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Computer Topology and Its Applications*

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Abstract

Recently, the generalized digital (k_0, k_1)-continuity and its properties are investigated. Furthermore, the k-type digital fundamental group for digital image has been studied with the generalized k-adjacencies. The main goal of this paper is to find some properties of the k-type digital fundamental group of Boxer and to investigate some properties of minimal simple closed k-curves with relation to their embeddings into some spaces in $\mathbb{Z}^n (2 \leq n \leq 3)$.

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1. Introduction

For digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$ and $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$, one concept of digital (k_0, k_1)-continuity was established via $\varepsilon - \delta$ definition for the case, $(k_0, k_1) = (2n_0, 2n_1)$ [2, 7]. Another was introduced in terms of the transformation from

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each k_0-connected subimage of X into k_1-connected one, $k_i \in \{2n_i(n_i \geq 1), 3^{n_i} - 1(n_i \geq 2), 18(n_i = 3)\}, i \in \{0, 1\}$ [cf. 1].

Recently the notion of digital (k_0, k_1)-continuity from the image X in $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$ into Y in $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$ has been studied with the generalized k_i-adjacencies, where $k_i \in \{2n_i(n_i \geq 1), 3^{n_i} - 1(n_i \geq 2), 3^{n_i} - \sum_{t=0}^{r-2} C_t^{n_i} 2^{n_i - t} - 1(r = n_i - 1, n_i \geq 3)\}, i \in \{0, 1\}$ [cf. 3, 4]. In this paper we investigate some digital images with generalized graph (k_0, k_1)-continuity. Furthermore, we calculate the k-type digital fundamental group of minimal simple closed k-curves in $\mathbb{Z}^n (2 \leq n \leq 3)$.

2. Preliminaries

We briefly overview some notations and terminologies. Let $\mathbb{Z}^n$ be the set of points in the Euclidian n-dimensional space with integer coordinates. Two functions on $\mathbb{Z}^n$ are assumed as follows: $d_n, d_* : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{N} \cup \{0\}$ with $d_n(p, q) = \sum_{i=1}^{n} |p_i - q_i|$ and $d_*(p, q) = max\{|p_i - q_i|\}_{i \in M}$, $M = \{1, 2, \cdots, n\}$, $p = (p_1, p_2, \cdots, p_n), q = (q_1, q_2, \cdots, q_n) \in \mathbb{Z}^n$ and $\mathbb{N}$ is the set of natural numbers.

Actually, the k-adjacency of a digital image is established from two functions, d_n and d_*. In fact, in $\mathbb{Z}^n$, some k-adjacencies have been used for the study of n-dimensional images, where $k \in \{2n(n \geq 1), 3^n - 1(n \geq 2), 18(n = 3)\}$ [1]. For $\{a, b\} \subset \mathbb{Z}$ with $a \lneq b$, $[a, b]_{\mathbb{Z}} = \{a \leq n \leq b | n \in \mathbb{Z}\}$ is called a digital interval.

Since the notion of generalized k-connectivity for n-dimensional digital image in $\mathbb{Z}^n (n \geq 1)$ are very useful in image processing, image synthesis, image analysis, computer vision, and so forth, it was induced from $(1) \sim (n)$ below [cf. 3, 4]:

(1) p and q are called $(3^n - 1)$-adjacent if $q \in N_{3^n - 1}(p)$, $n \geq 2$, where $N_{3^n-1}(p) = \{q \in \mathbb{Z}^n | d_n(p, q) \leq n, d_*(p, q) = 1\}$ and $3^n - 1$ is the cardinality of $N_{3^n - 1}(p)$.

Generally, for $l \in [2, n-1]_{\mathbb{Z}}, n \geq 3$, the following is established.

(l) p and q are called $(3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1)$-adjacent if $q \in N_{3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1}(p)$, where $N_{3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1}(p) = \{q \in \mathbb{Z}^n | d_n(p,q) \le n - l + 1, d_*(p,q) = 1\}$, where C_t^n stands for the combination of n objects taken t and $3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1$ is the cardinality of $N_{3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1}(p), r = n-1$. And finally,
(n) p and q are called $2n$-adjacent if $q \in N_{2n}(p)$, $n \ge 1$, where $N_{2n}(p) = \{q \in \mathbb{Z}^n | d_n(p,q) = 1\}$ and $2n$ is the cardinality of $N_{2n}(p)$.

