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A C^2 SURFACE EXTENSION METHOD USING SEVERAL CONTROL FUNCTIONS

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ABSTRACT. We suggest a method of C^2 surface extension with the aid of well-controlled functions. The extended surface is C^2 continuous along the old boundary. The function of the extension surface is obtained by replacing the monomials in the quadratic Taylor polynomial of the given surface-representing function by other functions subject to some boundary conditions. We present several sets of control functions. In order to illustrate our suggestion, it is shown that surfaces with a circular boundary and a square boundary can be extended using several base functions.

1. INTRODUCTION

In manufacturing or modeling of a surface, it is sometimes helpful or necessary to extend an existing surface across its boundary. A surface may be given explicitly by a function which can be defined on the whole plane, and there are many ways to extend or extrapolate the underlying surface depending on one's need and situation. The simplest way to extend the surface is to use the same function for a larger domain, but surface-representing functions may fluctuate more widely for larger values of variables and it becomes unacceptable for some purposes. So we are in need of other extrapolation method with some constraints. Extrapolation theory is not well developed because it is not unique outside the considered domain, while interpolation theory is well developed in curve and surface. For more details, we refer to [2, 3, 5].

From the viewpoint of practical application, one requires a smooth extension for machining in NC(numerical control) machine. In the past, we generated data outside the useful area and refitted those data using least squares method. However, this method has some drawbacks which have some errors within the useful area and require much computational time in the case of a surface using B-spline basis. So we need a

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surface having this property for machining. We keep the area and extend the useless area by a function which is smooth across the boundary.

In engineering design or modeling of parts, the required curve or surface usually has geometric shape and there is no universal solution for the extension problem. It is therefore practically a case-by-case problem and there there are some works in various directions (see [1], [4], and [6]).

In this work, we modify quadratic Taylor's formulas to give a suggestion of C^2 surface extension. Taylor's theorem provides "higher-order" applications to a function generalizing the line and the quadratic approximation based on the first and the second derivative of the function, respectively. Also one uses Taylor's theorem in several variables to derive tests for different type of extrema. There are other important applications of this theorem as well. Quadratic Taylor's formula gives a polynomial approximation of degree two called the Taylor polynomial. And we can use the polynomial approximation to extend the surface. Since the values of a polynomial become larger as the values of variables increase, it is somehow unsuitable for our purpose. So we replace monomials in the Taylor polynomials by some other functions called control functions which satisfy some conditions as in (2.2). It is our main idea and gives a method for C^2 surface extension (see, Theorem 2.2). In section 2, we suggest several sets of control functions and we illustrate our suggestion using the control functions in section 3.

2. A SURFACE EXTENSION METHOD

Taylor's theorem provides "higher-order" applications to a function generalizing the line approximation based on the first derivative of the function. Also Taylor's theorem is used in several variables to derive a test for different type of extrema. There are other important applications of this theorem as well.

Theorem 2.1. (Taylor's Formula for $f(x,y)$ at (a,b).) *Let D be a convex region $D \subset R^2$ containing a point (a,b) as an interior point and let f be a real valued function defined on D. If f has n-th derivative, then throughout D, we have*

$$\begin{aligned} f(x,y) =& f(a,b) + ((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y})f|_{(a,b)} \\ &+ \cdots + \frac{1}{(n-1)!}((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y})^{n-1} f|_{(a,b)} \\ &+ \frac{1}{n!}((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y})^{n} f|_{(x_a,y_b)} \end{aligned} \tag{2.1}$$

for some point (x_a, y_b) lying on the line segment joining (a,b) and (x,y).

