Kangweon-Kyungki Math. Jour. **10** (2002), No. 2, pp. 157–166

VANISHING OF CONTACT CONFORMAL CURVATURE TENSOR ON 3-DIMENSIONAL SASAKIAN MANIFOLDS

Keumseong Bang and JungYeon Kye

Abstract. We show that the contact conformal curvature tensor on 3-dimensional Sasakian manifold always vanishes. We also prove that if the contact conformal curvature tensor vanishes on a 3-dimensional locally φ-symmetric contact metric manifold M, then M is a Sasakian space form.

1. Introduction

The study of conformally invariant curvature tensors plays an important role in understanding various aspects of geometry. In 1949, S. Bochner introduced a curvature tensor, called the Bochner curvature tensor, on a Kähler manifold analogous to the Weyl curvature tensor on Riemannian manifolds. Recently, H. Kitahara, K. Matsuo and J. S. Pak ([4]) defined a new tensor field, which is a conformal invariance, on a hermitain manifold and studied some properties of this new tensor field.

Further, J. C. Jeong, J. D. Lee, G. H. Oh and J. S. Pak defined a new type of tensor field on Sasakian manifolds constructed from the conformal curvature tensor field by using the Boothby-Wang fibration. This curvature tensor, called the contact conformal curvature tensor, seems to be fundamental in studying the spectral geometry of compact Sasakian manifolds ([5]). Regarding the results of research on this field, Tanno proved that every conformally flat K-contact manifold is a space form, and Blair and Koufogiorgos improved this result by showing that every conformally flat contact metric manifold with $Q\varphi = \varphi Q$ is a

Received August 8, 2002.
2000 Mathematics Subject Classification: 53C15.
Key words and phrases: contact conformal curvature tensor, Sasakian manifold.
This work was supported by a research fund of the Catholic University of Korea, 2002.

space form, where Q is the Ricci operator. Moreover, J. S. Pak and Y. J. Shin ([7]) gave a geometric characterization of a contact metric manifold with vanishing contact conformal curvature tensor by showing that;

For $n > 2$, every $(2n+1)$-dimensional contact metric manifold with vanishing contact conformal curvature tensor is a Sasakian space form.

In this paper, we shall give a partial extension of Pak and Shin's result to 3-dimensional locally φ-symmetric contact metric manifold M, and also show that the contact conformal curvature tensor on 3-dimensional Sasakian manifold always vanishes.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M^{2n+1} is called a contact manifold if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Given a contact form η, there exists a unique vector field ξ, called the characteristic vector field of η, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for all vector fields X.

A differentiable manifold M^{2n+1} is said to have an almost contact structure $(\varphi,\ \xi,\ \eta)$ on M if it admits a field φ of endomorphisms of tangent spaces satisfying;

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \varphi\xi = 0, \qquad \eta \circ \varphi = 0 \qquad \text{and} \qquad \eta(\xi) = 1$$

where I denotes the identity transformation. We also call an almost contact structure (φ, ξ, η) satisfying $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$ for any vector field X, Y tangent to M, an almost contact metric structure.

Suppose that a $(2n+1)$-dimensional manifold M has an almost contact metric structure. Then we define a 2-form Φ on M by

$$\Phi(X,Y) = g(\varphi X, Y)$$

An almost contact metric structure $(\varphi,\ \xi,\ \eta,\ g)$ with $\Phi = d\eta$ is called a contact metric structure.

For the Lie differentiation L and the curvature tensor R, we define the operators l and h by

$$lX = R(X,\xi)\,\xi \quad \text{and} \quad h = \frac{1}{2} L_\xi \varphi \tag{2.1}$$

The (1,1)-type tensors h and l are symmetric and satisfy

$$h\xi = 0, \quad l\xi = 0, \;\; \mathrm{Tr}\, h = 0, \;\; \mathrm{Tr}\, h\varphi = 0 \quad \text{and} \quad h\varphi = -\varphi h$$