Hereafter, the n-dimensional digital image X is considered in a digital picture $(\mathbb{Z}^n, k, \bar{k}, X)$ with one of the following cases: $(k, \bar{k}) \in \{(2n, \bar{k}), (k, 2n)\}$, where $k, \bar{k} \in \{2n(n \ge 1), 3^n - 1(n \ge 2), 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1(r = n-1, n \ge 3)\}$.

The case $k = \bar{k}$ should not be taken because of the digital connectivity paradox in the digital Jordan theorem [5, p. 266] except for $k = 2n$.

For example, in $\mathbb{Z}^2$, k-adjacency is considered, $k \in \{4, 8\}$; in $\mathbb{Z}^3$, k-adjacency is also assumed, $k \in \{26, 18, 6\}$ [5]. Furthermore, from the formulas $(1) \sim (n)$, in $\mathbb{Z}^n (n \ge 4)$, a digital picture $(\mathbb{Z}^4, k, \bar{k}, X)$ is assumed with one of the following cases, $k \in \{80, 64, 32, 8\}$; a digital picture $(\mathbb{Z}^5, k, \bar{k}, X)$ is also considered with one of the following cases, $k \in \{242, 210, 130, 50, 10\}$. And further, a digital picture $(\mathbb{Z}^n, k, \bar{k}, X)(n \ge 6)$ is also obtained from (l) above.

For a digital image $X \subset \mathbb{Z}^n$, two points $x (\neq) y \in X$ are called k-connected [6] if there is a k-path $f : [0, m]_{\mathbb{Z}} \to X$ which the image is a sequence $(x_0, x_1, \cdots, x_m)$ from the set of points $\{f(0) = x_0 = x, f(1) = x_1, \cdots, f(m) = x_m = y\}$ such that x_i and x_{i+1} are k-adjacent, $i \in [0, m-1]_{\mathbb{Z}}, m \ge 1$. The length of a k-path is the number m above [cf. 2, 6]. And a simple k-curve is considered as a sequence $(x_0, x_1, \cdots, x_m)$ of an image of the k-path such that x_i and x_j are k-adjacent if and only if $j = i \pm 1 (\text{mod} m)$[cf. 1].

Now the digital continuity of [1, 2] is restated with relation to the local property and the generalized k-adjacencies, which is helpful to study a pointed digital (k_0, k_1)-homotopy theory in $\mathbb{Z}^n$.

Definition 2.1. In two digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$ and $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$, we say a map $f : X \to Y$ is digitally (k_0, k_1)-continuous at $x_0 \in X$ if f satisfies the following: The image under the map f of every k_0-connected subset containing x_0, $O_{k_0}(x_0)$, is k_1-connected.

If f is digitally (k_0, k_1)-continuous at any point $x \in X$, f is called a digitally (k_0, k_1)-continuous map where $k_i \in \{3^{n_i} - 1 (n_i \geq 2), 3^{n_i} - (\sum_{t=0}^{r-2} C_t^{n_i} 2^{n_i - t}) - 1 (2 \leq r \leq n_i - 1), 2n_i\}, i \in \{0, 1\}$.

3. Some Propertoes of the Boxer's K-Type Digital Fundamental Group

A digital (k_0, k_1)-homotopy can be established in terms of the digital (k_0, k_1)-continuity above. Specifically, digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$ and $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$, let $f, g : X \to Y$ be digitally (k_0, k_1)-continuous functions. And suppose that there is a positive integer m and a function, $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

(1) for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$,

(2) for all $x \in X$, the induced map $F_x : [0, m]_{\mathbb{Z}} \to Y$ defined by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$ is digitally $(2, k_1)$-continuous,

(3) for all $t \in [0, m]_{\mathbb{Z}}$, the induced map F_t which is defined by $F_t(x) = F(x, t) : X \to Y$ is digitally (k_0, k_1)-continuous for all $x \in X$.

