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ON THE RANKS OF SUBFORMS OF SOME LINEAR FORMS WHICH LEFT FROBENIUS NUMBERS INVARIANT

Byeong Moon Kim

Abstract. Suppose linear forms whose coefficients are arithmetic progressions are given. In this paper, we will find the upper and lower bounds of the minimal ranks of subforms of these forms which left the Frobenius numbers invariant. This is an improvement of Ritter's bounds.

1. Introduction

Let $a_1 < a_2 < \ldots < a_n$ be positive integers with $(a_1, a_2, \ldots, a_n) = 1$. The Frobenius number of a linear form $f = f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ is the largest integer m such that the equation $f = m$ has no non-negative integer solutions and it is denoted by $G(f)$. A linear form $g = g(x_1, x_2, \ldots, x_m) = b_1x_1 + b_2x_2 + \ldots + b_mx_m$ is a subform of f, if $\{b_1, b_2, \ldots, b_m\} \subset \{a_1, a_2, \ldots, a_n\}$. The rank of $f = f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ is the number of variables n and it is denoted by $R(f)$. For $a, d, n \in \mathbb{Z}^+$, such that $2 \le n \le a$ and $(a, d) = 1$. we define $f_{a,d,n} = ax_0 + (a + d)x_1 + \ldots + (a + (n-1)d)x_{n-1}$. Roberts [2] proved $G(f_{a,d,n}) = [\frac{a-2}{n-1}]a + (a-1)d$. Let $H(a, d, n) = \min\{R(g) |\ g \text{ is a subform of } f_{a,d,n} \text{ and } G(g) = G(f_{a,d,n})\}$. It is easy to see that if $n \ge 3$, $H(a, d, n) \ge 3$. In 1977, Selmer [3] proved if $d > a(a-2)$, $H(a, d, a) = 3$. In fact, he proved if $d > a(a-2)$, $G(f_{a,d,a}) = G(ax_0 + (a+d)x_1 + (a+(a-1)d)x_{a-1})$. In 1999, Ritter [1] proved if $3 \le n < a-1$, $3 \le H(a, d, n) \le 4\sqrt{n}$. He also proved if $n = a-1 > 7$ and $d = 1$, $H(a, d, n) \ge \frac{\sqrt{n}}{2}$. In this paper, we will find upper and lower bounds of $H(a, d, n)$ and improve Ritter's bounds as a Corollary.

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2. Main Result

Theorem 1. *Let* $a, d, n \in \mathbb{Z}^+$, $(a, d) = 1$, $3 \leq n < a$ *and* $[\frac{a-2}{n-1}]+1 \geq r$. *Then,*

$$H(a, d, n) \leq (2r-1)\sqrt[r]{n} + 2.$$

Proof. Let $q = [\frac{a-2}{n-1}]+1$, $t = [\frac{a-2}{q}]+1$, $(s-1)^r \leq t < s^r$ and $p = [\frac{t}{s^{r-1}}]$. Note that $n \geq t$. So $\sqrt[r]{n} + 1 \geq \sqrt[r]{t} + 1 \geq s$ and $p \leq s-1$. Let

$$A_1 = \{0, 1, \ldots, s-1\},$$

$$A_2 = \{s, 2s, \ldots, s(s-1)\},$$

$$A_3 = \{s^2, 2s^2, \ldots, s^2(s-1)\},$$

$$\ldots,$$

$$A_{r-1} = \{s^{r-2}, 2s^{r-2}, \ldots, s^{r-2}(s-1)\},$$

$$B_1 = \{t, t-1, \ldots, t-s+1\},$$

$$B_2 = \{t-s, t-2s, \ldots, t-s(s-1)\},$$

$$\ldots,$$

$$B_r = \{t-s^{r-1}, t-2s^{r-1}, \ldots, t-ps^{r-1}\},$$

$$T = A_1 \cup A_2 \cup \ldots \cup A_{r-1} \cup B_1 \cup B_2 \cup \ldots \cup B_r$$

and

$$S = \{a + bi | i \in T\} = \{b_1, b_2, \ldots, b_u\}.$$

Then,

$$\begin{aligned} u &\leq s + (r-2)(s-1) + s + (r-2)(s-1) + p \\ &\leq (2r-3)(s-1) + 2 \\ &\leq (2r-1)\sqrt[r]{n} + 2. \end{aligned}$$

