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On Intuitionistic Fuzzy Alpha-Continuous Mappings

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Abstract

The notion of intuitionistic fuzzy α-continuity is introduced, and its characterization is given.

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1. Introduction

After the introduction of the notion of fuzzy sets by L. A. Zadeh [7], several researches were conducted on the generalizations of the notion of fuzzy sets. As a generalization of fuzzy sets, the idea of intuitionistic fuzzy sets was first introduced by K. Atanassov [1]. Using the notion of intuitionistic fuzzy sets, D. Çoker [4] constructed intuitionistic fuzzy topological spaces. The concept of α-continuity in topological spaces is discussed by A. S. Mashhour et al. [6]. In this paper, we consider the intuitionistic fuzzification of α-continuity. We give a characterization of an intuitionistic fuzzy α-continuous mapping in an intuitionistic fuzzy topological space.

2. Preliminaries

Throughout this paper, I will denote the unit interval $[0, 1]$ of the real line. For any set X, let I^X denote the collection of all mappings from X into I. A member λ of I^X is called a *fuzzy set* of X. *Union* and *intersection* of fuzzy sets are denoted by $\vee$ and $\wedge$, respectively, and are defined by

$$\vee\lambda_i = \sup\{\lambda_i(x) \mid i \in \Lambda \text{ and } x \in X\},$$

$$\wedge\lambda_i = \inf\{\lambda_i(x) \mid i \in \Lambda \text{ and } x \in X\}.$$

For any two members λ and μ of I^X, $\mu \leq \lambda$ if and only if $\mu(x) \leq \lambda(x)$ for each $x \in X$, and in this case μ is said to be contained in λ. A constant fuzzy set taking value $\alpha \in [0, 1]$ will be denoted by $\underline{\alpha}$.

We present the fundamental definitions given by Atanassov.

Definition 2..1 (Atanassov [1]) An *intuitionistic fuzzy set* (IFS for short) A in X is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in X\}$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Definition 2..2 (Atanassov [1]) Let A and B be IFSs of the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in X\}$ and $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle \mid x \in X\}$. Then

(a) $A \subseteq$B if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$,

(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,

(c) $\bar{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle \mid x \in X\}$,

(d) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle \mid x \in X\}$,

(e) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle \mid x \in X\}$.

One can generalize the operations of intersection and union in Definition 2..2 to arbitrary family of IFSs as follows:

Definition 2..3 (Çoker [4]) Let $\{A_i \mid i \in J\}$ be an arbitrary family of IFSs in X. Then

- $\cap A_i = \{\langle x, \wedge\mu_{A_i}(x), \vee\gamma_{A_i}(x)\rangle \mid x \in X\}$,
- $\cup A_i = \{\langle x, \vee\mu_{A_i}(x), \wedge\gamma_{A_i}(x)\rangle \mid x \in X\}$.

For the sake of simplicity, we shall use the notation $A = \langle x, \mu_A, \gamma_A\rangle$ instead of $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in X\}$. The IFSs $0_\sim$ and $1_\sim$ are defined to be $0_\sim = \langle x, \underline{0}, \underline{1}\rangle$ and $1_\sim = \langle x, \underline{1}, \underline{0}\rangle$, respectively. Let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. An *intuitionistic fuzzy point* (IFP for short), written $\frac{p}{(\alpha,\beta)}$, is defined to be an IFS of X given by

$$\frac{p}{(\alpha,\beta)}(x) := \begin{cases} (\alpha,\beta) & \text{if } x = p, \\ (0,1) & \text{otherwise.} \end{cases}$$

3. Intuitionistic Fuzzy Alpha-Continuity

Çoker [4] generalized the concept of fuzzy topological space, first initiated by Chang [3], to the case of intuitionistic fuzzy sets as follows:

Definition 3..1 (Çoker [4, Definition 3.1]) An *intuitionistic fuzzy topology* (IFT for short) on X is a family τ of IFSs in X satisfying the following axioms:

(T1) $0_\sim, 1_\sim \in \tau$,

(T2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,

(T3) $\cup G_i \in \tau$ for any family $\{G_i \mid i \in J\} \subseteq \tau$.

In this case the pair (X, τ) is called an *intuitionistic fuzzy topological space* (IFTS for short) and any IFS in τ is known as an intuitionistic fuzzy open set (IFOS for short) in X. The complement $\bar{A}$ of an IFOS A in IFTS (X, τ) is called an *intuitionistic fuzzy closed set* (IFCS for short) in X.