We call the polynomial in the right-hand side of the equation (2.1) except the last term *Taylor polynomial of f*. Throughout this work, if f is a function of two variable

x and y, we denote the partial derivatives of f at (a,b) alternatively by

$$\frac{\partial f}{\partial x}(a,b) = f_x(a,b) \quad \text{and} \quad \frac{\partial f}{\partial y}(a,b) = f_y(a,b)$$

and we use the following notation

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x}, \quad \text{and} \quad f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2}.$$

We may regard Taylor polynomial of f at (a,b) as the expansion :

$$\begin{aligned} f(x,y) =& f(a,b) + f_x(a,b)A(x-a,y-b) + f_y(a,b)B(x-a,y-b) \\ &+ f_{xx}(a,b)C(x-a,y-b) + f_{xy}(a,b)D(x-a,y-b) + f_{yy}(a,b)E(x-a,y-b) \\ &+ R(x,y) \end{aligned}$$

where $R(x,y)$ is the remainder and the functions $A, B, C, D,$ and E have the property :

$$(2.2) \quad \begin{cases} A(0,0) = B(0,0) = C(0,0) = D(0,0) = E(0,0) = 0 \\ \frac{\partial A}{\partial x}(0,0) = 1 \quad \text{and} \quad \frac{\partial B}{\partial x}(0,0) = \frac{\partial C}{\partial x}(0,0) = \frac{\partial D}{\partial x}(0,0) = \frac{\partial E}{\partial x}(0,0) = 0 \\ \frac{\partial B}{\partial y}(0,0) = 1 \quad \text{and} \quad \frac{\partial A}{\partial y}(0,0) = \frac{\partial C}{\partial y}(0,0) = \frac{\partial D}{\partial y}(0,0) = \frac{\partial E}{\partial y}(0,0) = 0 \\ \frac{\partial^2 C}{\partial x^2}(0,0) = 1 \quad \text{and} \quad \frac{\partial^2 A}{\partial x^2}(0,0) = \frac{\partial^2 B}{\partial x^2}(0,0) = \frac{\partial^2 D}{\partial x^2}(0,0) = \frac{\partial^2 E}{\partial x^2}(0,0) = 0 \\ \frac{\partial^2 D}{\partial x \partial y}(0,0) = 1 \quad \text{and} \quad \frac{\partial^2 A}{\partial x \partial y}(0,0) = \frac{\partial^2 B}{\partial x \partial y}(0,0) = \frac{\partial^2 C}{\partial x \partial y}(0,0) = \frac{\partial^2 E}{\partial x \partial y}(0,0) = 0 \\ \frac{\partial^2 E}{\partial y^2}(0,0) = 1 \quad \text{and} \quad \frac{\partial^2 A}{\partial y^2}(0,0) = \frac{\partial^2 B}{\partial y^2}(0,0) = \frac{\partial^2 C}{\partial y^2}(0,0) = \frac{\partial^2 D}{\partial y^2}(0,0) = 0. \end{cases}$$

As shall be seen in the following, these properties play a very important role in our suggestion to the extension problem.

Now, we attempt to expand a function f defined on D to Ω of which the interior contains D. In the useful area D, we have a given surface $z = f(x,y)$ and outside the area, we are looking for a C^2 function which is differentiable across the boundary :

$$(2.3) \qquad F(x,y) = \begin{cases} f(x,y) & \text{in } D, \\ g(x,y) & \text{in } \Omega - D, \end{cases}$$

where g is defined by

$$(2.4) \qquad \begin{aligned} g(x,y) =& f(x_b,y_b) + f_x(x_b,y_b)A(x,y) + f_y(x_b,y_b)B(x,y) \\ &+ f_{xx}(x_b,y_b)C(x,y) + f_{xy}(x_b,y_b)D(x,y) + f_{yy}(x_b,y_b)E(x,y) \end{aligned}$$

where (x_b, y_b) is chosen as a point in ∂D depending on the point (x,y) so that F is a C^2 extension of f. That is to say, the boundary conditions are satisfied so that for each $(x_b,y_b) \in \partial D$, we have

$$F(x_b,y_b) = f(x_b,y_b), \quad F_x(x_b,y_b) = f_x(x_b,y_b), \quad F_y(x_b,y_b) = f_y(x_b,y_b)$$

and

$$F_{xx}(x_b,y_b) = f_{xx}(x_b,y_b), \quad F_{xy}(x_b,y_b) = f_{xy}(x_b,y_b), \quad F_{yy}(x_b,y_b) = f_{yy}(x_b,y_b).$$

To the end, Taylor's formula suggests a method that we extend the surface by replacing the monomials appeared in the Taylor polynomial (2.1) of f by some other functions satisfying the conditions as in (2.2). The function g in (2.4) of this extension is similar to the Taylor expansion in the form. The functions A, B, C, D, and E play an important role in our suggestion and these functions satisfy the boundary conditions as given in (2.2) to guarantee the C^2 extension.

