Honam Mathematical J. **25** No.1. pp. 3–15

An Analogue of Robinson-Schensted Correspondence for Oscillating Generalized Tableaux*

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Abstract

We prove an analogue of the Robinson-Schensted correspondence between generalized biwords and oscillating semi-standard tableaux. We give a geometric construction of the correspondence and examine combinatorial properties of the correspondence.

1991 Mathematics Subject Classification: 05A15
Key words and phrases: generalized biword, oscillating semi-standard tableau, shadow diagram

1. Introduction

An oscillating tableau is a finite sequence of tableaux where each tableau except the first one is obtained from the previous tableau by an insertion or a deletion of a cell. Sundaram [12] used the oscillating tableaux to prove a bijection establishing the Cauchy identity for the symplectic group and it was

*Supported by Faculty Research Fund of Jeonju University, 2002.

followed by numerous works dealing with the combinatorial properties of the Robinson-Schensted correspondence for oscillating tableaux [2, 4,5,6, 7]. We can cite the papers [1], [7],[11], [9],[10],[13] for the Robinson-Schensted correspondence.

In this paper, we extend the Robinson-Schensted correspondence for oscillating standard tableaux to the correspondence for oscillating semi-standard tableaux. In the section 2, we give basic definitions on generalized biwords and oscillating semi-standard tableaux and in the section 3, we present the correspondence for oscillating semi-standard tableaux. Then we give a geometric version of the Robinson-Schensted correspondence for oscillating semi-standard tableaux and examine combinatorial properties of this correspondence.

2. Definitions and notations

Let $\lambda = (\lambda_1, ..., \lambda_k)$, $\lambda_1 \geq ... \geq \lambda_k$, be a partition of n such that $\sum_{i=1}^{k} \lambda_i = n$. The partition λ can be displayed a Ferrers diagram with the part λ_i in the row i. A semi-standard tableau S of shape λ is a labeling of the cells of λ with positive integers so that the rows are strictly increasing and the columns are weakly increasing. $\Omega(\lambda)$ denotes the set of semi-standard tableaux of shape λ.

We introduce the external insertion and the external deletion for a semi-standard tableau ([3],[7]). Let S be a semi-standard tableau of shape λ.

The external insertion is the insertion defined by the Knuth [7]. This algorithm inserts an integer x in a semi-standard tableau S in the following way:

(1) if x is greater than any other labels in the first row, then x is inserted in the end of the first row,

(2) else if a label y is the smallest element in the first row such that $y \geq x$, then x is inserted in the place of y and y is bumped in the next row and repeat (1) with $x = y$ in the next row.

(3) the bumping process ends when there is no remaining row in S.

We denote the new tableau obtained after the external insertion by $ExtI(S,x)$. The inverse process is called external deletion, denoted by $ExtD(S,(u,v),x)$ or simply $ExtD(S,(u,v))$, which ends with the expulsion of an integer x out of S. Moreover, we can simply attach or erase a cell without using an insertion algorithm and a deletion algorithm.

Example 2.1. The following tableau P is a semi-standard tableau. $ExtI(P,6)$ inserts 6 in P by the external insertion, and the cell $P(2,3)$ is deleted and 3 is excluded from the tableau P by the external deletion $ExtD(P,(2,3),3)$.

$$P = \begin{array}{llll} 6 & & & \\ 2 & 3 & & \\ 1 & 3 & 5 & \\ 1 & 2 & 3 & 7 \end{array}$$

$$\mathrm{ExtI(P,6)} = \begin{array}{llll} 6 & & & \\ 2 & 3 & & \\ 1 & 3 & 5 & 7 \\ 1 & 2 & 3 & 6 \end{array} \qquad \mathrm{ExtD(P,(2,3),3)} = \begin{array}{llll} 6 & & & \\ 2 & 3 & & \\ 1 & 3 & & \\ 1 & 2 & 5 & 7 \end{array}$$

Let $\mathbb{S}_n$ be the set of permutations of $[n]$ =1,2,...,n. A generalized biword π on $[m]$ is a sequence of vertical pairs of positive integers of $[m]$,
$\pi = \begin{pmatrix} u_1 & u_2 & \ldots & u_n \\ v_1 & v_2 & \ldots & v_n \end{pmatrix}$, where $u_1 \geq u_2 \geq \ldots \geq u_k$, $u_i \geq v_i$ for $i = 1, \ldots, k$, and $v_i \geq v_j$ if $u_i = u_j$. If all of the u_i's and v_i's are pairwise distinct, then π is a biword.