Definition 3..2 (Çoker [4, Definition 3.13]) Let (X, τ) be an IFTS and $A = \langle x, \mu_A, \gamma_A \rangle$ be an IFS in X. Then the *intuitionistic fuzzy interior* and *intuitionistic fuzzy closure* of A are defined by

$$\text{int}(A) = \cup\{G \mid G \text{ is an IFOS in } X \text{ and } G \subseteq A\},$$

$$\text{cl}(A) = \cap\{K \mid K \text{ is an IFCS in } X \text{ and } A \subseteq K\}.$$

Note that, for any IFS A in (X, τ), we have

$$\text{cl}(\bar{A}) = \overline{\text{int}(A)} \text{ and } \text{int}(\bar{A}) = \overline{\text{cl}(A)}.$$

Definition 3..3 An IFS $A = \langle x, \mu_A, \gamma_A \rangle$ of an IFTS (X, τ) is called an *intuitionistic fuzzy α-open set* if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$. An IFS whose complement is an intuitionistic fuzzy α-open set is called an *intuitionistic fuzzy α-closed set.*

Proposition 3..4 *Let (X, τ) be an* IFTS. *Then arbitrary union of intuitionistic fuzzy α-open sets is an intuitionistic fuzzy α-open set, and arbitrary intersection of intuitionistic fuzzy α-closed sets is an intuitionistic fuzzy α-closed set.*

Proof. Let $\{A_i = \langle x, \mu_{A_i}, \gamma_{A_i} \rangle \mid i \in \Lambda\}$ be a collection of intuitionistic fuzzy α-open sets. Then, for each $i \in \Lambda$, $A_i \subseteq \text{int}(\text{cl}(\text{int}(A_i)))$. It follows that

$$\begin{aligned} \bigcup A_i &\subseteq \bigcup \text{int}(\text{cl}(\text{int}(A_i))) \subseteq \text{int}(\bigcup \text{cl}(\text{int}(A_i))) \\ &= \text{int}(\text{cl}(\bigcup \text{int}(A_i))) \subseteq \text{int}(\text{cl}(\text{int}(\bigcup A_i))). \end{aligned}$$

Hence $\cup A_i$ is an intuitionistic fuzzy α-open set. The second part follows immediately from the first part by taking complements. □

Having shown that arbitrary union of intuitionistic fuzzy α-open sets is an intuitionistic fuzzy α-open set, it is natural to consider whether or not the intersection of intuitionistic fuzzy α-open sets is an intuitionistic fuzzy α-open set, and the following example shows that the intersection of intuitionistic fuzzy α-open sets is not an intuitionistic fuzzy α-open set.

Example 3..5 Let $X = [0,1]$ and let A, B, C, and D be IFSs in X defined by

$$\mu_A(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2}, \\ 1-x, & \frac{1}{2} \le x \le 1, \end{cases} \qquad \gamma_A(x) = \begin{cases} 1-x, & 0 \le x \le \frac{1}{2}, \\ x, & \frac{1}{2} \le x \le 1, \end{cases}$$

$$\mu_B(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{2}, \\ 0, & \frac{1}{2} < x \le 1, \end{cases} \qquad \gamma_B(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2}, \\ 1, & \frac{1}{2} < x \le 1, \end{cases}$$

$$\mu_C(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 0, & \frac{1}{2} < x \le 1, \end{cases} \qquad \gamma_C(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2}, \\ 2x, & \frac{1}{2} < x \le 1, \end{cases}$$

$$\mu_D(x) = x, \text{ and } \gamma_D(x) = 1 - x \text{ for } x \in [0,1].$$

Then the collection $\tau = \{0_\sim, 1_\sim, A\}$ is an IFT on X. The IFSs B and D are intuitionistic fuzzy α-open sets, but $B \cap D$ is not an intuitionistic fuzzy α-open set. In fact, $B \cap D$ is an IFS on X given by

$$\mu_{B\cap D}(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2}, \\ 0, & \frac{1}{2} < x \le 1, \end{cases} \text{ and } \gamma_{B\cap D}(x) = \begin{cases} 1-x, & 0 \le x \le \frac{1}{2}, \\ 1, & \frac{1}{2} < x \le 1, \end{cases}$$

and so $B \cap D \not\subseteq \text{int}(\text{cl}(\text{int}(A)))$.

It is clear that every IFOS (resp. IFCS) is an intuitionistic fuzzy α-open set (resp. an intuitionistic fuzzy α-closed set), but the converse is not true (see Example 3..5).