Theorem 2.2. *Let $D \subset \Omega$ be subsets in R^2 and D belong to the interior of a region $\Omega \in R^2$. Assume that f is a C^4 function on D such that f and its derivatives are continuous on ∂D. If F is a function on Ω given by :*

$$F(X) = \begin{cases} f(X) & in \quad D, \\ g(X) & in \quad \Omega - D, \end{cases} \tag{2.5}$$

where g is defined by

(2.6)
$$\begin{aligned} g(X) =& f \circ \pi(X) + f_x(\pi(X))A(X - \pi(X)) + f_y(\pi(X))B(X - \pi(X)) \\ &+ f_{xx}(\pi(X))C(X - \pi(X)) + f_{xy}(\pi(X))D(X - \pi(X)) + f_{yy}(\pi(X))E(X - \pi(X)). \end{aligned}$$

with C^2 functions A, B, C, D, E satisfying the initial conditions as in (2.2), and $\pi : (\Omega - D) \cup \partial D \to \partial D$ is a C^2 function with identity on ∂D. Then F is a C^2 extension of f.

Proof. Since g is C^2 in $\Omega - D$ and $f = g$ on ∂D, it suffices to show that for every point $X_0 = (x_0, y_0) \in \partial D$, we have

$$f_x(x_0, y_0) = g_x(x_0, y_0), \quad f_y(x_0, y_0) = g_y(x_0, y_0)$$

and

$$f_{xx}(x_0, y_0) = g_{xx}(x_0, y_0), \quad f_{xy}(x_0, y_0) = g_{xy}(x_0, y_0), \quad f_{yy}(x_0, y_0) = g_{yy}(x_0, y_0).$$

The proof is straightforward and we show only the equation $f_x = g_x$ on ∂D.

Let denote $\pi(X) = (u(X), v(X))$. Using chain rules, we obtain the first derivative g_x at a point $X \in \Omega - D$,

$$
\begin{aligned}
g_x(X) =& f_x \circ \pi(X) u_x(X) + f_y \circ \pi(X) v_x(X) \\
&+ [f_{xx} \circ \pi(X) u_x(X) + f_{xy} \circ \pi(X) v_x(X)] A(X - \pi(X)) \\
&+ f_x \circ \pi(X) [A_x(X - \pi(X))(1 - u_x(X)) + A_y(X - \pi(X))(-v_x(X))] \\
&+ [f_{yx} \circ \pi(X) u_x(X) + f_{yy} \circ \pi(X) v_x(X)] B(X - \pi(X)) \\
&+ f_y \circ \pi(X) [B_x(X - \pi(X))(1 - u_x(X)) + B_y(X - \pi(X))(-v_x(X))] \\
&+ [f_{xxx} \circ \pi(X) u_x(X) + f_{yxx} \circ \pi(X) v_x(X)] C(X - \pi(X)) \\
&+ f_{xx} \circ \pi(X) [C_x(X - \pi(X))(1 - u_x(X)) + C_y(X - \pi(X))(-v_x(X))] \\
&+ [f_{xyx} \circ \pi(X) u_x(X) + f_{yxy} \circ \pi(X) v_x(X)] D(X - \pi(X)) \\
&+ f_{xy} \circ \pi(X) [D_x(X - \pi(X))(1 - u_x(X)) + D_y(X - \pi(X))(-v_x(X))] \\
&+ [f_{xyy} \circ \pi(X) u_x(X) + f_{yyy} \circ \pi(X) v_x(X)] E(X - \pi(X)) \\
&+ f_{yy} \circ \pi(X) [E_x(X - \pi(X))(1 - u_x(X)) + E_y(X - \pi(X))(-v_x(X))].
\end{aligned}
$$

