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Fuzzy Positive Implicative Hyper K-ideals in Hyper K-algebras

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Abstract

The fuzzification of positive implicative hyper K-ideals in hyper K-algebras is considered, Relations between fuzzy positive implicative hyper K-ideal and fuzzy hyper K-ideal are given. Characterizations of fuzzy positive implicative hyper K-ideals are provided. Using a family of positive implicative hyper K-ideals, we make a fuzzy positive implicative hyper K-ideal. Using the notion of a fuzzy positive implicative hyper K-ideal, a weak hyper K-ideal is established.

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1. Itroduction

The hyper structure theory (called also multialgebra) was introduced in 1934 by Marty [7] at the 8th congress of Scandinavian Mathematiciens. Hyper

*an Executive Research Worker of Educational Research Institute in GSNU structures have many applications to several sectors of both pure and applied sciences. Recently, Jun et al. [6] and Borzoei et al. [2] applied the hyper structure to BCK/BCI-algebras. In [3], Borzoei and Zahedi introduced the notion of positive implicative hyper K-ideals of serval types. In this paper we consider the fuzzification of positive implicative hyper K-ideals in hyper K-algebras, and investigate relations between fuzzy positive implicative hyper K-ideal and fuzzy hyper K-ideal. We provide characterizations of fuzzy positive implicative hyper K-ideals. Using a family of positive implicative hyper K-ideals, we make a fuzzy positive implicative hyper K-ideal. Using the notion of a fuzzy positive implicative hyper K-ideal, we establish a weak hyper K-ideal.

2. Preliminaries

We include some elementary aspects of hyper K-algebras that are necessary for this paper, and for more details we refer to [2] and [9]. Let H be a non-empty set endowed with a hyper operation "$\circ$", that is, $\circ$ is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$.

By a *hyper I-algebra* we mean a non-empty set H endowed with a hyper operation "$\circ$" and a constant 0 satisfying the following axioms:

(H1) $(x \circ z) \circ (y \circ z) < x \circ y$,

(H2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(H3) $x < x$,

(H4) $x < y$ and $y < x$ imply $x = y$

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$ and $\exists b \in B$ such that $a < b$. If a hyper I-algebra $(H, \circ, 0)$ satisfies an additional condition:

(H5) $0 < x$ for all $x \in H$,

then $(H, \circ, 0)$ is called a *hyper K-algebra* (see [2]).

In a hyper I-algebra H, the following hold (see [2, Proposition 3.4]):

(p1) $(A \circ B) \circ C = (A \circ C) \circ B$,

(p2) $A \circ B < C \Leftrightarrow A \circ C < B$,

(p3) $A \subseteq B$ implies $A < B$

for all nonempty subsets A, B and C of H.

In a hyper K-algebra H, the following hold (see [2, Proposition 3.6]):

(p4) $x \in x \circ 0$ for all $x \in H$.

An element a of a hyper K-algebra H is said to be *left* (resp. *right*) *scalar* if $|a \circ x| = 1$ (resp. $|x \circ a| = 1$) for all $x \in H$. If $a \in H$ is both left and right scalar, we say that a is a *scalar element* of H.

Definition 2..1 [2] A nonempty subset I of a hyper K-algebra H is called a *weak hyper K-ideal* of H if it satisfies the following conditions:

(I1) $0 \in I$,

(I2) $\forall x, y \in H,\ x \circ y \subseteq I, y \in I \Rightarrow x \in I$.

Definition 2..2 [2] A nonempty subset I of a hyper K-algebra H is called a *hyper K-ideal* of H if it satisfies (I1) and

(I3) $\forall x, y \in H,\ x \circ y < I, y \in I \Rightarrow x \in I$.

Note that every hyper K-ideal is a weak hyper K-ideal, but the converse is not true (see [2, Proposition 4.6 and Example 4.7]).

In his classical paper [8], L. A. Zadeh introduced the notion of a fuzzy set in a set H as a mapping from H into $[0, 1]$. In the sequel, we place a bar over a symbol to denote a fuzzy set so $\bar{A}$, $\bar{B}$, $\bar{G}$, $\cdots$ all represent fuzzy sets in H.