Let $g = b_1x_1 + b_2x_2 + \ldots + b_nx_n$. Suppose $x > qa + (a-1)d = G(f_{a,d,n})$. There are $\alpha, \beta \in \mathbb{Z}$ such that $x = \alpha a + \beta d$ and $0 \leq \beta < a$. Since

$$\alpha a + \beta d - (qa + (a-1)d) = (\alpha - q)a + (\beta - a + 1)d > 0,$$

$\alpha \geq q \geq r$. If $0 \leq \beta < s^{r-1}(s-1)$, $\beta = c_1 + c_2s + \ldots + c_rs^{r-1}$ for some $c_1, c_2, \ldots, c_{r-1} \in \{0, 1, \ldots, s-1\}$ and $0 \leq c_r \leq s-2$. Then,

$$x = (a + c_1d) + (a + c_2sd) + \ldots + (a + c_rs^{r-1}d) + (\alpha - r)a.$$

So x is represented by g.

If $s^{r-1}(s-1) \le \beta < t$, $0 < t-\beta \le s^r - s^{r-1}(s-1) = s^{r-1}$. So $t-\beta = d_1+d_2s+\ldots+d_{r-1}s^{r-2}$ for some $d_1, d_2, \ldots, d_{r-1} \in \{0,1,\ldots,s-1\}$. Then, since

$$\begin{aligned}\beta &= t - d_{r-1}s^{r-2} - d_{r-2}s^{r-3} - \ldots - d_1\\ &= (t-(d_{r-1}+1)s^{r-2}) + (s-d_{r-2}+1)s^{r-3}\\ &\quad +(s-d_{r-3}+1)s^{r-4} + \ldots + (s-d_2+1)s + (s-d_1),\end{aligned}$$

$$\begin{aligned}x &= (a+(t-(d_{r-1}+1)s^{r-2})d) + (a+(s-d_1)d) + (a+(s-d_2+1)sd)\\ &\quad +(a+(s-d_3+1)s^2d) + \ldots + (a+(s-d_{r-2}+1)s^{r-3}d).\end{aligned}$$

So x is represented by g.

If $\beta \ge t$, $jt \le \beta < (j+1)t$, for $1 \le j \le q-1$. Then,

$$ks^{r-j-1} < (j+1)t - \beta \le (k+1)s^{r-j-1}$$

for some $0 \le k < (p+1)s^{j+1}$.

Since $0 \le \beta - (j+1)t + (k+1)s^{r-j-1} < s^{r-j-1}$,

$$\beta - (j+1)t + (k+1)s^{r-j-1} = e_1 + e_2s + \ldots + e_{r-j-1}s^{r-j-2}$$

for some $e_1, e_2, \ldots, e_{r-j-1} \in \{0,1,\ldots,s-1\}$.

If $k \le (p+1)s^{j+1} - 2$, $k+1 = f_1 + f_2s + \ldots + f_{j+1}s^j$ for some $f_1, f_2, \ldots f_j \in \{0,1,\ldots,s-1\}$ and $\le f_{j+1} \le p$. Then,

$$\begin{aligned}\beta &= (j+1)t - (k+1)s^{r-j-1} + e_1 + e_2s + \ldots + e_{r-j-1}s^{r-j-2}\\ &= (t-f_1s^{r-j-1}) + (t-f_2s^{r-j}) + \ldots + (t-f_{j+1}s^{r-1})\\ &\quad +e_1 + e_2s + \ldots + e_{r-j-1}s^{r-j-2}.\end{aligned}$$

So

$$\begin{aligned}x &= (a+(t-f_1s^{r-j-1})d) + (a+(t-f_2s^{r-j})d) + \ldots + (a+(t-f_{j+1}s^{r-1})d)\\ &\quad +(a+e_1d) + (a+e_2sd) + \ldots + (a+e_{r-j-1}s^{r-j-2}d) + (\alpha-r)a\end{aligned}$$

So x is represented by g.