GB denotes the set of generalized biwords. The length of π is the number of pairs of $\binom{u_i}{v_i}$, or $|\pi| = n$. GB_n denotes the set of generalized biwords of length n.

An oscillating semi-standard tableau of length n is a sequence of semi-standard tableaux $P = (P_0, P_1, \ldots, P_n)$ where P_k is obtained from P_{k-1} by an insertion or a deletion of a cell.

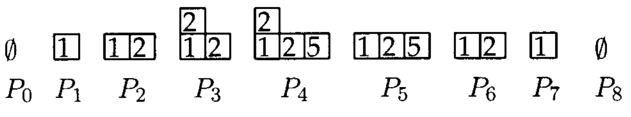
O_n denotes the set of oscillating semi-standard tableaux $P = (P_0, P_1, ..., P_n)$ of length n satisfying the following conditions:

(1) the shapes of P_0 and P_n are $\emptyset$,

(2) P_k is obtained from P_{k-1} by attaching a cell with a label (without using the insertion algorithms) or a deletion of a cell by external deletion.

(3) if $x_i, x_j, ..., x_m$ are inserted respectively in $P_i, P_j, ..., P_m$, $i < j < ... < m$, then $x_i \leq x_j \leq ... \leq x_m$.

The following tableaux belongs to O_5. 1 is inserted in P_0 to obtain P_1, 2 is inserted in P_1 to obtain P_2, 2 is inserted in P_2 to obtain P_3 and 5 is inserted in P_3 to obtain P_4.



For a $P \in O_n$, we define a set of nondecreasing sequences of positive integers in relation to P, $I(P) = \cup I_a$, where $I_a = \{a_0, a_1, a_2, ..., a_n\}$, $a_0 = 0 \leq a_1 \leq ... \leq a_n$ and $a_k = x$ if $P_k = P_{k-1} + (u, v)$ with $P_k(u, v) = x$ for $1 \leq k \leq n$. An I_a of the example above is $\{0, 1, 2, 2, 5, a_5, a_6, a_7, a_8\}$, where a_5, a_6, a_7, a_8 can be any positive integers satisfying $5 \leq a_5 \leq a_6 \leq a_7 \leq a_8$.

$\overline{O}_n$ denotes the set of oscillating tableaux of length n, $Q = (Q_0, Q_1, ..., Q_n)$, satisfying the following conditions:

(1) the shapes of Q_0 and Q_n are $\emptyset$,

(2) Q_k is obtained from Q_{k-1} by erasing of a labelled cell (without using the deletion algorithms) or an insertion of a cell by external insertion.

(3) if $x_i, x_j, ..., x_m$ are deleted respectively from $Q_i, Q_j, ...Q_m$, $i < j < ... < m$, then $x_i \geq x_j \geq ... \geq x_m$.

We know that $P = (P_0, P_1, ..., P_n) \in O_n$ if and only if $\overline{P} = (P_n, P_{n-1}, ..., P_0) \in \overline{O}_n$.

We define a set of nonincreasing sequences of positive integers in relation to $Q \in \overline{O}_n$, $J(Q) = \cup J_b$, $J_b = \{b_1, b_2, ..., b_n\}$ satisfying:

(1) $b_1 \geq b_2 \geq ... \geq b_n$

(2) if $Q_{k+1} = Q_k - (u, v)$ with $Q_k(u, v) = y$, then $b_k = y$.

3. Oscillating semi-standard tableaux

Let $\mathbb{N}$ be a set of positive integers. We consider a new alphabet $\mathbb{N}^* = \mathbb{N} \cup \{j^{(h)} : j, h \in \mathbb{N}\}$ such that

$$\ldots < j < j^{(1)} < j^{(2)} < \ldots < j+1 < (j+1)^{(1)} < (j+1)^{(2)} < \ldots .$$

Definition 1 *(i) Two line array* $\begin{pmatrix} u_1 & u_2 & \ldots & u_k \\ v_1 & v_2 & \ldots & v_k \end{pmatrix}$ *is a biword on* $\mathbb{N}^*$ *if, for* $i = 1, ..., k$, $u_i > v_i$, $u_i, v_i \in \mathbb{N}^*$, $u_1 > u_2 > ... > u_k$, *and all of the* u_i*'s and* v_i*'s are pairwise distinct.*

(ii) A standard tableau A *on* $\mathbb{N}^*$ *of shape* λ *is a labeling of the cells of* λ *with alphabets of* $\mathbb{N}^*$ *so that the rows and columns are strictly increasing.*