Definition 2..3 [5] A fuzzy set $\bar{A}$ in a hyper K-algebra H is called a *fuzzy hyper K-ideal* of H if it satisfying the following conditions:

(F1) $\forall x, y \in H,\ 0 \in x \circ y \Rightarrow \bar{A}(x) \geq \bar{A}(y)$.

(F2) $\bar{A}(x) \geq \min\{\inf_{a \in x \circ y} \bar{A}(a),\ \bar{A}(y)\},\ \forall x, y \in H$.

Definition 2..4 [5] A fuzzy set $\bar{A}$ in a hyper K-algebra H is called a *fuzzy weak hyper K-ideal* of H if it satisfies

$$\bar{A}(0) \geq \bar{A}(x) \geq \min\{\inf_{a \in x \circ y} \bar{A}(a),\ \bar{A}(y)\},\ \forall x, y \in H.$$

3. Fuzzy positive implicative hyper K-ideals

In what follows, let H denote the hyper K-algebra unless otherwise specified.

Definition 3..1 [3] Let I be a nonempty subset of H that satisfies (I1). Then I is called a *positive implicative hyper K-ideal of*

(i) *type* 1 if $\forall x, y, z \in H,\ (x \circ y) \circ z \subseteq I, y \circ z \subseteq I \Rightarrow x \circ z \subseteq I$,

(ii) *type* 2 if $\forall x, y, z \in H,\ (x \circ y) \circ z < I, y \circ z \subseteq I \Rightarrow x \circ z \subseteq I$,

(iii) *type* 3 if $\forall x, y, z \in H,\ (x \circ y) \circ z < I, y \circ z < I \Rightarrow x \circ z \subseteq I$.

Definition 3..2 A fuzzy set $\bar{A}$ in H is called a *fuzzy positive implicative hyper K-ideal* of H if it satisfies (F1) and

(F3) $\inf_{a \in x \circ z} \bar{A}(a) \geq \min\{\inf_{b \in (x \circ y) \circ z} \bar{A}(b),\ \inf_{c \in y \circ z} \bar{A}(c)\},\ \forall x, y, z \in H$.

Example 3..3 Let $H = \{0, a, b\}$ be a hyper K-algebra with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0, a, b\}$	$\{0, a, b\}$
a	$\{a\}$	$\{0, b\}$	$\{0, a, b\}$
b	$\{b\}$	$\{b\}$	$\{0, a, b\}$

Define a fuzzy set $\bar{A}$ in H by $\bar{A}(0) = \bar{A}(a) = 0.6$ and $\bar{A}(b) = 0.3$. It is easy to verify that $\bar{A}$ is a fuzzy positive implicative hyper K-ideal of H. But the fuzzy set $\bar{B}$ in H given by $\bar{B}(0) = \bar{B}(b) = 0.8$ and $\bar{B}(a) = 0.3$ is not a fuzzy positive implicative hyper K-ideal of H since $0 \in a \circ b$ and $\bar{B}(a) < \bar{B}(b)$.

Theorem 3..4 *Let H be a hyper K-algebra in which 0 is a right scalar element of H. Then every fuzzy positive implicative hyper K-ideal of H is a fuzzy hyper K-ideal of H.*

Proof. Let $\bar{A}$ be a fuzzy positive implicative hyper K-ideal of H and let $x, y \in H$. Taking $z = 0$ in (F3) induces that

$$\begin{aligned}\bar{A}(x) &= \inf_{a \in x \circ 0} \bar{A}(a) \\ &\geq \min\{\inf_{b \in (x \circ y) \circ 0} \bar{A}(b),\ \inf_{c \in y \circ 0} \bar{A}(c)\} \\ &= \min\{\inf_{b \in x \circ y} \bar{A}(b),\ \bar{A}(y)\}.\end{aligned}$$

Hence $\bar{A}$ is a fuzzy hyper K-ideal of H. □

Example 3..5 Let $H := \{0, a, b, c\}$ be a set with the following Cayley table:

$\circ$	0	a	b	u
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{0\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{0, a\}$	$\{c\}$	$\{0, a, c\}$