Definition 3..6 The *intuitionistic fuzzy α-closure* of an IFS A in an IFTS (X, τ), written $\text{cl}_\alpha(A)$, is defined by

$$\text{cl}_\alpha(A) := \cap\{B_i \mid B_i \text{ is an intuitionistic fuzzy } \alpha\text{-closed set and } A \subseteq B_i\}.$$

Proposition 3..7 *In an* IFTS (X, τ), *an* IFS A *is intuitionistic fuzzy α-closed if and only if* $A = \text{cl}_\alpha(A)$.

Proof. Assume that A is an intuitionistic fuzzy α-closed set. Obviously,

$$A \in \{B_i \mid B_i \text{ is an intuitionistic fuzzy } \alpha\text{-closed set and } A \subseteq B_i\},$$

and so

$$\begin{aligned} A &= \cap\{B_i \mid B_i \text{ is an intuitionistic fuzzy } \alpha\text{-closed set and } A \subseteq B_i\} \\ &= \mathrm{cl}_\alpha(A). \end{aligned}$$

Conversely suppose that $A = \mathrm{cl}_\alpha(A)$, which shows that

$$A \in \{B_i \mid B_i \text{ is an intuitionistic fuzzy } \alpha\text{-closed set and } A \subseteq B_i\}.$$

Hence A is an intuitionistic fuzzy α-closed set. □

Proposition 3..8 *In an* IFTS (X, τ), *the following hold for intuitionistic fuzzy α-closure:*

(i) $\mathrm{cl}_\alpha(0_\sim) = 0_\sim$.

(ii) $\mathrm{cl}_\alpha(A)$ *is intuitionistic fuzzy α-closed in* (X, τ) *for every* IFS A *in* X.

(iii) $\mathrm{cl}_\alpha(A) \subseteq \mathrm{cl}_\alpha(B)$ *whenever* $A \subseteq B$ *for every* IFS A *and* B *in* X.

(iv) $\mathrm{cl}_\alpha(\mathrm{cl}_\alpha(A)) = \mathrm{cl}_\alpha(A)$ *for every* IFS A *in* X.

Proof. The proof is easy. □

Definition 3..9 Let (X, τ_X) and (Y, τ_Y) be IFTSs. A mapping $f : X \to Y$ is said to be *intuitionistic fuzzy α-continuous* if the inverse image of each IFOS of Y is an intuitionistic fuzzy α-open set in X.

Example 3..10 Let A and B be IFSs of $X = [0, 1]$ given by

$$\mu_A(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2}, \\ 2x - 1, & \frac{1}{2} \le x \le 1, \end{cases} \qquad \gamma_A(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{2}, \\ -2x + 2, & \frac{1}{2} \le x \le 1, \end{cases}$$

$$\mu_B(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{4}, \\ -4x+2, & \frac{1}{4} \le x \le \frac{1}{2}, \\ 0, & \frac{1}{2} \le x \le 1, \end{cases} \qquad \gamma_B(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{4}, \\ 4x-1, & \frac{1}{4} \le x \le \frac{1}{2}, \\ 1, & \frac{1}{2} \le x \le 1, \end{cases}$$

respectively. Then the collection $\tau = \{0_\sim, 1_\sim, B\}$ is an IFT on X. Let $g : X \to X$ be defined by $g(x) = \frac{1}{2}x$ for all $x \in X$. Then $g^{-1}(0_\sim) = 0_\sim$, $g^{-1}(1_\sim) = 1_\sim$ and $g^{-1}(B) = \bar{A}$, which are intuitionistic fuzzy α-open sets in (X, τ). Hence g is an intuitionistic fuzzy α-continuous mapping.

Lemma 3..11 *[5] Let A be an IFS in X. Then* $A = \bigcup_{\frac{p}{(\alpha,\beta)} \subseteq A} \frac{p}{(\alpha,\beta)}$.

