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NOTE ON T-PARTS OF BCI-ALGEBRAS RELATIVE TO SUBSETS

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Abstract. In this paper, we introduce the notion of T-part of BCI-algebras relative to a subset. We show that if A is a subalgebra of a BCI-algebra X, then so is the T-part $T_A(X)$ of X relative to A. We provide equivalent conditions that the T-part of X relative to A is an ideal.

1. Introduction

The notion of BCK-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki [I] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Recently, X. Zhang and Y. B. Jun [ZJ] introduced the notion of T-part in a BCI-algebra X and showed that the T-part of X is a subalgebra of X.

In this paper, we introduce the notion of T-part of BCI-algebras relative to subsets. We give equivalent conditions that the T-part of X relative to A is an ideal.

We begin with some definitions and properties that will be useful in our results.

Definition 1.1. A BCI-*algebra* is an algebra $(X;*,0)$ of type (2,0) is satisfying the following axioms for all $x, y, z \in X$:

(a) $((x*y)*(x*z))*(z*y)=0$
(b) $(x*(x*y))*y=0$
(c) $x*x=0$
(d) $x*y=0$ and $y*x=0$ imply $x=y$.

For any BCI-algebra X, the relation $\leq$ defined by $x \leq y$ if and only if $x*y=0$ is a partial order on X.

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A BCI-algebra X has the following properties for all $x, y, z \in X$:

(1) $x * 0 = x$

(2) $(x * y) * z = (x * z) * y$

(3) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

Definition 1.2. A non-empty subset I of a BCI-algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$.

It is well known that an ideal I of a BCI-algebra X need not be a subalgebra.

Definition 1.3. Let X be a BCI-algebra and let A be a non-empty subset of X. The set

$$T_A(X) = \{(0 * x) * x \mid x \in A\}$$

is called the *T-part of X relative to A*. If $A = X$, then $T_X(X) := T(X)$ is called the *T-part* of X ([ZJ]).

Clearly, if $0 \in A$, then $0 \in T_A(X)$.

Example 1.4. If X is an associative BCI-algebra or a BCK-algebra, then $T_A(X) = \{0\}$, for every nonempty subset A of X. If X is a BCI-algebra, then $T_{G(X)}(X) = \{0\}$, where $G(X) = \{x \in X \mid 0 * x = x\}$.

Definition 1.5. [MX] An element a of X is called an *atom* if, for all $x \in X$, $x * a = 0$ implies $x = a$. The set $L(X) := \{a \in X | a \text{ is an atom of } X\}$ is called the *p-semisimple part* of X. For any $a \in L(X)$, the set $V(a) := \{x \in X | a \leq x\}$ is called a *branch* of X. The set $X_+ := \{x \in X | 0 \leq x\}$ is called the *BCK-part* of X.

Clearly, 0 is an atom of X. We know that $(L(X), *, 0)$ is a p-semisimple BCI-algebra and $(X_+, *, 0)$ is a BCK-algebra. It is clear that $V(0) = X_+$ ([MX]).

2. T-parts relative to subsets

Proposition 2.1. *If A is a subalgebra of a BCI-algebra X, then so is $T_A(X)$.*

Proof. Let $x, y \in T_A(X)$. Then $x = (0 * a) * a$ and $y = (0 * b) * b$ for some $a, b \in A$. Hence

$$\begin{aligned} x * y &= ((0 * a) * a) * ((0 * b) * b) \\ &= ((0 * ((0 * b) * b)) * a) * a \\ &= (((0 * (0 * b)) * (0 * b)) * a) * a \\ &= (((0 * a) * (0 * b)) * a \\ &= ((0 * (a * b)) * (0 * b)) * a \\ &= ((0 * (0 * b)) * (a * b) * a \\ &= ((0 * a) * (0 * b)) * (a * b) \\ &= (0 * (a * b)) * (a * b). \end{aligned}$$

Thus $x * y \in T_A(X)$. This completes the proof. □

Corollary 2.2. *([ZJ]) If X is a BCI-algebra, then $T(X)$ is a subalgebra of X.*

Proposition 2.3. *If $A \subset B$, then $T_A(X) \subset T_B(X)$.*

Proof. Let $x \in T_A(X)$. Then $x = (0 * a) * a$ for some $a \in A$. Since $A \subset B$, $x \in T_B(X)$. Thus, $T_A(X) \subset T_B(X)$. □

