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A Survey of Asymptotic Associated, Attached, and Coassocited Primes Relative to Some Modules

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Abstract

Recently there has been a large body of interest research on asymptotic behaviour of ideals relative to some modules. The purpose of this paper is to provide a survey of these new results.

1. Introduction

Throughout this paper R (resp. A) will denote a commutative (resp. a commutative Noetherian) ring. Both of them are assumed to have a non-zero identity.

Let M be an R-module. A prime ideal P of R is said to be an associated (resp. a weakly associated) prime of M if there exists an element $x \in M$ such that $P = (0 :_R Rx)$ (resp. P is minimal over $(0 :_R Rx)$). (See [5] and [16].) The set of assosiated (resp. weakly associated) primes of M is denoted by $Ass_R(M)$ (resp. by $W.Ass_R(M)$).

We shall follow Macdonald's terminology (see [15]) concerning secondary representation. So whenever an R-module M has a secondary representation, then the set of attached primes of M, which is uniquely determined, is denoted by $Att_R(M)$.

The concept of coassociated prime ideals was introduced by L. Chambles, H. Zoschinger, and S. Yassemi in different ways. However, these concepts are equivalent (see [36, (1.6) and (1.7)]). The concept of coassociated (resp. weakly coassociated) prime ideals is introduced in terms of cocyclic modules: An R module L is cocyclic (see [36] and [37]) if $L \subseteq E(R/P)$ for some maximal ideal P of R (for an R-module X, we will use $E(X)$ to denote the injective envelope of X). Also a prime ideal P of R is said to be a coassociated (resp. weakly coassociated)prime of M if there exists a cocyclic homomorphic image L of M such that $P = (0 :_R L)$ (resp. P is a minimal among the prime ideal containing $(0 :_R L)$). The set of coassociated (resp. weakly coassociated)primes of M is denoted by $Coass_R(M)$ (resp. $W.Coass_R(M)$).

L. J. Ratliff in [25] provided a nice survey of asymptotic prime divisors and gave valuable information about high powers of ideals, their integral closures, and their associated primes. Recently S. Yassemi in [39] provided a survey that gives valuable information about the new results concerning associated and coassociated primes. Our main purpose in this paper is to give some view about the new results concerning the asymptotic stability of of some sequences in which the terms are the sets of associated, attached, or coassociated prime ideals of certain modules.

2. Notation and terminology

Notation 2.1. Throughout this paper I will denote an ideal of R or of A. Also we use N to denote the set of positive integers and we write "finite" for "finitely generated".

Definition 2.2 (see [23] and [28, (1.1)]). Let J be another ideal of A. Then I is a reduction of J if $I \subseteq J$ and there exists an integer $s \in N$ such that $IJ^s = J^{s+1}$. An element x of A is said to be integrally dependent on I if there exists $s \in N$ and elements $c_1, ..., c_s \in A$ with $c_i \in I^i$ for $i = 1, ..., s$ such that

$$x^s + c_1 x^{s-1} + c_{s-1} x + c_s = 0.$$

The set of all elements of A which are integrally on I is an ideal of A and denoted by $(I)^-$. In fact it is the largest ideal of A which has I as a reduction.

Definition 2.3. (See [35, (1.2)].) Let L be a Noetherian R-module. We say that I is a reduction of the ideal J of R relative L if $I \subseteq J$ and there exists $s \in N$ such that $IJ^sL = J^{s+1}L$. An element x of R is said to be integrally dependent on I relative to L if there exists $s \in N$ such that

$$x^s L \subseteq \sum_{i=1}^{s} x^{s-i} I^i L.$$

The set of elements of A which are integrally dependent on I relative to L is an ideal of R, called the integral closure of I relative to L, denoted by $I^{-(L)}$, and is the largest ideal of R which has I as a reduction

Definition 2.4 (see [4] and [33]). Let J be another ideal of A and let M be an Artinian or an injective A-module. Then I is said to be a reduction of J relative to M if $I \subseteq J$ and there exists a positive integer s such that

$$(0 :_M IJ^s) = (0 :_M J^{s+1}).$$

An element x of A is said to be integrally dependent on I relative to M if there exists a positive integer s such that

$$(0 :_M \sum_{i=1}^{s} x^{s-i} I^i) \subseteq (0 :_M x^s).$$

Furthermore, the set of elements of A which are integrally dependent on I relative to M is an ideal of A, called the integral closure of I relative to M, denoted by $I^{\star(M)}$, and it is the largest ideal of A which has I as a reduction relative to M. Note that in the case of Artinian modules our ring is assumed to be commutative (not necessarily Noetherian).

