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On the Property of Harmonic Vector Field on the Sphere S^{2n+1}

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Abstract

In this paper we study the property of harmonic vector fields. We call a vector fields ξ harmonic if it is a harmonic map from the manifold into its tangent bundle with the Sasaki metric. We show that the characteristic polynomial of operator $A = \nabla\xi$ in S^{2n+1} is $(x^2+1)^n$.

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1. Introduction

What are the optimal unit vector fields on a round unit sphere of odd-dimension? Since a vector field on a Riemannian manifold M is a map from M to its tangent bundle TM as a graph, we can think about the best vector fields on M in two ways; the volume[2] and energy[9], [5]. In the case of volume the optimal unit vector field ξ means that ξ has a minimum volumes as a submanifold of the unit tangent bundles. On a flat torus, the optimal unit vector fields are parallel ones. On the round $(2n+1)$-sphere S^{2n+1}, no parallel vector field exist, but we can seek the best organized vector fields. The first result on this problem are due to Gluck and Ziller, who showed [2] that the unit vector fields of minimum volume on S^3 are precisely the Hopf vector fields, and no others. But in $S^{2n+1}(n \geq 2)$ the Hopf vector fields are only unstable ciritical points of volume, and thus no longer optimal by Johnson [8].

By the another way, the Hopf vector fields on S^{2n+1} are harmonic maps from the sphere into the unit tangent bundle US^{2n+1}, i.e. the critical points of the energy functional [5]. However these are not energy minimizer since harmonic maps from spheres to compact manifolds are unstable.

In [6] we define harmonic gauss map as Gauss map of m-dimensional distribution on a Riemannian manifold M which is a harmonic map from the manifold into its Grassmann bundle $G_m(TM)$ of m-dimensional tangent subspace. We show that the Hopf fibrations on S^{4n+3} are the harmonic gauss map of 3-dimensional distribution.

In [13] he propose the following conjecture: are there any harmonic sections of US^{2n+1} apart from Hopf vector field. We do not know whether this is still true for higher dimensional sphere. Hence in this paper we study the property of harmonic vector field on S^{2n+1}. This property is very similar to the property of Hopf vector fields.

2. Harmonic vector fields

Let M be a compact Riemannian manifold. For a point $(p, v) \in TM$ and let $V, W \in T_{(p,v)}TM$ be two tangent vectors in the tangent bundle TM at (p, v). Consider two curves in TM

$$\alpha : t \to (p(t), v(t)), \qquad \beta : s \to (q(s), w(s)),$$

such that $p(0) = q(0) = p, \quad v(0) = w(0) = v$, and $V = \alpha'(0), W = \beta'(0)$. We define an inner product on $T_{(p,v)}TM$ by

$$\langle V, W\rangle_{(p,v)} = \langle \pi_*(V), \pi_*(W)\rangle_p + \langle \frac{Dv}{dt}(0), \frac{Dw}{ds}(0)\rangle_p,$$

where π_* is the differential of the projection map $\pi : TM \to M$, and $\frac{Dv}{dt}$ is the covariant derivative of the vector field $v(t)$ along the curve $p(t)$. The metric on the tangent bundle TM defined this way is called the Sasaki metric ([11], [12]), and is a natural metric on TM induced from the Riemannian metric on M. A vector at $(p, v) \in TM$ that is perpendicular to the fiber $\pi^{-1}(p)$ is called a horizontal vector and a vector that is tangent to the fiber is called a vertical vector. For example the curve $\gamma(t) = (p(t), v(t))$ in TM is horizontal if the vector field $v(t)$ is parallel along the curve $p(t)$ in M.

A harmonic vector field ϕ_X as a section of tangent bundle with the Sasaki metric is a critical point of the energy functional

$$E(\phi_X) = \int_M e(\phi_X)dv,$$

where $e(\phi_X)$ is the energy density of ϕ_X, and dv is the volume form of M[3], [4]. For any orthonormal basis $\{e_1, \cdots, e_n\}$ of T_pM, the energy density associated with the vector field ϕ_X in M is

$$\begin{aligned} e(\phi_X)_p &= \frac{1}{2}||d\phi_X||^2_{TM} \\ &= \frac{1}{2}\sum_{i=1}^{n}\langle e_i + \nabla_{e_i}\phi_X, e_i + \nabla_{e_i}\phi_X\rangle_{TM} \\ &= \frac{1}{2}(n + \sum_{i=1}^{n}\langle \nabla_{e_i}\phi, \nabla_{e_i}\phi\rangle_M). \end{aligned}$$

Now we calculate the tension field of vector field ϕ_X. The indices $i, j, k, \ldots$ run over the range $\{1, \ldots, n\}$ and the indices $A, B, C, \ldots$ the range$\{1, \ldots, n, \ldots, 2n\}$. and also $i^* = n + i$.