We also have the following formulas for contact metric manifolds:

$$\nabla_X \xi = -\varphi X - \varphi h X \quad \text{and hence} \quad \nabla_\xi \xi = 0 \tag{2.2}$$

$$\nabla_\xi \varphi = 0 \tag{2.3}$$

$$\mathrm{Tr}\, l = g(Q\xi, \xi) = 2n - \mathrm{Tr}\, h^2 \tag{2.4}$$

$$\varphi l \varphi - l = 2(\varphi^2 + h^2) \tag{2.5}$$

$$\nabla_\xi h = \varphi - \varphi\, l - \varphi\, h^2 \tag{2.6}$$

where Q is the Ricci operator and ∇ the Riemannian connection of g. For the formulas (2.2)-(2.5), refer to [1] and (2.6) to [3], respectively.

A contact metric manifold for which ξ is Killing is called a K-contact manifold. A contact metric structure $(\varphi,\ \xi,\ \eta,\ g)$ is called a normal contact structure if it satisfies $(\nabla_X \varphi)Y = \eta(Y)X - g(X, Y)\xi$. Also, the normality condition is equivalent to $[\,\varphi, \varphi\,] + 2\, d\eta \otimes \xi = 0$. A manifold with a normal contact metric structure is called a Sasakian manifold. Thus a Sasakian manifold is K-contact, but the converse is not true except in dimension 3 ([1]). A 3-dimensional contact manifold is Sasakian if and only if $h = 0$ ([1]). On a Sasakian manifold, the Ricci operator Q commutes with φ ([1]). Moreover, the following propositions are well known:

Proposition 1 ([1]). *On a contact metric manifold M^{2n+1}, the followings are equivalent:*

(1) *The manifold is a K-contact manifold.*
(2) *The sectional curvature of plane section containing ξ is equal to* 1.
(3) *The Ricci curvature in the direction of ξ is $2n$.*

Proposition 2 ([3]). *Let M be a* 3*-dimensional contact metric manifold with $Q\varphi = \varphi Q$. Then the function* $\mathrm{Tr}\, l$ *is constant on M.*

PROPOSITION 3 ([6]). *On any 3-dimensional Sasakian manifold,*

$$\begin{aligned}R(X,Y)Z = &\frac{k+3}{4}(g(Y,Z)X - g(X,Z)Y)\\ &+ \frac{k-1}{4}\{g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z\\ &- \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y - \eta(X)g(Y,Z)\xi\\ &+ \eta(Y)g(X,Z)\xi\}\end{aligned}$$

where $k = \frac{1}{2}(s-4)$ *and* s *is a scalar curvature.*

The sectional curvature $K(X,\varphi X)$ of a plane section spanned by X and φX with X orthogonal to ξ is called a φ-sectional curvature. A Sasakian manifold of constant φ-sectional curvature is called a Sasakian space form.

We then consider, for a $(2n+1)$-dimensional contact metric manifold M, the following contact conformal curvature tensor C_0 of type (1,3) on M, which is defined([7]) by

(2.7)

$$\begin{aligned}C_0 =& R(X,Y)Z + \frac{1}{2n}\{Q_0(Y,Z)X - Q_0(X,Z)Y\\ &+ g(Y,Z)QX - g(X,Z)QY + \eta(X)\eta(Z)QY - \eta(Y)\eta(Z)QX\\ &+ \eta(Y)Q_0(X,Z)\xi - \eta(X)Q_0(Y,Z)\xi\\ &+ S_0(X,Z)\varphi Y - S_0(Y,Z)\varphi X + 2S_0(X,Y)\varphi Z\\ &+ \Phi(X,Z)SY - \Phi(Y,Z)SX + 2\Phi(X,Y)SZ\}\\ &+ \frac{1}{2n(n+1)}\{2n^2 - n - 2 + \frac{(n+2)s}{2n}\}\\ &\times\{\Phi(Y,Z)\varphi X - \Phi(X,Z)\varphi Y - 2\Phi(X,Y)\varphi Z\}\\ &+ \frac{1}{2n(n+1)}\{n+2 - \frac{(3n+2)s}{2n}\}\{g(Y,Z)X - g(X,Z)Y\}\\ &- \frac{1}{2n(n+1)}\{4n^2 + 5n + 2 - \frac{(3n+2)s}{2n}\}\\ &\times\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + \eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi\}\end{aligned}$$