If $k = (p+1)s^{j+1} - 1$, since

$$\begin{aligned}0 \ge jt-\beta &= (j+1)t - \beta - t > ks^{r-j-1} - t\\ &> ((p+1)s^{j+1}-1)s^{r-j-1}) - (p+1)s^{r-1} = -s^{r-j-1},\\ \beta - jt &= g_1 + g_2s + \ldots + g_{r-j-1}s^{r-j-2}\end{aligned}$$

for some $g_1, g_2, \ldots g_{r-j-1} \in \{0,1,\ldots,s-1\}$. Then,

$$\beta = jt + g_1 + g_2s + \ldots + g_{r-j-1}s^{r-j-2}.$$

So

$$x = j(a+td) + (a+g_1d) + (a+g_2)sd) + \ldots$$

$$+(a+g_{r-j-1}s^{r-j-2}d)+(\alpha-r+1)a.$$

So x is represented by g. Thus, $G(g)=qa+(a-1)d$. So

$$H(a,d,n)\le u\le(2r-1)\sqrt[r]{n}+2.$$

□

Corollary 1. *Let $a,d,n\in\mathbb{Z}^+$, $(a,d)=1$, $3\le n<a$. Then,*

$$H(a,d,n)\le\min_{2\le r\le q}\{(2r-1)\sqrt[r]{n}+2\},$$

where $q=[\frac{a-2}{n-1}]$.

Corollary 2. *Let $a,d,n\in\mathbb{Z}^+$, $(a,d)=1$, $3\le n<a$. Then,*

$$H(a,d,n)\le 3\sqrt{n}+2.$$

This is an improvement of the result of Ritter [1].

Theorem 2. *Let $a,d\in\mathbb{Z}^+$, $(a,d)=1$ and $3\le a$. Then, if $d=1$, $H(a,d,a)=a$ and if $d\ge 2$,*

$$[\ \frac{a}{d}\ ]\le H(a,d,a)\le[\ \frac{a}{d}\ ]+2[\ \sqrt{\frac{(d-1)}{d}a}\ ]+1.$$

Proof. For all i such that $a-[\ \frac{a}{d}\ ]<i\le a$,

$$a+id>(a-1)d.$$

So it is trivial that $[\ \frac{a}{d}\ ]\le H(a,d,a)$. For $0\le i\le a-1$, the linear form g is made from $f_{a,1,a}$ by deleting a term $(a+i)x_i$. Then, g can't represent $a+i$, which is larger than the Frobenius number $a-1$ of $f_{a,1,a}$. So $H(a,1,a)=1$.

Suppose $d\ge 2$. Let $t=[\frac{a}{d}]$ and $s=[\ \sqrt{\frac{(d-1)a}{d}}\ ]$. Let

$$A=\{a+(a-1)d,a+(a-2)d,\dots,a+(a-t-1)d\},$$
$$B=\{a+d,a+2d,\dots,a+(s-1)d\},$$
$$C=\{a+sd,a+2sd,\dots,a+s(s-1)d\},$$
$$D=A\cup B\cup C=\{a_1<a_2<\dots<a_n\}$$

and

$$g=a_1x_1+a_2x_2+\dots+a_nx_n.$$

Then,

$$n=t+2s+1=[\ \frac{a}{d}\ ]+2[\ \sqrt{\frac{(d-1)}{d}a}\ ]+1.$$

If $x > (a-1)d$ and $x = \alpha a + \beta d$ for $1 \le \beta < a$, since

$$x - (a-1)d = \alpha a + (\beta - a + 1)d > 0.$$

$\alpha \ge 1$. If $0 \le \beta \le a - t - 1$,

$$\alpha a > (a - 1 - \beta)d \ge (t+1)d.$$

Since

$$\alpha > (t+1)\frac{d}{a} = ([\frac{a}{d}] + 1)\frac{d}{a} \ge 1,$$

$\alpha \ge 2$. Then, since

$$0 \le \beta \le a - t - 2 \le a - \frac{a}{d} - 1 < \frac{(d-1)a}{d} \le s^2,$$