Definition 2.5 (see [7]). Let J be another ideal of A and let $R(I) = \oplus_{n\geq 0} I^n$ be the Rees ring with respect to I. Let M be an A-module and

$$M(I) = \oplus_{n\geq 0} I^n M$$

be the graded $R(I)$-module with respect to M. Further set

$$M_J(I) = \oplus_{n\geq 0} I^n M/JI^n M = M(I)/JM(I).$$

Then the analytic spread of I at J with respect to M is defined to be as $dim_{R(I)}(M_J(I))$ and denoted by $l_J(I, M)$. In the case that (A, J) is a local ring then $l_J(I, A)$ is the analytic spread $l(I)$ of I, introduced by Northcott and Rees in [23]. Further if M is a finite A-module, then the J-grade of M is defined to be the maximal length of M- sequences in J and denoted by $gr_J(M)$.

3. Asymptotic associated primes

M. Brodman in [6] showed that if M is a Noetherian R-module, Then the sequences of sets

$$Ass_R(M/I^n M),\ resp.\ Ass_R(I^n M/I^{n+1} M), n \in N$$

are ultimately constant. We will show the ultimate constant values of the above sequence respectively by $As^\star(I, M)$ and $Bs^\star(I, M)$. Also he showed that the sequence of sets

$$Ass_R(M/I^n M),\ n \in N$$

is not an increasing sequence which had been questioned by Ratliff. As a historical view, the works of Rees and Ratliff in [24] and [27] have been the main resources of inspiration for the above work.

Moreovere, there are other proofs (with different methods) given by S. McAdam and P. Schenzel when $M = A$. (see [17], and [30]). Furthermore, S. McAdam (see[17]) showed that

$$As^{\star}(I, A) \backslash Bs^{\star}(I, A) \subseteq Ass(A).$$

Later R. Y. Sharp (see [32]) applied the same method and showed that

$$As^{\star}(I, M) \backslash Bs^{\star}(I, M) \subseteq Ass_A(M).$$

D. Rush in [29] generalized the Brodmann's result as follows: Let M' be a submodule of a Noetherian R-module M. Then

$$Ass_R(M/I^n M'), \text{ and } Ass_R(I^n M/I^n M'), \ n \in N$$

are ultimately constant. The Rush's result was extended in turn by Divaani-Aazar and Tousi (see [8]) by putting in place of M, M' respectively $Hom_R(P, M)$ and $Hom_R(P, M')$, where P is a projective R-module. More recentely A. K. Kinsbury and R. Y. Sharp [12] have generalized this by showing that if

$$(a_n(1), ..., a_n(g)), \ n \in N \ ,$$

is a sequence of g-tuples of non-negative integers which is non decreasing in the sense that $a_i(j) \leq a_{i+1}(j)$ for all $j = 1, 2, ..., g$ and all $i \in N$, then

$$Ass_R(M/{I_1}^{a_n(1)} ... {I_g}^{a_n(g)} M'), \ n \in N$$

is ultimately constant. K. Divani-Aazar and M. Tousi [9] have further extended this by showing that if T is a covariant linear exact functor from the category of Noetherian R-modules to the category of R-modules and $I_1, ... I_g$

are ideals of R, then

$$Ass_R(T(M)/I_1{}^{a_n(1)}...I_g{}^{a_n(g)}T(M'),\ n \in N$$

and

$$Ass_R(I^nT(M)/I^nT(M')),\ n \in N$$

are ultimately constant.