Then H is hyper K-algebra in which 0 is a right scalar element of H. Define a fuzzy set $\bar{A}$ in H by $\bar{A}(0) = \bar{A}(b) = 0.8$ and $\bar{A}(a) = \bar{A}(c) = 0.4$. It is easy to verify that $\bar{A}$ is a fuzzy hyper K-ideal of H, but since

$$\inf_{a \in c \circ a} \bar{A}(a) = 0.4 \not\geq 0.8 = \min\{\inf_{b \in (c \circ a) \circ a} \bar{A}(b),\ \inf_{c \in a \circ a} \bar{A}(c)\},$$

we know that $\bar{A}$ is not a fuzzy positive implicative hyper K-ideal of H.

Combining Theorem 3..4 and [5, Theorem 3.8], we have the following corollary.

Corollary 3..6 *Let H be a hyper K-algebra in which 0 is a right scalar element of H. Then every fuzzy positive implicative hyper K-ideal of H is a fuzzy weak hyper K-ideal of H.*

The converse of Corollary 3..6 may not be true as seen in the following example.

Example 3..7 Let $H = \{0, a, b\}$ be a set with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Then H is a hyper K-algebra in which 0 is a right scalar element of H. Define a fuzzy set $\bar{A}$ in H by $\bar{A}(0) = \bar{A}(b) = 0.6$ and $\bar{A}(a) = 0.3$. It is easy to verify that $\bar{A}$ is a fuzzy weak hyper K-ideal of H. Note that $0 \in a \circ b$ and $\bar{A}(a) = 0.3 < 0.6 = \bar{A}(b)$. Hence $\bar{A}$ is not a fuzzy positive implicative hyper K-ideal of H.

Theorem 3..8 *If $\bar{A}$ is a fuzzy positive implicative hyper K-ideal of H, then the level set*

$$U(\bar{A}; \alpha) := \{x \in H \mid \bar{A}(x) \geq \alpha\}$$

is a positive implicative hyper K-ideal of type 1 when $U(\bar{A}; \alpha) \neq \emptyset$ for $\alpha \in [0, 1]$.

Proof. Assume that $U(\bar{A}; \alpha) \neq \emptyset$ for $\alpha \in [0, 1]$. Then there exists $x \in U(\bar{A}; \alpha)$, and so $\bar{A}(x) \geq \alpha$. Since $0 \in 0 \circ x$, it follows from (F1) that $\bar{A}(0) \geq \bar{A}(x) \geq \alpha$ so that $0 \in U(\bar{A}; \alpha)$. Let $x, y, z \in H$ be such that $(x \circ y) \circ z \subseteq U(\bar{A}; \alpha)$ and $y \circ z \subseteq U(\bar{A}; \alpha)$. Let $a \in x \circ z$. Then

$$\begin{aligned} \bar{A}(a) &\geq \inf_{b \in x \circ z} \bar{A}(b) \\ &\geq \min\{\inf_{c \in (x \circ y) \circ z} \bar{A}(c), \inf_{d \in y \circ z} \bar{A}(d)\} \\ &\geq \alpha, \end{aligned}$$

and so $a \in U(\bar{A};\alpha)$. This shows that $x \circ z \subseteq U(\bar{A};\alpha)$. Hence $U(\bar{A};\alpha)$ is a positive implicative hyper K-ideal of type 1. □

Lemma 3..9 *[3, Theorem 3.11] Let H be a hyper K-algebra in which 0 is a right scalar element of H. If a nonempty subset I of H is a positive implicative hyper K-ideal of type 2 or 3, then I is a hyper K-ideal of H.*

Theorem 3..10 *Let H be a hyper K-algebra in which 0 is a right scalar element of H. Let $\bar{A}$ be a fuzzy set in H such that $U(\bar{A};\alpha)$, $\alpha \in [0,1]$, is a nonempty positive implicative hyper K-ideal of type 1, 2, or 3. Then $\bar{A}$ is a fuzzy positive implicative hyper K-ideal of H.*