Now, we show how to standardize a generalized biword to a biword. For a given generalized biword $\pi = \begin{pmatrix} u_1 & u_2 & \ldots & u_n \\ v_1 & v_2 & \ldots & v_n \end{pmatrix}$, if $u_j = u_{j+1} = ... = u_{j+m} = v_{i_1} = v_{i_2} = ... = v_{i_k}$, $i_1 < i_2 < ... < i_k$, in π, then we change v_{i_k} into u_j, $v_{i_{k-1}}$ into $u_j^{(1)}$, ..., u_j into $u_j^{(m+k-1)}$. The new two line array is a biword on $\mathbb{N}^*$ and we denote it by τ. The transformation from π to τ, denoted by $\tau = \varphi(\pi)$, is bijective.

Example 3.1.

$$\pi = \begin{pmatrix} 7 & 7 & 6 & 5 & 4 & 4 \\ 5 & 5 & 1 & 3 & 4 & 2 \end{pmatrix} \overset{\varphi}{\longleftrightarrow} \tau = \begin{pmatrix} 7^{(1)} & 7 & 6 & 5^{(2)} & 4^{(2)} & 4^{(1)} \\ 5^{(1)} & 5 & 1 & 3 & 4 & 2 \end{pmatrix}$$

All of the contents of τ are pairwise distinct, so τ is a biword on the alphabet $\mathbb{N}^*$. Here we introduce algorithms that gives a bijection between a generalized biwords and oscillating semi-standard tableaux.

Algorithm 1.

The input is $\pi \in GB_n$. The output is (P, I) where $P = (P_0, P_1, ..., P_{2n})$ is an oscillating semi-standard tableau of length $2n$, $I = (i_0 = 0, i_1, i_2, ..., i_{2n})$ is an nondecreasing sequence where $i_k = x$ if $P_k = P_{k-1} + (u, v)$ with $P_k(u, v) = x$.

(i) Let $\tau = \varphi(\pi)$ and $J = (i_0 = 0, j_1, j_2, ..., j_{2n})$ be an increasing sequence such that $j_0 = 0 < j_1 < j_2 < ... < j_{2n}$, and $j_k \in \check{\tau}$ or $j_k \in \hat{\tau}$ for $1 \leq k \leq 2n$.

(ii) Let $T_{2n} = \emptyset$.

For k from $2n$ to 1 :

(a) if the pairs $\binom{j_k}{x}$ belong to τ, then $T_{k-1} = ExtI(T_k, x)$, and erase the pair $\binom{j_k}{x}$ from τ_k to obtain τ_{k-1}.

(b) else if there are cells $T_k(u, v) = j_k$, then erase the cell $T_k(u, v)$ to obtain and . T_{k-1} and $\tau_{k-1} = \tau_k$

So we find that if $T_k = T_{k-1} + (u, v)$ with $T_k(u, v) = a$ then $j_k = a$.

(iii) $P = (P_0, P_1, ..., P_{2n})$ is obtained from $T = (T_0 = \emptyset, ..., T_{2n} = \emptyset)$ by removing the exponent of each label if it exists and I is obtained from $J = \{j_0 = 0, j_1, j_2, ..., j_{2n}\}$ by removing the exponent of each content if it exists.

Algorithm 2.

The input is $P \in O_n$. $I = \{i_0 = 0, i_1, i_2, ..., i_{2n}\} \in I(P)$. The output is a generalized biword π of length n.

Let $\pi_0 = \emptyset$. For from 1 to $2n$: if $P_k = ExtD(P_{k-1}, x)$, then add the pair $\binom{i_k}{x}$ to obtain π_k, else $\pi_k = \pi_{k-1}$.

Finally, we obtain $\pi = \pi_n$.

In the following we have an oscillating semi-standard tableaux corresponding with biword τ given in Example 3.1.