The initial conditions on A, B, C, D, and E imply that for each point X_0 on ∂D we have

$$
\begin{aligned}
g_x(X_0) &= f_x(X_0) u_x(X_0) + f_y(X_0) v_x(X_0) + f_x(X_0)(1 - u_x(X_0)) + f_y(X_0)(-v_x(X_0)) \\
&= f_x(X_0).
\end{aligned}
$$

In the equation above, we used the continuity of these functions and their derivatives, and the condition $\pi(X) = X$ on ∂D as well. Hence we have shown that F is a C^2 extension of f to Ω. □

We call the functions A, B, C, D, and E given in Theorem 2.2 *the control functions.*

Remark 2.1. *Note that the condition of the smoothness of the projection mapping π implies also the smoothness of the boundary ∂D. Thus the existence of such a mapping π depends strongly upon the geometry of the boundary ∂D.*

We present several sets of control functions $A(x, y), B(x, y), C(x, y), D(x, y)$, and $E(x, y)$ satisfying the boundary conditions as given in (2.2).

Example 2.1.

$$A(x,y) = a\tan^{-1}(\frac{x}{a})$$
$$B(x,y) = a\tan^{-1}(\frac{y}{a})$$
$$C(x,y) = \frac{1}{2}(a\tan^{-1}(\frac{x}{a}))^2$$
$$D(x,y) = a^2\tan^{-1}(\frac{x}{a})\tan^{-1}(\frac{y}{a})$$
$$E(x,y) = \frac{1}{2}(a\tan^{-1}(\frac{y}{a}))^2$$

Example 2.2.

$$A(x,y) = x(e^{-\frac{x}{\beta}} + \frac{x}{\beta})$$
$$B(x,y) = y(e^{-\frac{y}{\beta}} + \frac{y}{\beta})$$
$$C(x,y) = \frac{1}{2}x^2e^{-\frac{x^2}{\beta}}$$
$$D(x,y) = xye^{-\frac{x^2+y^2}{\beta}}$$
$$E(x,y) = \frac{1}{2}y^2e^{-\frac{y^2}{\beta}}$$

The parameter β controls the shape of the curve and surface.

Example 2.3.

$$A(x,y) = x$$
$$B(x,y) = y$$
$$C(x,y) = \frac{1}{2}x^2$$
$$D(x,y) = xy$$
$$E(x,y) = \frac{1}{2}y^2$$

These functions give rise to the Taylor polynomial expansion but not enough for machining because they diverge as x and y become larger.

Example 2.4.

$$A(x,y) = \frac{x}{1+\alpha x^2}$$
$$B(x,y) = \frac{y}{1+\alpha y^2}$$
$$C(x,y) = \frac{1}{2}(\frac{x}{1+\alpha x^2})^2$$
$$D(x,y) = \frac{x}{1+\alpha x^2}\frac{y}{1+\alpha y^2}$$
$$E(x,y) = \frac{1}{2}(\frac{y}{1+\alpha y^2})^2$$

These functions behave nicely because they are similar to those functions given in Example 2.3 near the origin and decay with $O(\frac{1}{x})$ *rate at* ∞.

3. Applications to surface extension

3.1. Surface extension of a circular domain to a circular domain.

We consider the circular domain D,

$$D = \{(x,y) \in R^2 : x^2 + y^2 \leq 400\}$$

and we apply our suggestion to the domain Ω,

$$\Omega = \{(x,y) \in R^2 : 400 \leq x^2 + y^2 \leq 1600\}.$$

In this case, the projection mapping $\pi(x,y)$ is obviously defined by

$$\pi(x,y) = (\frac{20x}{\sqrt{x^2+y^2}}, \frac{20y}{\sqrt{x^2+y^2}}).$$

It is easy to see that π is C^2 in $(\Omega - D) \cup \partial D$. We illustrate Theorem 2.2 by extending the function $f(x,y) = x^2 + y^2$ using the control functions given in the previous section.