Then we use the notation $f \simeq_{d \cdot (k_0, k_1) \cdot h} g$.

Especially, for the case of digital (k, k)-homotopy, we call it a digital k-homotopy and use the notation: $f \simeq_{d \cdot k \cdot h} g$ instead of $f \simeq_{d \cdot (k,k) \cdot h} g$.

For the digital image X with k-adjacency and its subimage A, we call (X, A) a digital image pair with k-adjacency. Furthermore, if A is a singleton set $\{p\}$ then (X, p) is called a pointed digital image [1]. And the image X is called k-contractible if $1_X \simeq_k c_{\{x_0\}}$, where $c_{\{x_0\}}$ is a constant map for every $x_0 \in X$ [1].

A digitally (k_0, k_1)-continuous function $f : X \to Y$ is k_1-nullhomotopic in Y if f is digitally k_1-homotopic in Y to a constant function $c_{\{y_0\}}$, $y_0 \in Y$ [cf. 1].

In two digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, (X, A))$ and $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, (Y, B))$, we say $f : (X, A) \to (Y, B)$ is digitally (k_0, k_1)-continuous if $f : X \to Y$ is digitally (k_0, k_1)-continuous and $f(A) \subset B$, respectively.

The digital fundamental group was developed for a digital image in at most three dimensional image in $\mathbb{Z}^3$ by use of k-loops [6] and another was derived from an approach to algebraic topology with the standard k-adjacencies, where $k \in \{3^n - 1, 2n, 18\}$, $n \in \mathbb{N}$ [cf. 1]. We now generalize that of [1] with respect to the dimension and the k-adjacency of an image. A k-type digital fundamental group is induced in terms of the generalized pointed k-homotopy above. Namely, we study an image in $\mathbb{Z}^n$ with n kinds of k-adjacencies, $k \in \{3^n - 1 (n \geq 2), 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1 (2 \leq r \leq n-1), 2n\}$.

For a pointed digital image (X, p), a k-loop f based at p is a k-path, $f : [0, m]_{\mathbb{Z}} \to (X, p)$ such that $f(0) = p = f(m)$. The number m is not fixed, which depends on the k-loop on (X, p). And we put $F_1^k(X, p) = \{f | f$ is a k-loop based at $p\}$.

For maps $f, g \in F_1^k(X, p)$, *i.e.*, $f : [0, m_1]_{\mathbb{Z}} \to (X, p)$ with $f(0) = p = f(m_1)$ and $g : [0, m_2]_{\mathbb{Z}} \to (X, p)$ with $g(0) = p = g(m_2)$, a map $f * g : [0, m_1 + m_2]_{\mathbb{Z}} \to (X, p)$ is taken as follows: $f * g : [0, m_1 + m_2]_{\mathbb{Z}} \to (X, p)$ is defined by $f * g(t) = f(t), 0 \leq t \leq m_1$ and $g(t - m_1), m_1 \leq t \leq m_1 + m_2$. Then $f * g \in F_1^k(X, p)$.

We denote by $[f]$ a digital k-homotopy class of f. Obviously, the homotopy class $[f * g]$ only depends on the homotopy classes $[f]$ and $[g]$. Furthermore, for any $f_1, f_2, g_1, g_2 \in F_1^k(X, p)$ such that $f_1 \in [f_2], g_1 \in [g_2]$, $[f_1 * g_1] = [f_2 * g_2]$ [1]. In fact, the operation $*$ is due to Khalimsky [5]. Consequently, we put $\pi_1^k(X, p) = \{[f] | f \in F_1^k(X, p)\}$. And we take an operation $\cdot$ on $\pi_1^k(X, p)$ as follows: $[f] \cdot [g] = [f * g]$. The group structure on $\pi_1^k(X, p)$ is checked by the same method of [1] with respect to the digital $(2, k)$-continuity. Consequently, we get the group $\pi_1^k(X, p)$ with the operation $\cdot$ above, which is called a k-type digital fundamental group of a pointed digital image (X, p).

For digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$, $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$ and a digitally (k_0, k_1)-continuous based map $h : (X, p) \to (Y, q)$, the map h induces a digital fundamental group (k_0, k_1)-homomorphism, $\pi_1^{(k_0,k_1)}(h) = h_* : \pi_1^{k_0}(X,p) \to \pi_1^{k_1}(Y,q)$ by an equation $h_*([f_1]) = [h \circ f_1]$, where $[f_1] \in \pi_1^{k_0}(X,p)$, which is well defined. Particularly, if $k_0 = k_1$, we use the notation, $\pi_1^{k_0}(h)$.

For digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$, $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$ and $(\mathbb{Z}^{n_2}, k_2, \bar{k}_2, Z)$, let $f : X \to Y$ be digitally (k_0, k_1)-continuous based map and $g : Y \to Z$ be digitally (k_1, k_2)-continuous functions, then $\pi_1^{(k_0,k_2)}(g \circ f) = \pi_1^{(k_1,k_2)}(g) \circ \pi_1^{(k_0,k_1)}(f)$ is obviously followed. In particular, if $k_0 = k_1 = k_2$, $\pi_1^{k_0}(g \circ f) = \pi_1^{k_0}(g) \circ \pi_1^{k_0}(f)$ [cf. 1].

Actually, if a pointed image (X,p) is k-connected, for any point $q \in X$ there is a group isomorphism $\pi_1^k(X,p) \cong \pi_1^k(X,q)$, where $\cong$ means a group isomorphism.

For this reason, omitting the base point can be approved for a k-connected image with relation to a k-type digital fundamental group. If X is k-contractible, then $\pi_1^k(X,p)$ is trivial [cf. 1].

Definition 3.1. We say a digital image $X \subset \mathbb{Z}^n$ is simply k-connected if it is k-connected and $\pi_1^k(X, x_0)$ is trivial for every $x_0 \in X$, $k \in \{3^n - 1, 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1 (2 \le r \le n-1), 2n\}$.

4. Minimal Simple Closed K-curve and Their Digital Topological Properties

We now recall the minimal simple closed k-curves in $\mathbb{Z}^2$, $k \in \{4, 8\}$, MSC_8, MSC_4 and MSC_8', with relation to k-connectedness, and their digital topological properties with relation to k-contractibility were studied [cf. 3]. To be specific,

(1) MSC_8 is any set in $\mathbb{Z}^2$ of the form,

$\{(x_1, y_1), (x_1-1, y_1+1), (x_1-2, y_1), (x_1-2, y_1-1), (x_1-1, y_1-2), (x_1, y_1-1)\}$.

(2) MSC_4 is any set in $\mathbb{Z}^2$ of the form,

$\{(x_1, y_1), (x_1, y_1+1), (x_1-1, y_1+1), (x_1-2, y_1+1), (x_1-2, y_1), (x_1-2, y_1-1), (x_1-1, y_1-1), (x_1, y_1-1)\}$.

(3) Let MSC_8' be any set in $\mathbb{Z}^2$ of the form,

$\{(x_1, y_1), (x_1 - 1, y_1 + 1), (x_1 - 2, y_1), (x_1 - 1, y_1 - 1)\}$.

On the other hand, since each element of MSC'_8 is distinct with respect to the 4-connectedness, $\pi_1^4(MSC'_8, x_0)$ is trivial for any point $x_0 \in MSC'_8$ and further MSC'_8 is simply 8-connected [cf. 1].

It is possible to consider (MSC_4, p_0) as a subimage of $(\mathbb{Z}^2 - \{z_0\}, p_0)$ from $(*2)$ above, where $z_0 = (x_1 - 1, y_1)$. And (MSC_8, p_0) can be also considered as a subimage of $(\mathbb{Z}^2 - \{z_1, z_2\}, p_0)$ from $(*1)$ above, where $z_1 = (x_1 - 1, y_1), z_2 = (x_1 - 1, y_1 - 1)$.

Now some properties with relation to a k-type digital fundamental group homomorphism are followed.