We can put locally $X = \sum_{i=1}^{n} x^i e_i$. Define F_i^A by $\phi_X^*(\omega^A) = \sum_{i=1}^{A} \theta^i$ where $\{\theta^i\}$ is coframe of $\{e_i\}$ and ω_A is coframe of TM.

Then it holds

$$\phi_X^*(\omega^i) = \phi_X^*\pi^*(\theta^i) = \theta^i.$$

And

$$\phi_X^*(\omega^{i*}) = \sum_{k=1}^{n} X_k^i\theta_k$$

where X_k^i are components of the first covariant differential of X given by

$$\sum_{k=1}^{n} X_k^i\theta^k = dX^i + \sum_{j=1}^{n} X^j\theta_j^i.$$

Thus

$$F_j^i = \delta_j^i, F_j^{i*} = X_j^i.$$

The fundamental tensor F_{ij}^A of the mapping ϕ_X is defined to be

$$\sum_{j=1}^{n} F_{ij}^A\theta^j = dF_i^A + \sum_{B=1}^{2} nF_i^B\omega_B^A - \sum_{j=1}^{n} F_j^A\theta_i^j.$$

If $\sum_{i=1}^{n} F_{ii}^A = 0$, ϕ_X is called a harmonic map.

Proposition 2.1[7] *The component F_{ij}^A of the fundamental tensor of the map $\phi_X : M \to TM$ are given by*

$$\begin{aligned} F_{ij}^k &= \frac{1}{2}\sum_{l,h}(R_{ilh}^k X_j^h + R_{jlh}^k X_i^h)X^l \\ F_{ij}^{k*} &= X_{ij}^k + \frac{1}{2}\sum_{l=1}^{n}(R_{lij}^k X^i) \end{aligned}$$

where X_{ij}^k are the components of the second covariant differential of the vector field X.

Proposition 2.2[7] $\phi_X : M \to T(M)$ *is a harmonic map if and only if*

$$\tau(\phi_X)^H = \sum_{j,i=1}^{n} R_{ilj}^k X_i^j X^l = \text{trace}R(\nabla_* X,\ X)* = 0,$$

$$\tau(\phi_X)^V = \sum_{i=1}^{n} X_{ii}^k = \mathrm{trace}\nabla^2 X = 0.$$

In this case a vector field is harmonic, i.e., $\tau(\phi_X) = 0$ if and only if ϕ_X is parallel [9]. Therefore, it is quite difficult to find a harmonic vector field in TM. However, the situation is different in the case of harmonic vector fields into unit tangent bundle. A unit tangent bundle has the induced metric as a Riemannian submanifold of the tangent bundle with the Sasaki metric. Let X be a unit tangent vector field, which is harmonic as a mapping from the manifold to the unit tangent bundle UM. Then the energy functional is same as the tangent bundle case because we use the induced metric. However the variation of X is restricted to UM, and a unit vector field is harmonic if and only if the tension field $\tau(\phi_X)$ in TM is normal to UM. In other words,

$$\mathrm{trace}R(\nabla_* X,\ X)* = 0, \quad \mathrm{trace}\nabla^2 X = cX,$$

for some constant $c \in \mathbb{R}$[5].

When $M = S^{2n+1}$, $\tau(\phi_X)$ implies that

$$\nabla_X X = 0, \qquad \textstyle\sum_{i=1}^{2n} \langle \nabla_{e_i} X, e_i \rangle = 0,$$
$$\mathrm{trace}\nabla^2 X = cX.$$

In [5], we proved that the Hopf vector fields on S^{2n+1} are harmonic and harmonic vector field on S^3 is Hopf. But we do not know whether this is true for higher dimensional spheres. Hence in the next section we will study the property of harmonic vector fields.

3. Main Theorem

Consider the harmonic unit vector field ξ on the round sphere S^{2n+1}. Then the vector field ξ satisfies that

$$\mathrm{trace}R(\nabla_*\xi, \xi)* = 0, \quad \mathrm{trace}\nabla^2\xi = c\xi. \tag{1}$$

And the integral curve γ of ξ is geodesic in S^{2n+1}.

Let $\{e_0 = \xi, e_1, \cdots, e_{2n}\}$ be an orthonormal basis of T_pS^{2n+1} at p and parallel extend along γ. Consider the Jacobi tensor J along γ at p which has the initial condition

$$J(0) = Id, \quad J'(0)(e_i) = \nabla_{e_i}\xi$$

Let $A = \nabla\xi$ be the derivative operator of harmonic vector fields on round sphere.