Corollary 2.4. *If A is a non-empty subset of X, then $T_A(X) \subset T(X)$.*

Proposition 2.5. *If $f : X \to Y$ is a surjective homomorphism from a BCI-algebra X onto a BCI-algebra Y, then $f(T_A(X)) = T_{f(A)}(Y)$.*

Proof. Let $x \in T_{f(A)}(X)$. Then $x = (0 * y) * y$ for some $y \in f(A)$. Since $y = f(a)$, there exists $a \in A$ such that $y = f(a)$. It follows that $x = (0 * f(a)) * f(a) = f((0 * a) * a)$ since f is a homomorphism. Thus, $x \in f(T_A(X))$. Conversely, let $x \in f(T_A(X))$. Then $x = f(a)$ for some $a \in T_A(X)$. Hence there exists $b \in A$ such that $a = (0 * b) * b$. Thus $x = f(a) = f((0 * b) * b) = (0 * f(a)) * f(a)$. Since $f(a) \in f(A)$, we have $x \in T_{f(A)}(X)$. This completes the proof. □

Corollary 2.6. *If $f : X \to Y$ is a surjective homomorphism from a BCI-algebra X onto a BCI-algebra Y, then $f(T(X)) = T(Y)$.*

Theorem 2.7. *Let X be a BCI-algebra and let A be a non-empty subset of X. The following are equivalent :*

(i) $T_A(X)$ *is an ideal of* X,

(ii) $x * a = y * a$ *implies* $x = y$ *for all* $x, y \in X_+$ *and* $a \in T_A(X)$,

(iii) $x * a = 0 * a$ *implies* $x = 0$ *for all* $x \in X_+$ *and* $a \in T_A(X)$.

Proof. (i) ⇒ (ii) Assume that $T_A(X)$ is an ideal of X and let $x*a = y*a$ for all $x, y \in X_+$ and $a \in T_A(X)$. Then $(x*y)*a = (x*a)*y = (y*a)*y = (y*y)*a = 0*a \in T_A(X)$. Since $T_A(X)$ is an ideal of X, we have $x*y \in T_A(X)$, we have $x*y \in T_A(X)$. On the other hand, note that $x*y \in X_+$ and $X_+ \cap T_A(X) \subseteq X_+ \cap L(X) = \{0\}$. It follows that $x*y = 0$ and hence, $x \leq y$. Similarly, we get $y \leq x$, and therefore $x = y$.

(ii) ⇒ (iii) It is straightforward.

(iii) ⇒ (i) Suppose that (iii) holds. Assume that $x*y \in T_A(X)$ and $y \in T_A(X)$ for all $x, y \in X$. Denote $t = 0*(0*x)$. Then $t \in L(X)$. Since $t = 0*(0*x) \leq x$, it follows from (3) that $t*y \leq x*y$. Hence $x*y \in V(t*y)$. From $x*y \in T_A(X) \subseteq L(X)$, we have $x*y = t*y$. Thus $(x*t)*y = (x*y)*t = (t*y)*t = (t*t)*y = 0*y$. It follows from (iii) that $x*(0*(0*x)) = x*t = 0$, since $x*t \in X_+$. Thus $x = 0*(0*x) \in L(X)$. Since $T_A(X)$ is an ideal of $L(X)$, we have $x \in T_A(X)$, and hence $T_A(X)$ is an ideal of X. This completes the proof. □

Definition 2.8. ([Z]) A non-empty subset A of a BCI-algebra X is called a T-*ideal* of X if it satisfies :

(T1) $0 \in A$,

(T2) $x*(y*z) \in A$ and $y \in A$ imply $x*z \in A$ for all $x, y, z \in X$.

It is well known that every T-ideal of a BCI-algebra is an ideal (see, [Z, Theorem 1]), but not conversely ([ZJ]).

Lemma 2.9. *([Z]) Let A be a T-ideal of a BCI-algebra X. Then $(0*x)*x \in A$ for all $x \in X$.*

Proposition 2.10. *If A is a T-ideal of a BCI-algebra X, then we have $T_A(X) \subseteq A$.*

Proof. If $x \in T_A(X)$, then $x = (0*a)*a$ for some $a \in A$. By Lemma 2.9, we have $x = (0*a)*a \in A$. This completes the proof. □

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