Theorem 3..12 *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a mapping from an* IFTS (X, τ_X) *to an* IFTS (Y, τ_Y). *Then the following are equivalent:*

(i) *f is intuitionistic fuzzy α-continuous.*

(ii) *for each* IFP $\frac{p}{(\alpha,\beta)}$ *in X and each* IFOS *B in Y with $f(\frac{p}{(\alpha,\beta)}) \subseteq B$, there exists an intuitionistic fuzzy α-open set A in X such that $\frac{p}{(\alpha,\beta)} \subseteq A$ and $f(A) \subseteq B$.*

(iii) *if C is an* IFCS *in Y, then $f^{-1}(C)$ is an intuitionistic fuzzy α-closed set in X.*

Proof. Assume that f is intuitionistic fuzzy α-continuous. Since B is an IFOS in Y and $f(\frac{p}{(\alpha,\beta)}) \subseteq B$, we have $\frac{p}{(\alpha,\beta)} \subseteq f^{-1}(B)$ and $f^{-1}(B)$ is an intuitionistic fuzzy α-open set in X because f is intuitionistic fuzzy α-continuous. Taking $A = f^{-1}(B)$, then $\frac{p}{(\alpha,\beta)} \subseteq A$ and $f(A) = f(f^{-1}(B)) \subseteq B$. Conversely, let B be an IFOS in Y such that $\frac{p}{(\alpha,\beta)} \subseteq f^{-1}(B)$ and thus there exists an intuitionistic fuzzy α-open set $A_{\frac{p}{(\alpha,\beta)}}$ in X such that $\frac{p}{(\alpha,\beta)} \subseteq A_{\frac{p}{(\alpha,\beta)}}$ and $f(A_{\frac{p}{(\alpha,\beta)}}) \subseteq B$. Then $\frac{p}{(\alpha,\beta)} \subseteq A_{\frac{p}{(\alpha,\beta)}} \subseteq f^{-1}(B)$, and so $f^{-1}(B) = \bigcup_{\frac{p}{(\alpha,\beta)} \subseteq f^{-1}(B)} A_{\frac{p}{(\alpha,\beta)}}$ which is an intuitionistic fuzzy α-open set. Hence f is intuitionistic fuzzy α-continuous. Thus (i) and (ii) are equivalent. Assume that f is intuitionistic fuzzy α-continuous and let C be an IFCS in Y. Then $\bar{C}$ is an IFOS in Y, and so $\overline{f^{-1}(C)} = f^{-1}(\bar{C})$ is an intuitionistic fuzzy α-open set in X. Hence $f^{-1}(C)$ is an intuitionistic fuzzy α-closed set in X. Conversely let B be an IFOS in Y. Then $\bar{B}$ is an IFCS in Y, and thus $\overline{f^{-1}(B)} = f^{-1}(\bar{B})$ is an intuitionistic fuzzy α-closed set in X. Therefore $f^{-1}(B)$ is an intuitionistic fuzzy α-open set in X, and f is an intuitionistic fuzzy α-continuous mapping. This completes the proof. □

Since every IFOS is an intuitionistic fuzzy α-open set, it is clear that every intuitionistic fuzzy continuous mapping is an intuitionistic fuzzy α-continuous mapping. But the converse is not true because the mapping g in Example 3..10 is not intuitionistic fuzzy continuous. Hence the notion of an intuitionistic fuzzy α-continuous mapping is a generalization of an intuitionistic fuzzy continuous mapping.

Theorem 3..13 *Let* $f : (X, \tau_X) \to (Y, \tau_Y)$ *be a mapping from an* IFTS (X, τ_X) *to an* IFTS (Y, τ_Y). *If f is intuitionistic fuzzy α-continuous, then*

(i) $f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) \subseteq \mathrm{cl}(f(A))$ *for all* IFS A *in* X.

(ii) $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(f^{-1}(B)))) \subseteq f^{-1}(\mathrm{cl}(B))$ *for all* IFS B *in* Y.

Proof. Assume that f is an intuitionistic fuzzy α-continuous mapping. Let A be an IFS in X. Then $\mathrm{cl}(f(A))$ is an IFCS in Y, and thus $f^{-1}(\mathrm{cl}(f(A)))$ is an intuitionistic fuzzy α-closed set in X (see Theorem 3..12(iii)). It follows that

$$\begin{aligned}\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) &= \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{cl}(A)))) \\ &\subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(f^{-1}(\mathrm{cl}(f(A)))))) \\ &\subseteq f^{-1}(\mathrm{cl}(f(A)))\end{aligned}$$

so that $f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) \subseteq \mathrm{cl}(f(A))$, which proves (i). Now let B be an IFS in Y. Then $f^{-1}(B)$ is an IFS in X. Hence, by (i), we have

$$f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(f^{-1}(B))))) \subseteq \mathrm{cl}(f(f^{-1}(B))) \subseteq \mathrm{cl}(B),$$

and so $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(f^{-1}(B)))) \subseteq f^{-1}(\mathrm{cl}(B))$. This completes the proof. □

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