$\beta = b_1 + b_2 s$ for $0 \le b_1, b_2 \le s - 1$. So

$$x = \alpha a + \beta d = (a + b_1 d) + (a + b_2 s d) + (\alpha - 2)a.$$

So x is represented by g. If $a - t - 1 \le \beta \le a - 1$, $\beta \in A$. So

$$x = (a + \beta d) + (\alpha - 1)a.$$

So x is represented by g. So $H(a, d, a) \le [\ \frac{a}{d}\ ] + 2[\ \sqrt{\frac{(d-1)}{d}a}\ ] + 1$. □

Theorem 3. *Let $a, d \in \mathbb{Z}^+$, $(a, d) = 1$ and $3 \le a < d$. Then,*

$$H(a, d, a) \le [\log_{t+1}(a-2)] + 3$$

where $t = [\frac{d}{a}]$.

Proof. Let $s = [\log_{t+1}(a-2)]$,

$$A = \{a + (a-1)d, a + (a-2)d, \ldots, a + (a-t-2)d\}$$

$\cup\{a+(a-1-(t+1))d, a+(a-1-(t+1)^2)d, \ldots, a+(a-1-(t+1)^s)d\}$. Suppose $x = \alpha a + \beta d > (a-1)d$. Then, since $(t+1)^{s+1} \ge a+2$, $(a-1-(t+1)^{s+1} \le 2$. So β is 0, 1, $a-1$ or

$$a - 1 - (t+1)^{i+1} \le \beta < a - 1 - (t+1)^i$$

for some $i \in \{0, 1, \ldots, s\}$. Then, since $\alpha a > (a - 1 - \beta)d$,

$$\alpha > (t+1)^i \frac{d}{a} \ge (t+1)^i t.$$

If $i \le s - 1$,

$$x = (a + (a - 1 - (t+1)^{i+1})d)$$
$$+(\beta + (t+1)^{i+1} + 1 - a)(a + d)$$
$$+(\alpha - 2 - \beta - (t+1)^{i+1} + a)a.$$

Since

$$0 \le \beta + (t+1)^{i+1} + 1 - a$$
$$< a + (t+1)^i - (a - 1 - (t+1)^{i+1}) = t(t+1)^i,$$
$$\alpha - 2 - \beta - (t+1)^{i+1} + a \ge \alpha - 1 - t(t+1)^i \ge 0.$$

So x is represented by g.

If $i = s$, since $a - 1 \le (t+1)^{s+1}$,

$$\beta < a - 1 - (t+1)^s + 1 \le (t+1)^{s+1} - (t+1)^s \le \alpha.$$

So x is represented by g.

If β is 0, 1, $a-1$, it is trivial that g represent x. □

Corollary 3. *If* $d \ge a[\sqrt[r-2]{a-2}]$, $H(a,d,a) \le r$.

Corollary 4. (**Selmer**)*If* $a, d \in \mathbf{Z}^+$, $(a,d) = 1$, $3 \le a$ *and* $d > a(a-2)$, $H(a,d,a) = 3$.

Corollary 5. *If* $a, d \in \mathbf{Z}^+$, $(a,d) = 1$, $3 \le a$ *and* $d > a\sqrt{a-2}$, $H(a,d,a) \le 4$.