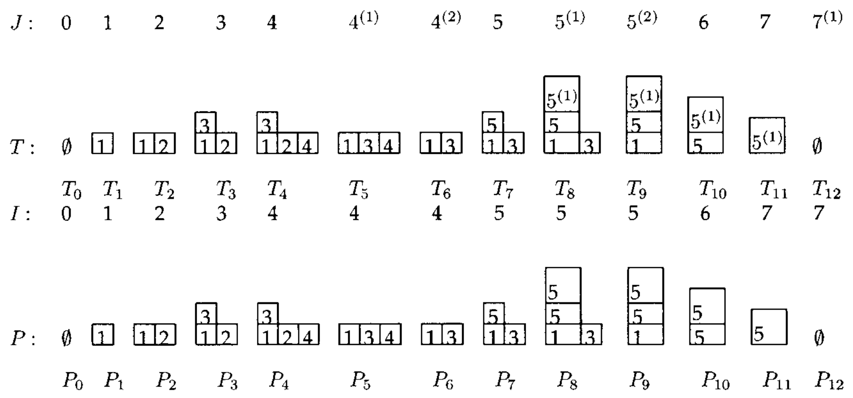


Figure 3.1

According to Algorithm 1 and Algorithm 2, we pronounce the following theorem.

Theorem 1 *There is a bijection Φ from π of GB_n to (P, I) with oscillating semi-standard tableaux P of O_{2n}, and $I = \{i_0, i_1, i_2, ..., i_{2n}\}$ being un increasing sequence of numbers in $\mathbb{N}$ such that $i_k = a$ when $P_k = ExtI(P_{k-1}, a)$.*

Theorem 2 *Let π be a generalized biword of length n. There is a bijection Φ_{RS} from $\pi \in GB_n$ to $\{(P, I_1), (Q, I_2))\}$ of $\cup_\beta[\{\Theta_n(\emptyset \to \beta) \times I(P)\} \times \{\overline{\Theta}_n(\emptyset \to \beta) \times J(Q)\}]$.*

Proof: According to Theorem 1, we get $(P_0 = \emptyset, ..., P_n P_{n+1} ..., P_{2n} = \emptyset)$ and $I = \{i_0, i_1, ...i_n, .., i_{2n}\}$. We have the result by taking $(P, I_1) = ((P_0, P_1, ..., P_n$ (of shape β), $\{i_0, i_1, ..., i_n\})$ with $I_1 \in I(P)$, and $(Q, I_2) = ((P_{2n}, P_{2n-1}, ..., P_n$ (of shape β), $\{i_{2n}, i_{2n-1}, \ldots, i_n\})$ with $I_2 \in J(Q)$. $\diamondsuit$

4. Geometric representation of a generalized biword

We represent a generalized biword in the the first quadrant of the Cartesian plane and we investigate the combinatorial properties of the geometric representation of a generalized biword.

For a given generalized biword $\pi = \begin{pmatrix} u_1 & u_2 & \dots & u_n \\ v_1 & v_2 & \dots & v_n \end{pmatrix}$, let $\tau = \varphi(\pi)$. we represent τ instead of π in the part $\{0, 1, 2, ..., n\} \times \{0, 1, ..., n\}$ of the Cartesian plane as follows:

- Define a map Ψ : abscissas x $(x = 0, 1, 2, ..., n) \to \{u_1 + 1\} \cup \hat{\tau}$ by

$$\Psi(x) = \begin{cases} u_1 + 1 & \text{if } x = 0 \\ x^{th} \text{ greatest element of } \hat{\tau} & \text{else} \end{cases}$$

- Define a map Γ : abscissas y $(y = 0, 1, 2, ..., n) \to \{0\} \cup \check{\tau}$ by

$$\Gamma(y) = \begin{cases} 0 & \text{if } y = 0 \\ y^{th} \text{ lowest element of } \check{\tau} & \text{else} \end{cases}$$

- We define valid domain which is the set of points (x, y) such that $\Psi(x) \geq \Gamma(y)$.

Definition 2 *The shadow $S(\tau)$ of a generalized biword τ on $\mathbb{N}^*$ is the set of points (x, y) such that there is a point (x', y') of the representation of τ with $x' \leq x$, $y' \leq y$.*

Shadow lines of τ are defined recursively. The first shadow line L_1 of τ is the boundary of $S(\tau)$. To construct the shadow line L_{i+1} of τ remove the points of the representation of τ lying on L_i and construct the shadow line of the remaining points. This procedure ends when there is no remaining point on the plane. The SW-corners of a shadow line are the points of the representation of τ located on this line. The NE-corners of a shadow line are the points (x, y) of the shadow line such that $(x + 1, y)$ and $(x, y + 1)$ are not a part of this shadow line [8].