where Q_0 and s denote the Ricci tensor and the scalar curvature respectively, i.e.,

$$Q_0(X,Y) = g(QX,Y), \quad s = \operatorname{Tr} Q, \quad SX = Q(\varphi X)$$

and

$$S_0(X,Y) = g(SX,Y)$$

3. Main Results

We now study contact conformal curvature tensor on Sasakian manifolds. First of all, we recall the known result that gives a geometric characterization of a contact metric manifold with vanishing contact conformal curvature tensor.

THEOREM 4 ([7]). *For $n > 2$, every $(2n+1)$-dimensional contact metric manifold with vanishing contact conformal curvature tensor is a Sasakian space form.*

We note that Theorem 4 holds only for dimension $(2n+1) > 5$. In fact, we have a partial extension of it to 3-dimensional manifold. We recall that a contact metric structure (φ, ξ, η, g) is said to be locally φ-symmetric if $\varphi^2(\nabla_W R)(X,Y,Z) = 0$ for all vectors W, X, Y, Z orthogonal to ξ. Then, we have a theorem due to D. Blair.

THEOREM 5 ([3]). *Let M be a 3-dimensional contact metric manifold with $Q\varphi = \varphi Q$. Then M is locally φ-symmetric if and only if the scalar curvature of M is constant.*

Using this theorem, we have an extension of Theorem 4 as mentioned above.

COROLLARY 6. *Let M be a 3-dimensional locally φ-symmetric contact metric manifold. If the contact conformal curvature tensor C_0 vanishes on M, then M is a Sasakian space form.*

Proof. Suppose that the contact conformal curvature tensor C_0 vanishes identically on M. Then, from (2.7), we can get

(3.1)
$$\begin{aligned} QX =& \frac{1}{4n-5}\{-3\varphi Q\varphi X - 3\eta(QX)\xi - 2\eta(X)Q\xi\} \\ &+ \frac{2}{n(4n-5)}\{n(n-2)s - 2n(n-2)\}X \\ &+ \frac{2}{n(4n-5)}\{2n(2n^2+n-2) - (n-2)s\}\eta(X)\xi \end{aligned}$$

Letting $X = \xi$ in (3.1), we obtain

$$Q\xi = 2n\xi. \tag{3.2}$$

Thus, by Proposition 1, we know that a contact metric manifold with $C_0 \equiv 0$ is a K-contact manifold. But, since every 3-dimensional K-contact manifold is Sasakian, M is Sasakian. So, it remains to show that the φ-sectional curvature is constant.

Now, we substitute (3.2) into (3.1), we have

(3.3)
$$\begin{aligned} QX =& \frac{-3}{4n-5}\varphi Q\varphi X + \frac{2(n-2)}{n(4n-5)}(s-2n)X \\ &+ \frac{2}{n(4n-5)}\{n(4n^2-3n-4)-(n-2)s\}\eta(X)\xi \end{aligned}$$

Applying the operator φ to this identity and using (3.2), we also get

$$g(\varphi QX, Y) = \frac{3}{4n-5}g(Q\varphi X, Y) + \frac{2(n-2)(s-2n)}{n(4n-5)}g(\varphi X, Y) \tag{3.4}$$

since Q is a symmetric endomorphism.

