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FIXED POINTS OF SUMS OF NONEXPANSIVE MAPS AND COMPACT MAPS

Jongsook Bae and Daejong An

Abstract. Let X be a Banach space satisfying Opial's condition, C a weakly compact convex subset of X, $F : C \to X$ a nonexpansive map, and let $G : C \to X$ be a compact and demiclosed map. We prove that $F + G$ has a fixed point in C if $F + G : C \to X$ is a weakly inward map.

1. Introduction

Let C be a nonempty subset of a Banach space X. A map $T : C \to X$ is called nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$. A map $T : C \to X$ is said to be a Lipschitzian if there exists $k \geq 0$ such that for all $x, y \in C$, $\|T(x) - T(y)\| \leq k\|x - y\|$; it is a contraction if $0 < k < 1$ and is a compact if it is continuous and maps bounded sets to relatively compact sets.

In 1955, Krasnoselskii proved the following theorem which can be found in [1].

Theorem 1. *Let C be a nonempty bounded closed convex set in a Banach space X. Let $F : C \to X$ be a contraction map, and let $G : C \to X$ be a compact map. If $(F + G)(C) \subseteq C$, then $F + G$ has a fixed point in C.*

Let X be a Banach space and $\mathcal{B}$ the family of its bounded subsets. Then $\alpha : \mathcal{B} \to [0, \infty]$, defined by

$$\alpha(B) = \inf\{d > 0 | B \text{ admits a finite cover by sets of diameter } \leq d\ \},$$

is called the Kuratowski measure of noncompactness ([2]). It is not only natural but also useful since α has interesting properties, some of which are listed in Proposition 2 ([4]).

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Proposition 2. *Let X be a Banach space, $\mathcal{B}$ the family of all bounded sets of X, and let $\alpha : \mathcal{B} \to [0, \infty)$ be the Kuratowski measure of noncompactness. Then*

(a) *$\alpha(B) = 0$ iff $\bar{B}$ is compact.*

(b) *α is a seminorm, i.e.,*

$$\alpha(\lambda B) = |\lambda|\alpha(B) \text{ and}$$
$$\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2).$$

(c) *$B_1 \subseteq B_2$ implies $\alpha(B_1) \leq \alpha(B_2)$;*
$\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}$.

(d) *$\alpha(convB) = \alpha(B)$.*

(e) *α is continuous with respect to the Hausdorff metric H, defined by $H(B_1, B_2) = \max\{\sup_{B_1} d(x, B_2), \sup_{B_2} d(x, B_1)\}$; in particular $\alpha(\bar{B}) = \alpha(B)$.*

For $D \subset X$, $T : D \to X$ is called condensing if $\alpha(T(B)) < \alpha(B)$ for any bounded subset B of D with $\alpha(B) > 0$. Let K be a convex subset of a Banach space X and $x \in K$. The inward set $I_K(x)$ of K at x is defined by

$$I_K(x) = \{r(y - x) + x | y \in K, r \geq 1\}.$$

A map $T : K \to X$ is called inward if for all $x \in K$, $T(x) \in I_K(x)$, and T is said to be weakly inward if for all $x \in K$, $T(x) \in \overline{I_K(x)}$.

In 1979, Deimling proved the following theorem which can be found in [3].

Theorem 3. *Let X be a Banach space, $D \subset X$ closed bounded convex, $F : D \to X$ a continuous, condensing and weakly inward map. Then F has a fixed point.*

Remark. Let C be a nonempty bounded closed convex set in a Banach space X. Let $F : C \to X$ be a contraction map and $G : C \to X$ a compact map. Then $T = F + G$ is a continuous and condensing map. Indeed, if B is a subset of C and $\alpha(B) > 0$, then we have

$$\begin{aligned} \alpha(T(B)) &= \alpha(F(B) + G(B)) \\ &\leq \alpha(F(B)) + \alpha(G(B)) \\ &\leq \alpha(F(B)) + \alpha(\overline{G(B)}) \\ &= \alpha(F(B)) \\ &< \alpha(B). \end{aligned}$$

From Theorem 3 and Remark, Theorem 1 can be restated as follows; Let C be a nonempty bounded closed convex set in a Banach space X. Let $F : C \to X$ be a contraction map, and let $G : C \to X$ be a compact map. If $T = F + G : C \to X$ is a weakly inward map, then T has a fixed point.

The purpose of this paper generalizes the above result by replacing nonexpansive maps instead of contraction maps.

2. The results

A Banach space X is said to satisfy Opial's condition if whenever a sequence $\{x_n\}$ in X converges weakly to x_0, then for $x \neq x_0$,

$$\lim_{n\to\infty} \inf \|x_n - x_0\| < \lim_{n\to\infty} \inf \|x_n - x\| \quad ([5]).$$

Let D be a subset of a Banach space X. A map $T : D \to X$ is said to be demiclosed if for any sequence $\{x_n\}$ in D the following implication holds:

$$w - \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} \|T(x_n) - w\| = 0$$

implies

$$x \in D \text{ and } T(x) = w.$$

Theorem 4. *Let X be a Banach space which satisfies Opial's condition, and C a nonempty weakly compact convex subset of X. Let $F : C \to X$ be a nonexpansive map and $G : C \to X$ a compact and demiclosed map. If $T = F + G : C \to X$ is a weakly inward map, then T has a fixed point.*

Proof. Without loss of generality we may assume $0 \in C$. Let $0 < k < 1$. Then by the same way as in Remark we can show that, $kT = kF + kG$ is continuous, k-condensing. Also it can be easily shown that $kT : C \to X$ is a weakly inward map (See [3]). From Theorem 3, kT has a fixed point, i.e., there exists x_k in C such that $kTx_k = x_k$. In this case we have

$$\|x_k - Tx_k\| = \|x_k - \frac{x_k}{k}\| = \frac{1-k}{k}\|x_k\| \to 0 \ as \ k \to 1^-.$$

Since C is weakly compact and G is a compact map we can take a sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to x for some $x \in C$, $\|Tx_n - x_n\| \to 0$ and $G(x_n) \to y$ for some $y \in X$. Then we have

$$\begin{aligned}\|Tx_n - Fx - y\| &\leq \|Fx_n - Fx\| + \|Tx_n - Fx_n - y\| \\ &\leq \|x_n - x\| + \|Tx_n - Fx_n - y\|.\end{aligned}$$

Hence we have

$$\begin{aligned}&\liminf \|x_n - (Fx + y)\| \\ &\leq \liminf(\|x_n - Tx_n\| + \|Tx_n - (Fx + y)\|) \\ &\leq \liminf(\|x_n - Tx_n\| + \|x_n - x\| + \|Tx_n - (Fx_n + y)\|) \\ &= \liminf \|x_n - x\|.\end{aligned}$$

Since X satisfies Opial's condition, we have $Fx + y = x$. And since G is demiclosed $Gx = y$ so that $Tx = Fx + Gx = x$. Hence T has a fixed point. □

COROLLARY 5. *Let X be a reflexive Banach space which satisfies Opial's condition, and C a bounded closed convex subset of X. Let $F : C \to X$ be a nonexpansive map and $G : C \to X$ a compact and demiclosed map. If $T = F + G : C \to X$ is a weakly inward map, then T has a fixed point.*

Applications of our results will be given sufficient conditions so that there exist solutions of difference equations which are asypmtotically constant ([6]).

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Department of Mathematics
College of Science
Myong Ji University
Yongin-Si, 449-728, Korea