There is another extension of Brodmann's result by S. Yassemi (see [36, 1.20]) as follows: Let M and F be respectively a finite and a flat A-module. Then

$$Ass_A(M/I^nM \otimes F), n \in N$$

and

$$Ass_A(I^nM/I^{n+1}M \otimes F), n \in N$$

are ultimately constant. Also the ultimate constant values of the above sequences are described in terms of $As^\star(I,A)$, $Bs^\star(I,A)$, and $Coass_A(M \otimes F)$. Further we have

$$As^\star(I, M{\otimes}F)\backslash Bs^\star(I, M{\otimes}F) \subseteq \{P \in Ass(A) : P \subseteq Q \; for\; some\; Q \in Coass_A(F)\}.$$

I. Nishitani in [22] showed that if M is an Artinian R-module and E is an injective R-module, then

$$Ass_R(Hom_R(M,E)/I^nHom_R(M,E))$$

and

$$Ass_R(I^nHom_R(M,E)/I^{n+1}Hom_R(M,E))$$

are ultimately constant.

L. Melkerson and P. Schenzel in [19] proved that if M is a Noetherian R-module then the sequences of sets

$$Ass_R(Tor_i^R(R/I^n, M)), n \in N$$

and

$$Ass_R(Tor_i^R(I^n/I^{n+1}, M)), n \in N$$

are ultimately constant.

The above results can be extended by putting $Hom_R(P, M)$ (here P is a projective R-module) or $M \otimes_A F$, where F is a flat A-module, in place of M. (See [8, Proposition 1.10] and [1, Proposition 3.7] .) Also the results remains true if we change M with the module $T(M)$, where T is a covariant linear exact functor frome the category of Notherian R-modules to the category of R-modules. (See [9, 1.9].)

In [1], Ansari-Toroghy showed that if M is a finite A-module then the sequence of sets

$$Ass_A((M \otimes L)/I^n(M \otimes L)), n \in N$$

and

$$Ass_A(I^n(M \otimes L)/I^{n+1}(M \otimes L)), n \in N$$

are ultimately constant in the following cases:
(i) L is a finite A-module;
(ii)L is an injective A-module;
(iii)L is an Artinian A-module;
(iv)L is a flat A-module.

L. J. Ratlif in [26] showed that the sequence of sets

$$Ass_A(A/(I^n)^-), n \in N$$

is ultimately costant (here $\bar{I}$ denotes the integral closure of the ideal I in A). He first proved the above result for ideals of height ≥ 1 and then showed this is true in the case that I is an arbitrary ideal (see [24], and [26]). Later, this was proved by different method by S. McAdam and P. Schenzel (see [17], and [30]). We denote the ultimate constat value of the above sequence by $\bar{As}^\star(I, A)$.

Suppose that I contains a non -zerodivisor on A then the sequence of sets

$$Ass_A((I^n)^-/I^n),\ n \in N$$

is ultimately constant (see [17, 11.16]). Furthermore, if we denote the ultimate value of the above sequence by $Cs^\star(I, A)$, then [17, 11.19] shows that

$$As^\star(I, A) = As^\star(I, A) \cup \bar{As}^\star(I, A).$$

In [34] Sharp and Tiras showed that if A is a complete semi-local Noetherian ring and M is a non-zero Artinian A-module, then the sequence of sets

$$Ass(A/(I^n)^{\star(M)}),\ n \in N$$

is ultimately constant.

Now let

$$(a_n(1), ..., a_n(g)),\ n \in N\ ,$$

be a sequence of g-tuples of non-negative integers which is non decreasing in the sense that $a_i(j) \leq a_{i+1}(j)$ for $j = 1, 2, ..., g$ and all $i \in N$. In [11] D. Katz, S. McAdam and Ratliff showed that if $I_1, ..., I_g$ are regular ideals of A, then the sequence of sets

$$Ass_A(A/\overline{I_1{}^{a_n(1)}...I_g{}^{a_n(g)}}),\ n \in N$$

is ultimately constant.

Let M be an R-module and L be a sumodule of M. In [20] G. Naude and C. Naude introduced the concept of closure of I relative to L in M by means of integer valued filtration. The second author in [21] employed the above concept and showed (among the other results) that if M is a Noetherian R-module and L is a submodule of M, the the sequences of sets

$$Ass_R(M/(I^n)^{-(L)}L),\ n \in N,$$

and

$$Ass_R((I^n)^{-(L)}L/(I^{n+1})^{-(L)}L),\ n \in N$$

are ultimately constant (see [21, 2.10 and 4.1]). It implies that when $L = M$, the sequences of sets

$$Ass_R(M/(I^n)^{-(M)}M),\ n \in N,$$

and

$$Ass_R((I^n)^{-(M)}M/(I^{n+1})^{-(M)}M),\ n \in N$$

are ultimately constant.

