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GRADATIONS OF SUPRAOPENNESS

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Abstract. We introduce the concept of gradation of supraopenness. With the concept of gradation of supraopenness, we invesigate the basic properties of H-fuzzy supratopological spaces, H-fuzzy suprainterior and H-fuzzy supraclosure.

1. Introduction

Fuzzy topological spaces were first introduced in the literature by Chang [2] who studied a number of the basic concepts including fuzzy continuous maps and compactness. And fuzzy toplogical spaces are a very natural generalization of topological spaces. In [3], R. N. Hazra et.al introduced a new fuzzy topology and fuzzy topological space in terms of lattices L and L', both of which were taken to be $I = [0, 1]$. In this paper, we will call the new fuzzy topology an H-fuzzy topology. 0_X and 1_X will denote the characteristic functions of the crisp sets $\emptyset$ and X, respectively. An H-fuzzy topological space [3] is a pair (X, τ), where X is a non-empty set and $\tau : I^X \to I$ is a mapping satisfying the following properties:

(O1) $\tau(0_X) = \tau(1_X) = 1$.

(O2) If $\tau(A) > 0, \tau(B) > 0$, then $\tau(A \cap B) > 0$, for $A, B \in I^X$.

(O3) For every subfamily $\{A_i : i \in J\} \subset I^X$, if $\tau(A_i) > 0$, then $\tau(\cup_{i\in J} A_i) > 0$.

Then the mapping $\tau : I^X \to I$ is called an H-fuzzy topology or a gradation of openness on X.

If the H-fuzzy topoloy τ on X has the following property:

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(O4) $\tau(I^X) \subset \{0,1\}$, then τ corresponds in a one to one way to a fuzzy topology in Chang's sense [2].

A mapping $\tau^* : I^X \to I$ is called an H-fuzzy cotopology or a gradation closedness [3] iff the following three conditions are satisfied:

(C1) $\tau^*(0_X) = \tau^*(1_X) = 1$.

(C2) If $\tau^*(A), \tau^*(B) > 0$, then $\tau^*(A \cup B) > 0$, for $A, B \in I^X$.

(C3) For every subfamily $\{A_i : i \in J\} \subset I^X$, if $\tau^*(A_i) > 0$, then $\tau^*(\cap_{i \in J} A_i) > 0$.

If τ is an H-fuzzy topology on X, then a mapping $\tau^* : I^X \to I$, defined by $\tau^*(A) = \tau(A^c)$ where A^c denotes the complement of A, is an H-fuzzy cotopology. Conversely, if τ^* is an H-fuzzy cotopology on X, then a mapping $\tau : I^X \to I$, defined by $\tau(A) = \tau^*(A^c)$, is an H-fuzzy topology on X [3].

Let (X, τ) be an H-fuzzy topological space and $A \in I^X$. Then the H-fuzzy closure of A, denoted by A^-, is defined by

$$A^- = \cap\{K \in I^X : \tau^*(K) > 0, A \subset K\},$$

where $\tau^*(K) = \tau(K^c)$ [3].

In 1987, M. E. Abd El-Monsef et al. introduced a fuzzy supratopology [1] as the following : A subclass τ of I^X is called a fuzzy supratopology for the set X if

(1) $0_X, 1_X \in \tau$.

(2) For every subfamily $\{A_i : i \in J\} \subset I^X, \cup_{i \in J} A_i \in \tau$.

And the pair (X, τ) is called a fuzzy supratopological space.

2. Gradations of supraopenness

Definition 2.1. A gradation of supraopenness τ on X is a map $\tau : L^X \to L'$ satisfying the following properties, where $L = L' = [0,1]$:

(S1) $\tau^*(0_X) = \tau^*(1_X) = 1$.

(S2) For every subfamily $\{A_i : i \in J\} \subset I^X$, if $\tau(A_i) > 0$, then $\tau(\cup_{i \in J} A_i) > 0$.

Then the pair (X, τ) is called an H-fuzzy supratopological space. If τ is a crisp (i.e. $L' = \{0,1\}$), then the τ is a classical fuzzy supratopology on X [1].

Definition 2.2. A mapping $\tau^* : I^X \to I$ is called a gradation of supraclosedness if the following two conditions are satisfied:

(C1) $\tau^*(0_X) = \tau^*(1_X) = 1$.

(C2) For every subfamily $\{A_i : i \in J\} \subset I^X$, if $\tau^*(A_i) > 0$, then $\tau^*(\cap_{i\in J} A_i) > 0$.

Obviously we get the following theorem from Definition 2.2.

Theorem 2.3. *(1) Let τ be a gradation of supraopenness on X and $\tau^* : I^X \to I$ be a mapping defined by $\tau^*(A) = \tau(A^c)$, where A^c is the complement of A. Then τ^* is a gradiation of supraclosedness on X.*

(2) Let τ^* be a gradation of supraclosedness on X and $\tau : I^X \to I$ be a mapping defined by $\tau(A) = \tau^*(A^c)$. Then τ is a gradation of supraopenness on X.