Following, we give a generalized biword π of GB_6 and $\tau = \varphi(\pi)$ in example 3.1. and their geometric representations with white circles. The geometric representation π is transformed bijectively into a geometric representation of $\tau = \varphi(\pi)$ by lengthening axis from the geometric representation of π. The white circles in geometric representation of τ are two by two disjoint and the only one circle lie on the line $x = k$ $(k = 1, ..., 6)$. The limit of valid domain, the dashed line, is slightly extended on the figure.

$$\pi = \begin{pmatrix} 7 & 7 & 6 & 5 & 4 & 4 \\ 5 & 5 & 1 & 3 & 4 & 2 \end{pmatrix} \overset{\varphi}{\longleftrightarrow} \tau = \begin{pmatrix} 7^{(1)} & 7 & 6 & 5^{(2)} & 4^{(2)} & 4^{(1)} \\ 5^{(1)} & 5 & 1 & 3 & 4 & 2 \end{pmatrix}$$

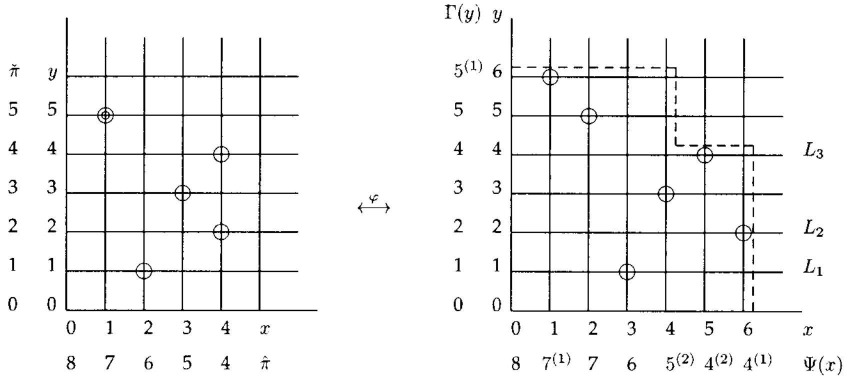


Figure 4.1. Geometric representation of π and $\tau = \varphi(\pi)$

The shadow lines L_1, L_2 and L_3 are described with thick lines. The white circles mark SW-corners of shadow lines.

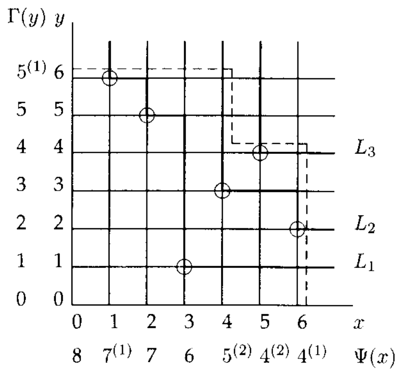


Figure 4.2. Shadow diagram of $\tau = \varphi(\pi)$

The following algorithms show how to make an oscillating semi-standard tableau from the geometric representation of a generalized biword.

Algorithm 1. (a) let $A = \{\Psi(x_i)\}_{i=1..n}$ and $B = \{\Gamma(y_i)\}_{i=0..n}$. We line up $\{\Psi(x_i)\}_{i=1..n}$ and $B = \{\Gamma(y_i)\}_{i=0..n}$ in increasing order and denote it by $J = \{j_0, ..., j_{2n}\}$.

(b) For k from $2n$ to 1:

if $j_k \in A$ and $(\Psi^{-1}(j_k), y)$ is SW-corner of a Shadow line, then $T_{k-1} = ExtI(T_k, \Gamma(y))$,

else if $j_k \in B$ and $(x, \Gamma^{-1}(j_k))$ is SW-corner of a Shadow line, then $T_{k-1} = T_k - (u, v)$ with $T_k(u, v) = j_k$.

Algorithm 2. The input is T an oscillating standard tableaux on $\mathbb{N}^*$ from $\emptyset$ to $\emptyset$ of length $2n$. $J = \{j_0 = 0, j_1, j_2, ..., j_{2n}\}$ an increasing sequence of length $2n$ such that $j_k = x$ when $T_k = T_{k-1} + (u, v)$, with $T_k(u, v) = x$.

The output is a biword τ on $\mathbb{N}^*$ of length n.

Let $\tau_0 = \emptyset$.

For k from 1 to $2n$:

if $T_k = ExtD(T_{k-1}, (u', v'), x)$, then add the pair (j_k, x) to τ_{k-1} to obtain τ_k.

Finally, we obtain $\tau = \tau_n$.