4. Asymptotic coassociated and attached primes

In [15] Macdonald introduced the theory of secondary representation and proved that an Artinian R-module has a secondary representation. This theory associates with an Artinian R-module M a finite set called the set of attached prime ideals of M and denoted by $Att_R(M)$.

In [31] R. Y. Sharp showed that if M is an Artinian R-module then the sequence of sets

$$Att_R((0:_M I^n)),\ n \in N,$$

and

$$Att_R((0:_M I^n)/(0:_M I^{n-1})),\ n \in N,$$

are ultimately constant. Further he showed that (see [32])

$$At^{\star}(I,M)\backslash Bt^{\star}(I,M) \subseteq Att_R(M)$$

(Here $At^{\star}(I,E)$ and $Bt^{\star}(I,E)$ denote the ultimately constant value of the above sequnces.) The above result can be regarded as a dual of Brodman's results concerning associated primes as mentioned in section one.

Later in [29] D. E. Rush expended upon this by showing that if $M'' \subseteq M'$

are submodules of an R-module M, then

$$Att_R(M' :_M I^n),\ n \in N$$

and

$$Att_R(M' :_M I^n / M'' :_M I^n),\ n \in N$$

are ultimately constant.

In [2] H. Ansari Toroghy and R. Y. Sharp showed that if E is an injective A-module then sequence of sets

$$Att_A((0 :_E I^n)),\ and\ Att_A((0 :_E I^n)/(0 :_E I^{n-1})),\ n \in N$$

are ultimately constant. Let denote the ultimate constant value of the above sequences respectively by $At^\star(I, E)$ and $Bt^\star(I, E)$. They described these ultimate constant values in terms of $At^\star(I, A)$, $Bt^\star(I, A)$, and $Ass_A(E)$. Further they showed that

$$At^\star(I, E) \backslash Bt^\star(I, E) \subseteq Att_A(E).$$

Later L. Melkerson and P. Schenzel (see [18]) improved slightly the above results by dropping the Noetherian property of A and showed that if M is a Noetherian R-module, then the above results remains true if we put R and $Hom_R(M, E)$ respectively in place of A and E. There is another generalization concerning the change of rings (this fact was pointed out to me by S. Yassemi) as follows: let $\phi : R \rightarrow A$ be a ring homomorphisim and let M be a finite A-module. Further let E be an R-module such that $W.Ass_R(E) = Ass_R(E)$. Let I be an ideal of A and set $L = Hom_R(M, E)$. Then the sequences of sets

$$W.Coass_A(Hom_A(A/I^n, L)),\ n \in N,$$

and

$$W.Coass_A(Hom_A(A/I^{n+1}, L)/Hom_A(A/I^n, L)),\ n \in N$$

are ultimately constant. Let denote their ultimate constant values respectively by $At_A(I,L)$ and $Bt_A(I,L)$. Then we have

$$At_A(I,L)\backslash Bt_A(I,L) \subseteq W.Coass_A(L) \cap V(I).$$

Here we give a sketch of proof as follows:

$$Hom_A(A/I^n, L) \cong Hom_R(A/I^n \otimes_A M, E) \cong Hom_R(M/I^nM, E).$$

Also we have

$$Hom_A(A/I^{n+1}, L)/Hom_A(A/I^n, L) \cong Hom_R(I^nM/I^{n+1}M, E).$$

Further if X is a finite A-module, then by [38, 2.11],

$$W.Coass_A(Hom_R(X,E)) = \{P \in Ass_A(X) : P^c \subseteq Q \; for\; some\, Q \in Ass_R(E)\}.$$

(Here P^c is the contraction of P with respect to ring homomorphism ϕ : $R \to A$.) Now the claim follows this and the fact that $Ass_A(M/I^nM)$, and $Ass_A(I^nM/I^{n+1}M)$ stablizes for large n.

Let L be a finite A-module. Then (see [1, 3.7]) the sequences of sets

$$Coass_A(0 :_{Hom_A(L,M)} I^n), \; n \in N,$$

and

$$Coass_A((0 :_{Hom_A(L,M)} I^n)/(0 :_{Hom_A(L,M)} I^{n-1})), \; n \in N,$$

are ultimately constant in the following cases:
(i) M is a finite A-module;
(ii) M is an injective A-module;
(iii) M is an Artinian A- module;
(iv) M is a flat A-module.