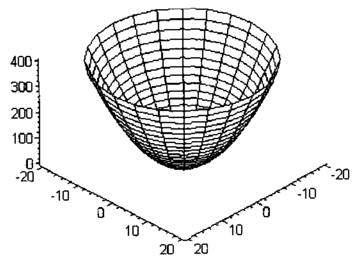

Figure 1. Graph of $f(x,y) = x^2 + y^2$

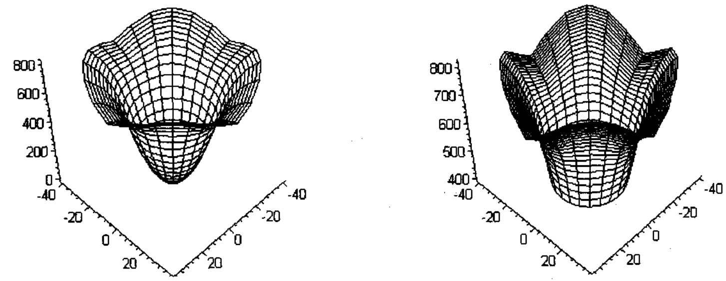

Figure 2. Graphs of $F(x,y)$ and $g(x,y)$ related to Example 2.1 with $a=1$

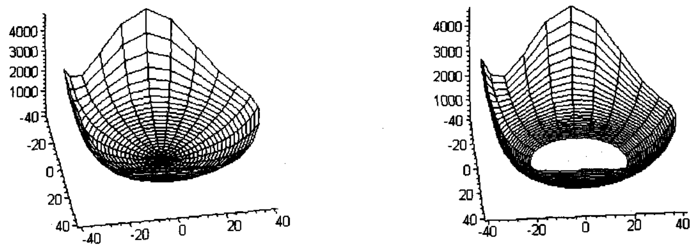

Figure 3. Graphs of $F(x,y)$ and $g(x,y)$ related to Example 2.2 with $\beta=10$

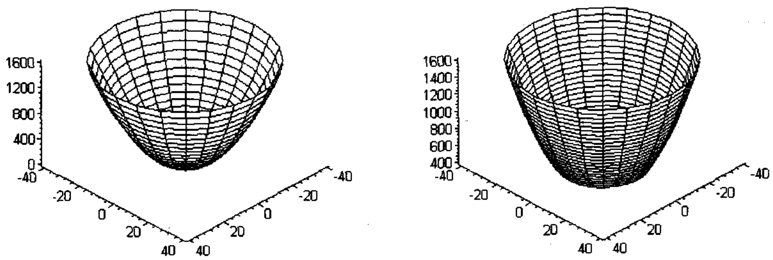

Figure 4. Graphs of $F(x,y)$ and $g(x,y)$ related to Example 2.3

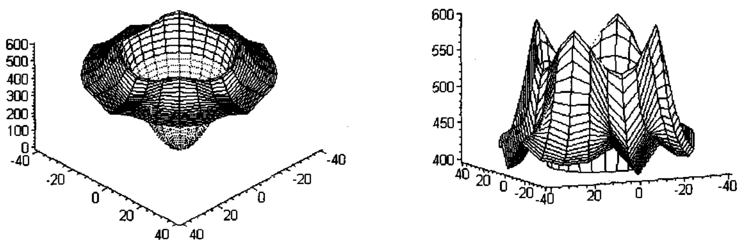

Figure 5. Graphs of $F(x,y)$ and $g(x,y)$ related to Example 2.4 with $\alpha=10^{-3}$

Remark 3.1. *(i) Since $f(x,y) = x^2 + y^2$ is a polynomial of degree 2, the Taylor polynomial of degree 2 of f is equal to itself, so that the extension $F(x,y)$ is the same as f (see Figure 4.)*

(ii) The control functions given in Example 2.4 decay with a rate of $1/\sqrt{x^2+y^2}$, thus values $F(x,y)$ tend to $f\circ\pi(x,y)$, the values of f at boundary points $\pi(x,y)$ as variables increase. So this extension works nicely in the case when f behaves slowly near the boundary (see Figure 10). However, when f fluctuates near the boundary, also F fluctuates near the boundary ∂D and gets plain rapidly in the outer domain (see Figure 4 and Figure 10 below).