This Jacobi tensor satisfies

$$J'' + J = 0,$$

and therefore

$$J''' = -J'.$$

Lemma *Let ξ be a harmonic vector fields on S^{2n+1} and $A = \nabla\xi$. Then*

$$\mathrm{trace}A^m = \begin{cases} (-1)^{[m/2]}2n, & \text{if } m = \text{even}, \\ 0, & \text{otherwise}. \end{cases}$$

Proof By the harmonic equation $\mathrm{div}(\xi) = 0$, ξ preserves volume. Hence for any t, $\det(J(t)) = 1$.

$$0 = \det(J(t))' = \mathrm{trace}(J'(t)J^{-1}(t))\det(J(t))$$

Hence $\mathrm{trace}(J'(t)J^{-1}(t)) = 0$. At $t = 0$ $\mathrm{trace}(A \cdot I) = \mathrm{trace}A = 0$. Also by $J''(0) = -J(0)$, we obtain $A^2 + I = 0$.

Let $U(t) = J'(t)J^{-1}(t)$, then $\mathrm{trace}U(t) = 0$. Since $J' = UJ, J'' = -J = U'J + UJ'$,

$$U' = -UJ'J^{-1} - I.$$

Hence $U(t)$ satisfies the following Riccati equation.

$$U' + U^2 + I = 0$$

Also

$$\text{trace}U'(t) = \text{trace}(-U^2 - I) = 0$$

At $t = 0$

$$0 = \text{trace}U'(0) = -\text{trace}(A^2) - 2n.$$

Therefore $\text{trace}(A^2) = -2n$.

In the case of $m = 3$,

$$\text{trace}U''(t) = \text{trace}(-UU' - U'U) = \text{trace}(-2U^3 - 2U) = 0$$

At $t = 0$

$$0 = \text{trace}U''(0) = -2\text{trace}(A^3) - \text{trace}(A).$$

Therefore $\text{trace}(A^3) = 0$. In the case of $m = 4$,

$$\text{trace}U'''(t) = \text{trace}(-2U^3 - 2U) = \text{trace}(6U^4 + 8U^2 + 2Id) = 0$$

At $t = 0$

$$0 = \text{trace}U'''(0) = 6\text{trace}(A^4) + 8\text{trace}(A^2) + 4n.$$

Therefore $\text{trace}(A^4) = 2n$.

Hence by induction generally

$$\text{trace}A^{4k} = 2n, \qquad \text{trace}A^{4k+1} = 0,$$
$$\text{trace}A^{4k+2} = -2n, \qquad \text{trace}A^{4k} = 0$$

□

Now we can find the characteristic polynomial of $A = \nabla\xi$.

Theorem *The characteristic polynomial of $A = \nabla\xi$ is*

$$(x^2 + 1)^n$$

Proof. Let $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$ be eigenvalues of A. Since trace$A = \sum_{i=0}^{2n} \alpha_i$ and $\alpha_1^m, \alpha_2^m, \cdots, \alpha_{2n}^m$ are eigenvalues of A^m, by lemma 1 we can show that

$$\sum {\alpha_i}^{4k+1} = 0, \sum {\alpha_i}^{4k+2} = -2n$$
$$\sum {\alpha_i}^{4k+3} = 0, \sum {\alpha_i}^{4k+4} = 2n$$

Let

$$s_0 = 1, s_l = \sum_{i_1 < i_2 < \cdots < i_l} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_l}$$
$$p_0 = k, p_l = \sum_i (\alpha_i)^l,$$

then by Newton's formula

$$\sum_{i=0}^{k} (-1)^i s_i p_{k-i} = 0$$

But by the condition of A,

$$\begin{aligned} p_{4k} &= \sum (\alpha_i)^{4k} = 2n \\ p_{4k+1} &= \sum (\alpha_i)^{4k+1} = 0 \\ p_{4k+2} &= \sum (\alpha_i)^{4k+2} = -2n \\ p_{4k+3} &= \sum (\alpha_i)^{4k+3} = 0 \end{aligned}$$

Hence

$$\begin{aligned} s_{odd} &= 0, \\ s_{4k+2} &= \frac{2n-4k}{4k+2} s_{4k} = \binom{n}{2k+1} \end{aligned}$$

Therefore the characteristic polynomial is

$$\begin{aligned}\sum_{i=0}^{2n}(-1)^i s_i x^{2n-i} &= \sum_{i=0}^{n}\binom{n}{i}x^{2i} \\ &= (x^2+1)^n\end{aligned}$$

□

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