Theorem 4. *Let* $q \ge 3$, $q^2 | k$, $n = n(k,q) = \frac{1}{q}\binom{k}{q} + 1$ *and* $a = a(k,q) = \binom{k}{q} + 2$. *Then,*

$$H(a,1,n) \ge k+1.$$

Proof. Suppose there is a subset

$$A = \{i_1, i_2, \ldots, i_k\} \subset \{0, 1, \ldots, n-1\},$$

such that if $g = g(x_1, x_2, \ldots, x_k) = (a+i_1)x_1 + (a+i_2)x_2 + \ldots + (a+i_k)x_k$,

$$G(g) = qa - 1 = G(a,1,n).$$

Then, for all $j \in \{0, 1, \ldots, a-1\} = B$, there are non-negative integers $x_0, x_1, \ldots, x_k$ such that

$$qa + j = (a+i_1)x_1 + (a+i_2)x_2 + \ldots + (a+i_k)x_k$$
$$= (x_1 + x_2 + \ldots + x_k)a + (i_1x_1 + i_2x_2 + \ldots + i_kx_k).$$

Since $i_1 + i_2 + \ldots + i_k \equiv j \pmod{a}$,

$$i_1 + i_2 + \ldots + i_k \ge j.$$

So $x_1 + x_2 + \ldots + x_k \le q$. Since

$$i_1 + i_2 + \ldots + i_k \le qn \le a - 1,$$

$i_1 + i_2 + \ldots + i_k = j$. So

$$\begin{aligned} j \in C &= \{i_1x_1 + i_2x_2 + \ldots + i_kx_k | x_1 + x_2 + \ldots + x_k \le q\} \\ &= \{l_1 + l_2 + \ldots + l_q | l_1, l_2, \ldots, l_q \in A\}. \end{aligned}$$

But, since $C \supset B$,

$$|C| = \binom{k}{q} \ge |B| = a = \binom{k}{q} + 2.$$

This gives a contradiction. So $H(a, 1, n) \ge k + 1$. □

Corollary 6. *Let* $4|k$, $n = \frac{k(k+1)+4}{4}$ *and* $a = \frac{k(k+1)+4}{2}$. *Then,*

$$H(a, 1, n) \ge k + 1 = \frac{1 + \sqrt{16n - 15}}{2}.$$

This improves the lower bound of Ritter [1].

3. Open problems

Let

$$\mathcal{F}(r) = \{a | (a, d) = 1, a \ge rn - n - r + 3\},$$
$$H_r(d, n) = \max_{a \in \mathcal{F}(r)} H(a, d, n),$$
$$H_r^+(d) = \limsup_{n \to \infty} \frac{H(d, n)}{\sqrt[r]{n}}$$

and

$$H_r^-(d) = \liminf_{n \to \infty} \frac{H(d, n)}{\sqrt[r]{n}}.$$

Trivially, $H_1^+(1) = H_1^-(1) = 1$. The Ritter's bound is

$$H_2^+(d) \le 4.$$

From Theorem 1, we obtain if $r \ge 2$,

$$H_r^+(d) \le 2r - 1.$$

Especially, from Corollary 2 and 6, we obtain

$$H_2^+(d) \le 3.$$

From Theorem 4, we obtain

$$\sqrt[r]{r \cdot r!} \le H_r^+(1).$$

For $r=2$, we improved Ritter's lower bound $H_2^+(1) \geq \frac{1}{2}$ to $H_2^+(1) \geq 2$. From Theorem 2, we obtain

$$H_1^-(d) = H_1^+(d) = \frac{1}{d} = \frac{H_1^+(1)}{d}.$$

The basic open problem is

1. Compute $H_r^+(d)$ and $H_r^-(d)$.

But it seems to take a long time to solve it completely. So we want to answer the following problems at first.

2. Improve the bounds of $H_r^+(d)$ and $H_r^-(d)$.
3. Is $H_r^+(d) = H_r^-(d)$?
4. Is $H_r^+(d) = \frac{H_r^+(1)}{d}$ and $H_r^-(d) = \frac{H_r^-(1)}{d}$?

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Department of Mathematics,
Kangnung National University
Kangnung, Korea.
E-mail: kbm@knusun.kangnung.ac.kr