Proof. Assume that $U(\bar{A};\alpha) \neq \emptyset$ is a positive implicative hyper K-ideal of type 1, 2, or 3. Then $U(\bar{A};\alpha)$ is a hyper K-ideal of H by Lemma 3..9. In particular, $U(\bar{A};\bar{A}(y))$ is a hyper K-ideal of H for every $y \in H$. Let $x, y \in H$ be such that $0 \in x \circ y$. Since $0 \in U(\bar{A};\bar{A}(y))$, it follows from (H3) that $x \circ y < U(\bar{A};\bar{A}(y))$. Note that $y \in U(\bar{A};\bar{A}(y))$. Hence $x \in U(\bar{A};\bar{A}(y))$ by (I3), and so $\bar{A}(x) \geq \bar{A}(y)$. For every $x, y, z \in H$, let

$$\beta = \min\{\inf_{b\in(x\circ y)\circ z} \bar{A}(b),\ \inf_{c\in y\circ z} \bar{A}(c)\}.$$

Then $\bar{A}(u) \geq \inf_{b\in(x\circ y)\circ z} \bar{A}(b) \geq \beta$ for all $u \in (x\circ y)\circ z$ and $\bar{A}(v) \geq \inf_{c\in y\circ z} \bar{A}(c) \geq \beta$ for all $v \in y \circ z$. It follows that $u \in U(\bar{A};\beta)$ and $v \in U(\bar{A};\beta)$, i.e.,

$$(x \circ y) \circ z \subseteq U(\bar{A};\beta) \text{ and } y \circ z \subseteq U(\bar{A};\beta). \tag{1}$$

In the case of type 1, we have $x \circ z \subseteq U(\bar{A};\beta)$. In the case of type 2, the inclusion $(x \circ y) \circ z \subseteq U(\bar{A};\beta)$ implies $(x \circ y) \circ z < U(\bar{A};\beta)$ by (p3). Hence $x \circ z \subseteq U(\bar{A};\beta)$. In the case of type 3, the inclusions (1) imply that $(x \circ y) \circ z < U(\bar{A};\beta)$ and $y \circ z < U(\bar{A};\beta)$. Thus $x \circ z \subseteq U(\bar{A};\beta)$. Consequently,

$$\inf_{a\in x\circ z} \bar{A}(a) \geq \beta = \min\{\inf_{b\in(x\circ y)\circ z} \bar{A}(b),\ \inf_{c\in y\circ z} \bar{A}(c)\}.$$

This completes the proof. □

Theorem 3..11 *If $\bar{A}$ is a fuzzy positive implicative hyper K-ideal of H, then the set*

$$I := \{x \in H \mid \bar{A}(x) = \bar{A}(0)\}$$

is a positive implicative hyper K-ideal of type 1.

Proof. Clearly $0 \in I$. Let $x, y, z \in H$ be such that $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$. For every $u \in x \circ z$, we have

$$\begin{aligned} \bar{A}(u) &\geq \inf_{a \in x \circ z} \bar{A}(a) \\ &\geq \min\{\inf_{b \in (x \circ y) \circ z} \bar{A}(b), \inf_{c \in y \circ z} \bar{A}(c)\} \\ &= \bar{A}(0). \end{aligned}$$

Since $\bar{A}(0) \geq \bar{A}(x)$ for all $x \in H$, it follows that $\bar{A}(u) = \bar{A}(0)$, that is, $u \in I$. Hence $x \circ z \subseteq I$, ending the proof. □

Theorem 3..12 *Let H be a hyper K-algebra in which* 0 *is a right scalar element and let $\{I_\alpha \mid \alpha \in \Lambda \subseteq [0,1]\}$ be a collection of positive implicative hyper K-ideals of type* 1 *such that $H = \bigcup_{\alpha \in \Lambda} I_\alpha$ and for all $\alpha, \beta \in \Lambda$, $\beta > \alpha$ if and only if $I_\beta \subset I_\alpha$. Define a fuzzy set $\bar{A}$ in H by*

$$\bar{A}(x) = \sup\{\alpha \in \Lambda \mid x \in I_\alpha\}, \ \forall x \in H.$$

Then $\bar{A}$ is a fuzzy positive implicative hyper K-ideal of H.