Moreover, since φ is a skew-symmetric endomorphism, (3.4) implies

$$g(Q\varphi X, Y) = \frac{3}{4n-5}g(\varphi QX, Y) + \frac{2(n-2)(s-2n)}{n(4n-5)}g(\varphi X, Y)$$

This together with (3.4) shows that $g(\varphi QX, Y) = g(X, \varphi Y)$, that is,

$$Q\varphi = \varphi Q \tag{3.5}$$

Here, we use the identity given on p.98, [1];

$$s = \frac{1}{2}\{n(2n+1)(c+3) + n(c-1)\}$$

where s is the scalar curvature and c is a φ-sectional curvature. Since M is a 3-dimensional manifold, the φ-sectional curvature is constant by Theorem 5. Thus, M is a Sasakian space form. □

We recall that the curvature tensor of a 3-dimensional Riemannian manifold is also given ([3]) by

(3.6)
$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X \\ & - g(QX,Z)Y - \frac{s}{2}\{ g(Y,Z)X - g(X,Z)Y \} \end{aligned}$$

We now present our main theorem. It is, in fact, a partial converse of Theorem 4 and Corollary 6.

THEOREM 7. *Let M be a 3-dimensional Sasakian manifold. Then, the contact conformal curvature tensor C_0 on M vanishes.*

Proof. Since M is Sasakian manifold, the Ricci operator Q commutes with φ and $h = 0$. So, $\nabla_\xi h = 0$. Using $Q\varphi = \varphi Q$, $\varphi\xi = 0$, and (2.4), we have

(3.7) $$Q\xi = (\operatorname{Tr} l)\,\xi$$

Using (2.1) and (3.7), we have from (3.6), for any tangent vector field X,

(3.8)
$$\begin{aligned} lX &= QX - \eta(X)Q\xi + (\operatorname{Tr} l)X - g(QX,\xi)\xi - \frac{s}{2}(X - \eta(X)\xi) \\ &= QX + (\operatorname{Tr} l - \frac{s}{2})X + \eta(X)(\frac{s}{2} - \operatorname{Tr} l)\xi - g(QX,\xi)\xi \end{aligned}$$

and hence, $Q\varphi = \varphi Q$ and $\varphi\xi = 0$ imply

(3.9) $$\varphi l = l\varphi$$

By virtue of (3.9), (2.5), and (2.6), we obtain

(3.10) $$-l = \varphi^2$$

From this, we get $g(lX, \xi) = 0$ and $g(lX, \varphi X) = 0$ for any X orthogonal to ξ. Thus, lX is parallel to X for any X orthogonal to ξ. So, we may write $lX = \alpha X$ for such X. Using (3.8), we have

(3.11) $$QX + (\operatorname{Tr} l - \alpha - \frac{s}{2})X = 0$$

Now, we let $\{ X, \varphi X, \xi \}$ be a φ-basis. Taking X to be a unit vector field, the scalar curvature can be computed as

$$\begin{aligned} s &= g(QX, X) + g(Q\varphi X, \varphi X) + g(Q\xi, \xi) \\ &= 2\, g(QX, X) + \operatorname{Tr} l \\ &= 2\,\alpha - \operatorname{Tr} l + s \end{aligned}$$

So, $\alpha = \frac{1}{2} \operatorname{Tr} l$. Using (3.11) and (3.7), we get

$$QX = \frac{1}{2}\,(s - \operatorname{Tr} l\,)X + \frac{1}{2}\,(3 \operatorname{Tr} l - s)\, \eta(X)\xi \tag{3.12}$$

for any tangent vector field X. Substituting (3.12) in (3.7), we also have

$$\begin{aligned} R(X,Y)Z =& \{a\, g(Y,Z) + b\, \eta(Y)\eta(Z)\}X \\ &- \{a\, g(X,Z) + b\, \eta(X)\eta(Z)\}Y \\ &+ b\,\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\xi \end{aligned} \tag{3.13}$$

where $a = \frac{s}{2} - \operatorname{Tr} l$ and $b = \frac{1}{2}\,(3 \operatorname{Tr} l - s)$. For $Z = \xi$, (3.13) gives

$$R(X,Y)\xi = \frac{\operatorname{Tr} l}{2}(\eta(Y)X - \eta(X)Y) \tag{3.14}$$

Since M is a Sasakian, M is a contact metric manifold and $Q\varphi = \varphi Q$. Thus, by Proposition 2, the function $\operatorname{Tr} l$ is constant on M. And, we have

$$R(X,Y)\xi = k\,(\eta(Y)X - \eta(X)Y) \tag{3.15}$$

where k is a constant.