Theorem 4.1 There exist some digital topological properties of MSC_4, MSC_8 and $\mathbb{Z}^2 - \{z_1, z_2, \cdots, z_n\}$ with relation to their embeddings into some spaces in $\mathbb{Z}^n (2 \leq n \leq 3)$.

(1) The inclusion map $i : (MSC_4, p_0) \to (\mathbb{Z}^2 - \{z_0\}, p_0)$ induces a k-type digital fundamental group isomorphism $i_* : \pi_1^k(MSC_4, p_0) \to \pi_1^k(\mathbb{Z}^2 - \{z_0\}, p_0)$, $k \in \{4, 8\}$.

(2) The inclusion map $j : (MSC_8, p_0) \to (\mathbb{Z}^2 - \{z_1, z_2\}, p_0)$ induces an 8-type digital fundamental group isomorphism: $j_* : \pi_1^8(MSC_8, p_0) \to \pi_1^8(\mathbb{Z}^2 - \{z_1, z_2\}, p_0)$, where $z_i (\neq) z_j$ are 4-adjacent. But the inclusion map j does not induce a 4-type digital fundamental group isomorphism, *i.e.*, $j_* : \pi_1^4(MSC_8, p_0) \to \pi_1^4(\mathbb{Z}^2 - \{z_1, z_2\}, p_0)$ is not an isomorphism.

(3) If any two elements of the set $\{q_i\}_{i \in M}$ are not 4-adjacent, $M = \{1, 2, \cdots, n\}$, $\mathbb{Z}^2 - \{q_1, q_2, \cdots, q_n\}$ is simply 8-connected. On the other hand, if at least two elements of $\{q_i\}_{i \in M}$ are 4-adjacent, $\mathbb{Z}^2 - \{q_1, q_2, \cdots, q_n\}$ is not simply 8-connected.

Proof. (1) The minimal simple 4-curve MSC_4 can be assumed as a subimage of $\mathbb{Z}^2 - \{z_0\}$. Since any k-loops based at p_0 in $(\mathbb{Z}^2 - \{z_0\}, p_0)$ are k-homotopic to a k-loop based at p_0 in (MSC_4, p_0), $\pi_1^k(MSC_4, p_0)$ is isomorphic to $\pi_1^k(\mathbb{Z}^2 - \{z_0\}, p_0)$, $k \in \{4, 8\}$. Furthermore, MSC_4 can be assumed in this way:

$\{p_0 = (x_1, y_1), p_1 = (x_1, y_1+1), p_2 = (x_1-1, y_1+1), p_3 = (x_1-2, y_1+1), p_4 = (x_1 - 2, y_1), p_5 = (x_1 - 2, y_1 - 1), p_6 = (x_1 - 1, y_1 - 1), p_7 = (x_1, y_1 - 1)\}$.

Then there is a digital 8-homotopy, $H : MSC_4 \times [0,3]_{\mathbb{Z}} \to MSC_4$ as follows:

(1) $H(p_i, 0) = p_i$, for any $p_i \in MSC_4$,

(2) $H(p_{2i+1}, 1) = p_{2i}, H(p_{2i}, 1) = p_{2i}, i \in [0,3]_{\mathbb{Z}}$,

(3) $H(p_i, 2) = p_0, i \in [0,3]_{\mathbb{Z}}$ and $H(p_j, 2) = p_6, j \in [4,7]_{\mathbb{Z}}$,

(4) $H(p_i, 3) = p_0, i \in [0,7]_{\mathbb{Z}}$.

Thus $1_{MSC_4} \simeq_8 c_{\{p_0\}}$. Therefore $\pi_1^8(\mathbb{Z}^2 - \{z_0\}, p_0) \simeq \pi_1^8(MSC_4, p_0) \simeq 0$. Namely, if $k = 8$, i_* is a trivial group isomorphism.

On the other hand, since any 4-loops based at p_0 in (MSC_4, p_0) are not 4-nullhomotopic, $\pi_1^4(MSC_4)$ is not trivial.