Proof. (1). (C1) Obvious. (C2) For every subfamily $\{A_i : i \in J\} \subset I^X$, if $\tau^*(A_i) > 0$, then $\tau(A_i^c) > 0$. Since τ is a gradation of supraopenness, we get $\tau^*(\cap_{i\in J} A_i) = \tau(\cup_{i\in J} {A_i}^c) > 0$.

(2). Similar to (1). □

Definition 2.4. Let τ and σ be gradations of supraopenness on X. We say that τ is finer than σ or σ is coarser than τ (denoted by $\tau > \sigma$) is $\tau(A) \geq \sigma(A)$ for every $A \in I^X$.

Remark 2.5. Let X be a non-empty set. Let $\tau_0, \tau_1 : I^X \to I$ be defined by the rule:

$\tau_0(0_X) = \tau_0(1_X) = 1$.

$\tau_0(A) = 0$, for all $A \in I^X - \{0_X, 1_X\}$.

$\tau_1(A) = 1$, for all $A \in I^X$.

Then τ_0 and τ_1 are two gradations of supraopenness on X such that for any gradation of supraopenness τ on X,

$$\tau_0 \leq \tau \leq \tau_1.$$

Let $G_s(X)$ be the set of all gradations of supraopenness on X. Then we have:

Theorem 2.6. *$(G_s(X), \leq)$ is a complete lattice.*

Definition 2.7. Let (X,τ) be an H-fuzzy supratopological space and $A \in I^X$. Then

(1) The H-fuzzy supraclosure of A, denoted by $sl(A)$, is defined by

$$sl(A) = \cap\{K \in I^X : \tau^*(K) > 0, A \subset K\},$$

where $\tau^*(K) = \tau(K^c)$.

(2) The H-fuzzy suprainterior of A, denoted by $si(A)$, is defined by

$$si(A) = \cup\{K \in I^X : \tau(K) > 0, K \subset A\}.$$

Theorem 2.8. *Let (X,τ) be an H-fuzzy supratopogical space. Then for each $A, B \in I^X$*

(1) $si(1_X) = 1_X$,

(2) $si(A) \subset A$,

(3) $A \subset B \Rightarrow si(A) \subset si(B)$,

(4) $si(si(A)) = si(A)$,

(5) $si(A \cap B) \subset si(A) \cap si(B)$.

Proof. (1),(2) and (3) can be obtained from Definition 2.7.

(4) For each $A \in I^X$, we get $\tau(si(A)) = \tau(\cup\{K \in I^X : \tau(K) > 0, K \subset A\})$ by Definition 2.7. Since τ is a gradation of supraopenness on X, we can say $\tau(si(A)) > 0$. Consequently we have $si(si(A)) = si(A)$ from the concept of H-fuzzy suprainterior.

(5) From (2) we obtain $si(A \cap B) \subset si(A)$ and $si(A \cap B) \subset si(B)$, and so easily (5) is obtained. □

The following example shows that the equality of Theorem 2.8(5) is not true in general.

Example 2.9. Let $X = I$ and $\tau : I^X \to I$ be defined by

$$\tau(A) = \begin{cases} 0, & \text{if } A(x) \leq 1/2 \text{ for all } x \in X, \\ 1, & \text{otherwise,} \end{cases}$$

for each $A \in I^X - \{0_X\}$ and $\tau(0_X) = 1$. Now we consider two fuzzy sets A, B defined as the following:

$$A(x) = x, \text{ for all } x \in X,$$
$$B(x) = 1 - x, \text{ for all } x \in X.$$

Then $\tau(A) = \tau(B) = 1$, but $\tau(A \cap B) = 0$. Thus τ is a gradation of supraopennes. And we have $si(A) \cap si(B) = A \cap B$ and $si(A \cap B) = 0_X$, from the gradation τ of supraopenness. Thus $si(A) \cap si(B)$ is not equal to $si(A \cap B)$.

THEOREM 2.10. *Let* (X, τ) *be an H-fuzzy supratopological space and* $A, B \in I^X$. *Then*
(1) $sl(1_X) = 1_X$,
(2) $A \subset sl(A)$,
(3) $A \subset B \Rightarrow sl(A) \subset sl(B)$,
(4) $sl(A) = sl(sl(A))$,
(5) $sl(A) \cup sl(B) \subset sl(A \cup B)$.

Proof. Similar to Theorem 2.8. □

The following example shows the equality of Theorem 2.10(5) is not true in general.

EXAMPLE 2.11. Let $X = I$ and $\tau : I^X \to I$ be a gradation of suraopenness defined as Example 2.9. Let two fuzzy sets A, B be defined as the following:

$$A(x) = x, \qquad \text{for all } x \in X,$$
$$B(x) = 1 - x, \qquad \text{for all } x \in X.$$

Since $\tau^*(A) = \tau(A^c) = \tau(B) > 0$ and $\tau^*(B) = \tau(B^c) = \tau(A) > 0$, we have $A = sl(A)$ and $B = sl(B)$. Let C be a fuzzy set such that $C \neq 1_X$ and $A \cup B \subset C$. Then $\tau^*(C) = \tau(C^c) = 0$, and so $sl(A \cup B) = 1_X$. Thus we have $sl(A) \cup sl(B) \neq sl(A \cup B)$.