It is shown that (see [1, 3.7]) if L, M,F, and E are respectively a finite, an Artinian, a flat, and an injective A-module, then for $i \geq 0$, the sequences of sets

$$Att_A(Ext^i_A(L/I^nL, M)),\ n \in N \tag{1}$$

and

$$Att_A(Ext^i_A(I^nL/I^{n+1}L, M)),\ n \in N \tag{2}$$

$$Coass_A(Ext^i_A(A/I^n, Hom_A(L, E))),\ n \in N$$

and

$$Coass_A(Ext^i_A(I^n/I^{n+1}, Hom_A(L, E))),\ n \in N,$$

are ultimately constant.

Also H. Ansari-Toroghy showed that (see [1, 4.2]) if M, F are respectively an Artinian and a flat A-modules and M', M'' are submodules of M such that $M'' \subseteq M'$, and if we set

$$L_n := (Hom_A(F, M') :_{Hom_A(F,M)} I^n),$$

and

$$T_n := (Hom_A(F, M') :_{Hom_A(F,M)} I^n)/(Hom_A(F, M'') :_{Hom_A(F,M)} I^n),$$

Then, for each $n \in N$, L_n and T_n have secondary representations and the sequences of sets $Att_A(L_n)$ and $Att_A(T_n)$, $n \in N$ are ultimately constant. Nearly the same time, Divaani-Azar and M. Tousi proved the above results for a projective and a flat R- module with a completely different arguments (see [8] and [10]).

L. Melkerson and P. Schenzel in [19] showed that if M is an Artinian over a Noetherian R-module then for a given $i \geq 0$ the sequences of sets

$$Att_R(Ext^i_R(R/I^n, M))),\ n \in N$$

and

$$Att_R(Ext^i_R(I^n/I^{n+1}, M)),\ n \in N$$

are ultimately constant.

In [1] Ansari Toroghy have further expended upon this by showing that if L, M, and F are respectively a finite, an Artinian, and a flat A-module, then for a given $i \geq 0$,

$$Att_A(Ext^i_A(L/I^nL, Hom_A(F, M))),\ n \in N \tag{i}$$

and

$$Att_A(Ext^i_A(I^n/I^{n+1}, Hom_A(F, M))),\ n \in N$$

$$Att_A(Ext^i_A(F/I^nF, M)),\ n \in N \tag{ii}$$

and

$$Att_A(Ext^i_A(I^nF/I^{n+1}F, M)),\ n \in N$$

are ultimately constant. Both cases (i) and (ii) extend the Melkerson and Schenzel's results.

In [9] K. Divani-Aazar and M. Tousi have shown that if T is a contravariant linear exact functor from the category R-modules to the category of R-modules and $M' \subseteq M$ are Noetherian R-modules and if

$$(a_n(1), ..., a_n(g)),\ n \in N\ ,$$

is a sequence of g-tuples of non-negative integers which is non decreasing in the sense that $a_i(j) \leq a_{i+1}(j)$ for all $j = 1, 2, ..., g$ and all $i \in N$, then for given ideals $I_1, ...I_g$ of R and $i \geq 0$, the sequences of sets

$$Att_R(T(M/M') :_{T(M)} {I_1}^{a_n(1)}...{I_g}^{a_n(g)}),\ n \in N \tag{i}$$

and

$$Att_R((T(M/M') :_{T(M)} I^n)/(0 :_{T(M)} I^n)),\ n \in N$$

$$Att_R(Ext^i_R(R/I^n, T(M))),\ n \in N \tag{ii}$$

and

$$Att_R(Ext^i_R(I^n/I^{n+1}, T(M))),\ n \in N$$

are ultimately constant. The last assersion is in fact another extension for Melkerson and Schenzel's result as we mentioned above.

In [33] R. Y. Sharp and A. J. Taherizadeh have asked the following questions which can be regrded as a dual of Ratliff's result as we mentioned in section one. Let M be an Artinian R-module.