We compare (3.14) and (3.15) using $Q\xi = (\operatorname{Tr} l)\xi$, and get

$$Q\xi = 2k\xi$$

Since M is a Sasakian manifold, it is K-contact. Thus, by Proposition 1, $Q\xi = 2\xi$, *i.e.*, $k = 1$. So, from (3.6), we find

$$R(X,Y)\xi = \eta(Y)QX - \eta(X)QY + (2 - \frac{s}{2})(\eta(Y)X - \eta(X)Y) \tag{3.16}$$

Comparing (3.15) and (3.16), we get

$$\eta(Y)\{QX + (1 - \frac{s}{2})X\} - \eta(X)\{QY + (1 - \frac{s}{2})Y\} = 0$$

Taking $X = \xi$, we have

$$QY = (\frac{s}{2} - 1)Y + (3 - \frac{s}{2})\,\eta(Y)\xi \tag{3.17}$$

for any tangent vector field Y.

Using (3.17), we easily have

$$\begin{aligned} & g(QX, Y) = (\frac{s}{2} - 1)g(X, Y) + (3 - \frac{s}{2})\eta(X)\eta(Y) \\ & \varphi QX = (\frac{s}{2} - 1)\varphi X \\ \text{and}\quad & g(Y, Z)QX = (\frac{s}{2} - 1)g(Y, Z)X + (3 - \frac{s}{2})g(Y, Z)\eta(X)\xi \end{aligned} \tag{3.18}$$

Now, from the definition (2.7), we compute C_0 using Proposition 3, $Q\varphi = \varphi Q$, and (3.18) and finally get

$$\begin{aligned} C_0 =& R(X, Y)Z - (\frac{s}{8} + \frac{1}{4})(g(Y, Z)X - g(X, Z)Y) \\ &+ \{(\frac{s}{8} - \frac{3}{4})(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ &+ \eta(X)g(Y, Z)\xi - \eta(Y)g(X, Z)\xi \\ &+ g(\varphi X, Z)\varphi Y - g(\varphi Y, Z)\varphi X + 2g(\varphi X, Y)\varphi Z\} \\ &= 0 \end{aligned}$$

This completes the proof of our theorem. □

References

1. D. E. Blair, Contact manifold in Riemannian Geometry, Lecture Notes in Math., vol. 509, Springer-Verlag, Berlin, 1976.
2. D. E. Blair and T. Koufogiorgos, *When is the tangent sphere bundle conformally flat?*, J. Geom. **49** (1994), 55-66.

3. D. E. Blair, T. Koufogiorgos and R. Sharma, *A classification of 3-dimensional contact manifolds with* $Q\varphi = \varphi Q$, Kodai Math. J. **13** (1990), 391-401.
4. H. Kitahara, K. Matsuo and J. S. Pak, *A conformal curvature tensor on hermitian manifolds; Appendium*, J. Korean Math. Soc. **27** (1990), 7-17 ; Bull. Korean Math. Soc. **27** (1990), 27-30.
5. J. C. Jeong, J. D. Lee, G. H. Oh and J. S. Pak, *On the contact conformal curvature tensor*, Bull. Korean Math. Soc. **27** (1990), No.2, 133-142.
6. J. S. Pak, J. C. Jeong and W. T. Kim, *The contact conformal curvature tensor field and the spectrum of the Laplacian*, J. Korean Math. Soc. **28** (1991), 267-274.
7. J. S. Pak and Y. J. Shin, *A note on contact conformal curvature tensor*, Comm. Korean Math. Soc. **13** (1998), 337-343.

Keumseong Bang
Department of Mathematics
The Catholic University of Korea
Puchon, Kyonggi-Do 420-743, Korea
E-mail: bang@catholic.ac.kr

JungYeon Kye
Interdisciplinary Graduate Program
Mathematics and Education
University of Missouri at Kansas City
Kansas City, MO 644110, U. S. A.