = \frac{1}{8}\begin{bmatrix} 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 \end{bmatrix}$
3	$P_3 = \frac{1}{8}\begin{bmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$	$P_3 = \frac{1}{8}\begin{bmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$
4	$P_4 = \frac{1}{8}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}$	$P_4 = \frac{1}{8}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}$
5	$P_5 = \frac{1}{16}\begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix}$	$P_5 = \frac{1}{16}\begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix}$
6	$P_6 = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \end{bmatrix}$	$P_7 = \frac{1}{8}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
7	$P_8 = \frac{1}{8}\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix}$	$P_9 = \frac{1}{8}\begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix}$
8	$P_{10} = \frac{1}{8}\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}$	$P_{11} = \frac{1}{8}\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$
9	$P_{12} = \frac{1}{64}\begin{bmatrix} 7 & 9 & 9 & 7 \\ 3 & 13 & 13 & 3 \end{bmatrix}$	$P_{13} = \frac{1}{64}\begin{bmatrix} 3 & 13 & 13 & 3 \\ 7 & 9 & 9 & 7 \end{bmatrix}$
10	$P_{14} = \frac{1}{8}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$	$P_{15} = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix}$

4. Conclusions

1. The extreme points of $M_{NLOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $M_{NUOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ fall into three distinct categories. The first five extreme points are common to both the sets. Observe that $P_6 = \frac{1}{2}P_4 + \frac{1}{2}P_{15}$ $P_8 = \frac{1}{2}P_2 + \frac{1}{2}P_{15}$ $P_{10} = \frac{1}{2}P_3 + \frac{1}{2}P_{15}$ $P_{12} = \frac{3}{4}P_5 + \frac{1}{4}P_{15}$.
Consequently, $P_6, P_8, P_{10}, P_{12} \in M_{NUOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Also observe that $P_7 = \frac{1}{2}P_4 + \frac{1}{2}P_{14}$ $P_9 = \frac{1}{2}P_2 + \frac{1}{2}P_{14}$ $P_{11} = \frac{1}{2}P_3 + \frac{1}{2}P_{14}$ $P_{13} = \frac{3}{4}P_5 + \frac{1}{4}P_{14}$. Consequently $P_7, P_9, P_{11}, P_{13} \in M_{NLOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and $P_i \in M_{NLOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \cap M_{NUOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = M_{NOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ $for\ i = 1, 2, \cdots, 13$. The extreme point trivariate distribution P_{14} of $M_{NLOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is not NUOD, because of (3.11). The extreme point trivriate distributions P_{15} of $M_{NUOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is not NLOD, because of (3.7).

2. Because of the symmetry present in the probabilities $p_1 = \frac{1}{2} = p_2$, $q_1 = \frac{1}{2} = q_2$, and $r_1 = \frac{1}{2} = r_2$, the extreme points of $M_{NUOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ can be obtained from thoes of $M_{NLOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ by flipping 1 and 2 among the idices of P_{ijk}'s of P_i's, $i = 1, 2, 3, 4, 5, 6, 8, 10, 12, 14$.

3. The distributions $P_i's$, $i = 1, 2, \cdots, 12, 13$ are extreme points $M_{NLOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ $\cap M_{NUOD}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

4. If one wishes to construct a trivariate distribution P which is NLOD but not NUOD, one could use P_{14} as a building block. Look for convex combinations of P_{14} and some of all of $P_1, P_2, P_3, P_4, P_5, P_6, P_8, P_{10}, P_{12}$. For instance, any convex combination $\lambda P_1 + (1 - \lambda)P_{14}$ with $0 \leq \lambda < 1$ is NLOD but not NUOD, because of (3.11).

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