(i) Is it the case that

$$Att_R(0 :_R (I^n)^{\star(M)}) \subseteq Att_R(0 :_R (I^{n+1})^{\star(M)})\ forall\ n \in N?$$

(ii) Is the sequence of sets

$$Att_R(0 :_R (I^n)^{\star(M)}),\ n \in N$$

ultimately constant ? These questions are still open.

Recentely A. K. Kingsbury and R. Y. Sharp in [13] showed that in the special case when A is a complete semi-local commutative Noetherian ring and M is an Artinian A- module, the part (ii) of the above question is true. In fact they showed that for ideals $I_1, ..., I_g$ of A, and a non-decreasing sequence

$$(a_n(1), ..., a_n(g)),\ n \in N\ ,$$

the sequence of sets

$$Att_A(0 :_M ({I_1}^{a_n(1)}...{I_g}^{a_n(g)})^{\star(M)}),\ n \in N$$

is ultimately constant.

Now let E be an injective A-module. In [3] H. Ansari Toroghy and R. Y. Sharp have shown that the set of sequence of sets

$$Att_R(0 :_R (I^n)^{\star(E)}),\ n \in N$$

are ultimately constant. Furthermore they showed that (see [3]) if I contains a non-zerodivisor on R, then the sequence of sets

$$Att_A((0 :_R I^n)/(0 :_A (I^n)^{\star(E)}),\ n \in N$$

is ultimately constant. Let denote the ultimate constant values of the sequences of

$$Att_A(0:_A I^n),\ Att_A(0:_A (I^n)^{\star(E)}),\ and\ Att_A((0:_A I^n)/(0:_A (I^n)^{\star(E)}),$$

respectively by $At^{\star}(I,E), \bar{A}t^{\star}(I,E)$, and $Ct^{\star}(I,E)$. They described the ultimate constant values of the above seuences in terms of by $At^{\star}(I,A)$, $\bar{A}t^{\star}(I,A)$,and $Ass_A(E)$. Further they showed that

$$At^{\star}(I,E) = Ct^{\star}(I,E) \cup \bar{A}t^{\star}(I,E).$$

Let M be an Artinian R-module and let M' be a submodule of M. In [21] Cornelia Naude introduced the concept of the closure of I relative to M' by means of an integer valued filtration on the ring R and employed the above concept and among the other results showed that the sequences of sets

$$Att_R(M' :_M (I^n)^{\star(M')}),\ n \in N$$

is ultimately constant. (See [21, 3.7 and 4.1].)

5. Some further results

Now let A be a local ring. M. Brodmann showed that (see [17. 7.4]) if A is a local ring, then the sequence

$$grade(A/(I^n)),\ n \in N$$

is ultimately constant. Katz showed that if A is an analytically unramified (the ring A is called analytically unramified, if the completion $\hat{A}$ of A does not have non-zero nilpotent elements) or a complete local ring, then above result is true when I^n is replaced with $(I^n)^-$ (see [17]).

M. Brodmann in [7] has shown that if J is another ideal of A and M is an A-module of finite type, then

$$gr_J(M/I^nM),\ n \in N \tag{i}$$

is ultimately constant. It's constant value is denoted by $g_J(I, M)$.

(ii) If $I \subseteq J$, we have

$$l_J(I, M) \leq dim_R(M) - gr_J(I, M).$$

In [14] V. Kodiyalam has shown that if A is a local ring and M is a finite A-module and I is a proper ideal of A, then the sequences of sets

$$pd_A(M/I^nM),\ and\ id_A(M/I^nM),\ n \in N$$

are ultimately constant (possibly infinite). (Here for an R-module L, $pd_R(L)$ and $id_R(L)$ denote respectively projective and injective dimension of L.)

We end this section by the following questions.

Question 1. Let P be a maximal ideal of A and let M be an Artinian A-module such that $M = \cup_{n=1}^{\infty}(0 :_M P^n)$. Is

$$I^{\star(M)}\widehat{A_P} = (I\widehat{A_P})^{\star(M)}?$$

Question 2. Let in definion of 2.4, M be a projective or a flat A-module. Is

$$I^{\star(M)} = \{x \in A : x\ is\ integrally\ dependent\ on\ I\ relative\ M\}?$$

Qestion 3. Let M be a finite A-module and let $ht(I) \geq 0$. Is

$$((I^{n+1})^{-(M)}M :_M I) = (I^n)^{-(M)}M?$$

for all $n \geq 0$

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