3.2. Surface extension from a rectangle to a rectangle.

Now we consider the case of the rectangle domain D,

$$D = \{(x,y) \in R^2 : |x| \le 10, \quad |y| \le 10\}$$

and we apply our suggestion to the domain Ω,

$$\Omega = \{(x,y) \in R^2 : |x| \le 20, \quad |y| \le 20\}.$$

In this case, the projection mapping $\pi(x,y) = (u(x,y), v(x,y))$ is defined by

$$u(x,y) = \begin{cases} -10, & \text{if } x \le -10, \\ 10, & \text{if } x \ge 10, \\ x, & \text{otherwise} \end{cases} \quad \text{and} \quad v(x,y) = \begin{cases} -10, & \text{if } y \le -10, \\ 10, & \text{if } y \ge 10, \\ y, & \text{otherwise} \end{cases}$$

So π is not differentiable at the points in Γ,

$$\Gamma := \{(x,\pm 10) \in \Omega \ : \ 10 \le |x| \le 20\} \cup \{(\pm 10, y) \in \Omega \ : \ 10 \le |y| \le 20\}$$

and the extension g may not be differentiable on Γ. It comes from the fact that the boundary curve ∂D is not differentiable at the four vertices $(\pm 10, \pm 10)$. Nevertheless, we can see from the proof of Theorem 2.2 that g is C^2 on $\Omega - \Gamma$. We illustrate our suggestion with the function $f(x,y) = -\frac{xy}{16}e^{-\frac{x^2+y^2}{16}}$ defined on D by extending to Ω using the control functions given in the examples.

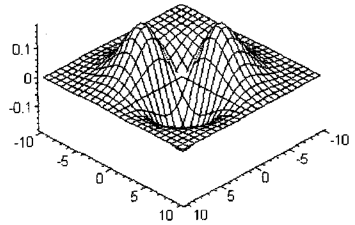

Figure 6. Graph of $f(x,y) = -\frac{xy}{16}e^{-\frac{x^2+y^2}{16}}$

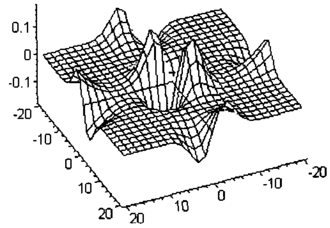

Figure 7. Graph of $F(x,y)$ related to Example 2.1 with $a = 100$

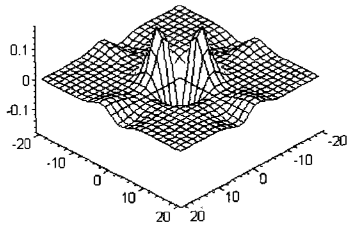

Figure 8. Graph of $F(x,y)$ related to Example 2.2 with $\beta = 100$

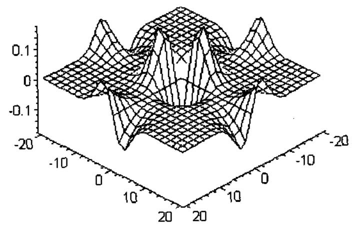

Figure 9. Graph of $F(x,y)$ related to Example 2.3

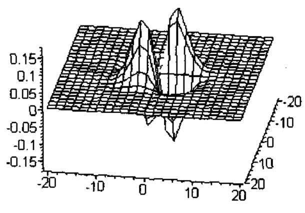

Figure 10. Graph of $F(x, y)$ related to Example 2.4 with $\alpha = 100$

For the calculation of Gaussian and mean curvature, we can derive the function values, derivative values, and second derivative values in D and Ω.

Taylor's series expansion may fluctuate more widely for larger values of variables and it becomes unacceptable for some purposes, so we can not use it because the machining is very difficult. In this situation, we use other functions instead of the control functions given in Example 2.3. The control functions presented in Examples 2.2 and 2.4 have very good behavior because they are smooth and good for matching. We have found C^2 extensionable surface functions.

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