(2) The minimal simple 8-curve MSC_8 can be assumed as a subimage of $\mathbb{Z}^2 - \{z_1, z_2\}$. Since any 8-loops based at p_0 in $(\mathbb{Z}^2 - \{z_1, z_2\}, p_0)$ are 8-homotopic to an 8-loop based at p_0 in (MSC_8, p_0), $\pi_1^8(\mathbb{Z}^2 - \{z_1, z_2\}, p_0) \cong \pi_1^8(MSC_8, p_0)$. In fact, any nontrivial 8-loops in (MSC_8, p_0) are not 8-nullhomotopic.

On the other hand, any nontrivial 4-loops surrounding the points p_1, p_2 in $(\mathbb{Z}^2 - \{p_1, p_2\}, p_0)$ are not 4-nullhomotopic. Thus $\pi_1^4(\mathbb{Z}^2 - \{p_1, p_2\}, p_0)$ is not trivial. However, since any 4-loops in MSC_8 are 4-nullhomotopic, $\pi_1^4(MSC_8, p_0)$ is trivial. Thus $\pi_1^4(\mathbb{Z}^2 - \{p_1, p_2\}, p_0)$ is not isomorphic to $\pi_1^4(MSC_8, p_0)$.

(3) We only prove that any nontrivial 8-loops in $(\mathbb{Z}^2 - \{q_1, q_2, \cdots, q_n\}, r_1)$ are 8-nullhomotopic, where any two elements of the set $\{q_i\}_{i \in M}$ are not 4-adjacent. First, we prove that any nontrivial 8-loops in $(\mathbb{Z}^2 - \{q_1, q_2\}, r_1)$ are 8-nullhomotopic, where q_1 and q_2 are not 4-adjacent. Specifically, W is assumed as a subimage of $(\mathbb{Z}^2 - \{q_1, q_2\})$, where $W = \{r_1 = (x_1, y_1), r_2 = (x_1 - 1, y_1 + 1), r_3 = (x_1 - 2, y_1), r_4 = (x_1 - 1, y_1 - 1), r_5 = (x_1 + 1, y_1 - 1), r_6 = (x_1, y_1 - 2), \}$, $q_1 = (x_1 - 1, y_1)$ and $q_2 = (x_1, y_1 - 1)$.

Actually, any 8-loops based at p_0 in $(\mathbb{Z}^2 - \{z_1, z_2\}, p_0)$ are 8-homotopic to an 8-loop based at p_0 in (W, p_0). And further, there is a digital 8-homotopy on W, *i.e.* $H : W \times [0,2]_{\mathbb{Z}} \to W$ such that

(1) $H(r_i, 0) = r_i$, for any $r_i \in W$,

(2) $H(r_3, 1) = H(r_2, 1) = r_2, H(r_4, 1) = H(r_1, 1) = r_1, H(r_6, 1) = H(r_5, 1) = r_5$,

(3) $H(r_i, 2) = r_1$, for any $r_i \in W$ and $i \in [1, 6]_{\mathbb{Z}}$.

Namely, W is 8-contractible from the digital 8-homotopy above. Moreover, $\pi_1^8(\mathbb{Z}^2 - \{q_1, q_2\})$ is group isomorphic to $\pi_1^8(W)$ as a trivial group.

Similarly, if any two elements of $\{q_i\}_{i \in M}$ are not 4-adjacent, $M = [1, n]_{\mathbb{Z}}$, any nontrivial 8-loops in $(\mathbb{Z}^2 - \{q_1, q_2, \cdots, q_n\}, r_1)$ are 8-nullhomotopic.

On the other hand, if at least two elements of $\{q_1, q_2, \cdots, q_n\}$ are 4-adjacent, any nontrivial 8-loops surrounding the points $q_1, q_2, \cdots, q_n$ in $\mathbb{Z}^2 - \{q_1, q_2, \cdots, q_n\}$ are not 8-nullhomotopic by the non 8-contractibility of the minimal simple closed 8-curve MSC_8. Thus $\pi_1^8(\mathbb{Z}^2 - \{q_1, q_2, \cdots, q_n\}, r_1)$ are not trivial. □

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