Proof. Using Theorems 3..8 and 3..10, it is sufficient to show that the nonempty level set $U(\bar{A}; \delta)$ of $\bar{A}$ is a positive implicative hyper K-ideal of type 1 for every $\delta \in [0, 1]$. We should consider two cases as follows:

$$\text{(i) } \delta = \sup\{\alpha \in \Lambda \mid \alpha < \delta\} \text{ and (ii) } \delta \neq \sup\{\alpha \in \Lambda \mid \alpha < \delta\}.$$

For the first case, we have

$$x \in U(\bar{A}; \delta) \Leftrightarrow x \in I_\alpha \text{ for all } \alpha < \delta \Leftrightarrow x \in \bigcap_{\alpha < \delta} I_\alpha,$$

and so $U(\bar{A};\delta) = \bigcap_{\alpha<\delta} I_\alpha$ which is a positive implicative hyper K-ideal of type 1. The second case implies that there exists $\varepsilon > 0$ such that $(\delta - \varepsilon, \delta) \cap \Lambda = \emptyset$. If $x \in \bigcup_{\alpha\geq\delta} I_\alpha$, then $x \in I_\alpha$ for some $\alpha \geq \delta$. It follows that $\bar{A}(x) \geq \alpha \geq \delta$ so that $x \in U(\bar{A};\delta)$. This proves that $\bigcup_{\alpha\geq\delta} I_\alpha \subset U(\bar{A};\delta)$. Assume that $x \notin \bigcup_{\alpha\geq\delta} I_\alpha$. Then $x \notin I_\alpha$ for all $\alpha \geq \delta$, which implies that $x \notin I_\alpha$ for all $\alpha > \delta - \varepsilon$, i.e., if $x \in I_\alpha$ then $\alpha \leq \delta - \varepsilon$. Thus $\bar{A}(x) \leq \delta - \varepsilon < \delta$ and so $x \notin U(\bar{A};\delta)$. Therefore $U(\bar{A};\delta) \subset \bigcup_{\alpha\geq\delta} I_\alpha$ and consequently $U(\bar{A};\delta) = \bigcup_{\alpha\geq\delta} I_\alpha$ which is a positive implicative hyper K-ideal of type 1. This completes the proof. □

Theorem 3..13 *If $\bar{A}$ is a fuzzy positive implicative hyper K-ideal of H, then for $\alpha \in \mathrm{Im}(\bar{A})$ the set*

$$\bar{A}_w := \{x \in H \mid x \circ w \subseteq U(\bar{A};\alpha)\}$$

is a weak hyper K-ideal of H for every right scalar element w of H.

Proof. Since w is a right scalar element of H, we have $0 \circ w = \{0\} \subseteq U(\bar{A};\alpha)$. Hence $0 \in \bar{A}_w$. Let $x, y \in H$ be such that $x \circ y \subseteq \bar{A}_w$ and $y \in \bar{A}_w$. Then $(x \circ y) \circ w \subseteq U(\bar{A};\alpha)$ and $y \circ w \subseteq U(\bar{A};\alpha)$. Since $U(\bar{A};\alpha)$ is a positive implicative hyper K-ideal of type 1 by Theorem 3..8, it follows that $x \circ w \subseteq U(\bar{A};\alpha)$ so that $x \in \bar{A}_w$. Therefore $\bar{A}_w$ is a weak hyper K-ideal of H. □

Theorem 3..14 *Assume that H satisfies the zero condition, that is, $0 \circ x = \{0\}$ for all $x \in H$. If $\bar{A}$ is a fuzzy positive implicative hyper K-ideal of H, then for $\alpha \in \mathrm{Im}(\bar{A})$ the set*

$$\bar{A}_w := \{x \in H \mid x \circ w \subseteq U(\bar{A};\alpha)\}$$

is a weak hyper K-ideal of H for all $w \in H$.

Proof. The proof is quite similar to the proof of Theorem 3..13. □

We now pose an open problem.

Open Problem 3..15 In Theorem 3..13 and/or Theorem 3..14, is the set $\bar{A}_w$ a hyper K-ideal of H?

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