THEOREM 2.12. *Let* (X, τ) *be an H-fuzzy supratopological space and* $A \in I^X$. *Then*
(1) $(si(A))^c = sl(A^c)$.
(2) $(sl(A))^c = si(A^c)$.

Proof. The proof is obtained from Definition 2.7. □

THEOREM 2.13. *Let (X, τ) be an H-fuzzy supratopological space and $A \in I^X$. Then*
(1) $\tau(A) > 0$ iff $A = si(A)$.
(2) $\tau^*(A) > 0$ iff $A = sl(A)$.

Proof. (1) Let $\tau(A) > 0$. Then $A \subset \cup\{K \in I^X : \tau(K) > 0, K \subset A\} = si(A)$. Hence we get $A = si(A)$ from Theorem 2.6(2).
For the converse, let $A = si(A)$. Then

$$\tau(A) = \tau(si(A)) = \tau(\cup\{K \in I^X : \tau(K) > 0, K \subset A\}).$$

Since τ is a gradation of supraopenness, we have $\tau(A) > 0$
(2) Similar to (1). □

DEFINITION 2.14. Let (X, τ) be an H-fuzzy topological space and τ_s be a gradation of supraopenness on X. We call τ_s an associated gradation of supraopenness with τ on X if for every $A \in I^X, \tau(A) \leq \tau_s(A)$.

Given a gradation of openness on X, we can find an associated gradation of supraopenness with τ as the following example shows.

EXAMPLE 2.15. Let (X, τ) be an H-fuzzy topological space. Define $\tau_s : I^X \to I$ as

$$\tau_s(A) = \begin{cases} \tau(si(A)), & \text{if } A \subset sl(si(A)), \\ 0, & \text{otherwise.} \end{cases}$$

Then clearly $\tau_s(0_X) = \tau_s(1_X) = 1$. For any index set J, let $\tau_s(A_i) > 0$ for all $i \in J$. Easily we can show $\cup A_i \subset sl(si(\cup A_i))$, and from Theorem 2.8 and Theorem 2.13, we get $\tau_s(\cup A_i) = \tau(si(\cup A_i)) > 0$. Thus τ_s is a gradation of supraopenness on X. Let $\tau(A) > 0$ for $A \in I^X$. Since $A \subset sl(si(A))$ and $\tau(A) = \tau(si(A))$, we have $\tau(A) \leq \tau_s(A)$ for each $A \in I^X$. Thus τ_s is an associated gradation of supraopenness with τ.

THEOREM 2.16. *Let (X, τ) be an H-fuzzy supratopological space and $Y \subset X$. Define a mapping $\tau_Y : I^Y \to I$ by*

$$\tau_Y(A) = \cup\{\tau(B) : B \in I^X, B/Y = A\},$$

where B/Y is the resriction of B on X. Then τ_Y is a gradation of supraopenness on X.

Proof. It is clear that $\tau_Y(0_X) = \tau_Y(1_X) = 1$.

For every subfamily $\{A_i : i \in J\} \subset I^Y$. Let $\tau_Y(A_i) > 0$. Since $\{\cup_{i\in J} C_i : C_i \in I^X, C_i/Y = A_i\} \subset \{C \in I^X : C/Y = \cup A_i\}$, we get $\tau(\cup_{i\in J} A_i) > 0$. □

Definition 2.17. The H-fuzzy supratopological space (Y, τ_Y) is called a subsupraspace of (X, τ) and τ_Y is called the induced gradation of supraopenness on Y from τ.

Theorem 2.18. *Let (Y, τ_Y) be an H-fuzzy subsupraspace of (X, τ) and $B \in I^Y$. Then*
(1)$\tau_Y{}^*(B) = \cup\{\tau^*(C) : C \in I^X, C/Y = B\}$.
(2) If $D \subset Y \subset X$, then $\tau_D = (\tau_Y)_D$.

Proof.

$$\begin{aligned}\tau_Y{}^*(B) &= \tau_Y(B^c)\\ &= \cup\{\tau(C) : C \in I^X, C/Y = B^c\}\\ &= \cup\{\tau(C^c) : C^c \in I^X, C^c/Y = B\}\\ &= \cup\{\tau^*(C^c) : C^c \in I^X, C^c/Y = B\}.\end{aligned}$$

$$\begin{aligned}(\tau_Y)_D(B) &= \cup\{\tau_Y(E) : E \in I^Y, E/D = B\}\\ &= \cup\{\cup\{\tau(C) : C \in I^X, C/Y = E\} : E \in I^Y, E/D = B\}\\ &= \cup\{\tau(C) : C \in I^X, C/D = B\}\\ &= \tau_D(B).\end{aligned}$$

□

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