Algorithm 1 and Algorithm 2 prove the bijection from the set of biwords of length n on $\mathbb{N}^*$ to the set of (T, J) with T an oscillating standard tableaux on $\mathbb{N}^*$ from $\emptyset$ to $\emptyset$ of length $2n$, $J = \{j_0 = 0, j_1, j_2, ..., j_{2n}\}$ an increasing sequence of length $2n$ and $j_k = x$ when $T_k = T_{k-1} + (u, v)$, with $T_k(u, v) = x$. So Algorithm 1 and Algorithm 2 prove again the Theorem 1 and Theorem 2 in the section 3 by using the geometric description. Here we investigate the combinatorial properties of the shadow diagram.

Definition 3 *Let* $A = \{\Psi(x_i)\}_{i=1..n}$*,* $B = \{\Gamma(y_i)\}_{i=0..n}$*. Define a function* $s : A \cup B \to \mathbb{N}$ *by* $s(x) = k$ *if* x *is* k^{th} *lowest element of* $A \cup B$*.*

Applying algorithm 1 to the shadow diagram in Figure 4.2, we find again the oscillating semi-standard tableau $T_{12}, T_{11}, ..., T_0$ in figure 3.1.

The shadow line L_1 in Figure 4.2 describes the behavior of the first cell of the first row during the construction of $T_{12}, T_{11}, ..., T_0$. The shadow line L_1 has three SW-corners at (1,6), (2,5) and (3,1). For the SW-corner (1,6), with $\Psi(1) = 7^{(1)}$ and $\Gamma(6) = 5^{(1)}$, followed by (2,5) with $\Psi(2) = 7$ and $\Gamma(5) = 5$. During the construction of the tableaux T_{12} to T_0, the first cell of first row is created during step $s(7^{(1)}) = 12$ with label $5^{(1)}$, this label is replaced during step $s(7) = 11$ by the label 5. The label 5 is replaced during step $s(6) = 10$ by the label 1, because $\Psi(3) = 6$ and $\Gamma(1) = 1$. The cell is deleted during step $s(1) = 1$.

Theorem 3 *The shadow lines of the shadow diagram describe the behavior of the first row of the tableaux T_{2n}, ...T_0 in the following rules:*

1. *a SW-corner (x, y) of L_i indicates that, during the step $s(\Psi(x))$, the i^{th} cell of the first row is labeled with $\Gamma(y)$,*
2. *if the line L_i leaves the valid domain through (x, y), i^{th} cell of the first row is deleted during the step $s(\Gamma(y))$,*
3. *otherwise, the cell in the first row remains unchanged.*

Proof: Induct on k, $1 \leq k \leq n$ for the line $x = k$. if $k = 1$, then a SW-corner $(1, y_1)$ of L_1 exists on the abscissa $x = 1$. $T_{2n} = \emptyset$ and during the step $s(\Psi(1))$, $\Gamma(y_1)$ is inserted in the first cell of the first row of T_{2n} to obtain T_{2n-1}. So the result holds for $k = 1$.

Assume that the result holds for the restriction of the shadow lines to the points having abscissa lower than or equal to $k - 1$ and consider the line $x = k$.

Let (k, y_k) be a SW-corner of the shadow line L_i. We have following two cases :

1. if the SW-corner $(k - 1, y_{k-1})$ on the line $x = k - 1$ is on the shadow line L_{i-1}, then, by assumption, the $i - 1^{th}$ cell of first row of $T_{(\Psi(k))}$ is

labeled with $\Gamma(y_{k-1})$, that is , $T_{(\Psi(k))} = ExtI(T_{s(\Psi(k))-1}, \Gamma(y_{k-1}))$. So we have the inequality $\Gamma(y_{k-1}) < \Gamma(y_k)$, which implies that $\Gamma(y_k)$ is inserted in the i^{th} cell of the first row $T_{\Psi(k)-1}$,

2. if the SW-corner $(k-1, y_{k-1})$ is on the shadow line L_i, then, by assumption, the i^{th} cell of first row $T_{s(\Psi(k))}$ is labeled with $\Gamma(y_{k-1})$. We have $\Gamma(y_{k-1}) > \Gamma(y_k)$ because $\Gamma(y_{k-1}) \in L_i$ and $\Gamma(y_k)Ł_i$. Therefore, $\Gamma(y_k)$ is inserted in the i^{th} cell of the first row to obtain $T_{\Psi(k)-1}$.

On the other hand, a shadow line L_i leaves the valid domain through a point (k, y), $k < n$, if and only if $\Psi(k+1) < \Gamma(y) < \Psi(k)$. So the only operation performed is the suppression of the cell having $\Gamma(y)$ during the step $s(\Gamma(y))$, by the Algorithm 1, to obtain $T_{\Psi(k)-1}$. $\diamondsuit$

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