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CONVERGENCE OF PREFILTER BASE ON THE FUZZY SET

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Abstract. In this paper, we investigate the prefilter base on a fuzzy set and fuzzy net φ on the fuzzy topological space (X, δ). And we show that the prefilter base $\mathcal{B}(\varphi)$ determines by the fuzzy net φ converge to a fuzzy point p iff the fuzzy net φ converge to a fuzzy point p. Also we prove that if the prefilter base $\mathcal{B}$ converge to a fuzzy point p, then the $\mathcal{B}$ has the cluster point p.

1. Introduction

The concept of a fuzzy set, which was introduced in [1], provides a natural framework for generalizing many of the concepts of general topology to what might be called fuzzy topological spaces. Throughout this paper, the symbol I will denote the unit interval. Let X be a non-empty set. A fuzzy set in X is a function with domain X and value in I, that is, an element of I^X.

2. Preliminaries

Definition 1. A fuzzy point p in X in a fuzzy set with membership function:

$$p(x) = \begin{cases} t_0, & \text{if } x = x_0 \\ 0, & \text{otherwise} \end{cases},$$

where $0 < t_0 < 1$. p is said to have support x_0 and value t_0, and is noted by $p(x_0, t_0)$ or even (x_0, t_0). We denote by $B_F(X)$ the collection of all fuzzy points in X.

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DEFINITION 2. Let $\{\mu_i \mid i \in \Lambda\}$ be a fuzzy sets in X. We define the following fuzzy sets:
(1) $\wedge\{\mu_i|i \in \Lambda\}(x) = \inf\{\mu_i(x)|i \in \Lambda\}$ for each $x \in X$.
(2) $\vee\{\mu_i|i \in \Lambda\}(x) = \sup\{\mu_i(x)|i \in \Lambda\}$ for each $x \in X$.
(3) $c_t \in I^X$, by $c_t(x) = t$ for each $x \in X$ and $t \in I$.

In 1968, C.L.Chang define a fuzzy topology on X as a subset $\delta \subset I^X$ such that
(1) $c_0, c_1 \in \delta$.
(2) If $\mu_1, \mu_2 \in \delta$, then $\mu_1 \wedge \mu_2 \in \delta$.
(3) If $\{\mu_\alpha|\alpha \in \Lambda\} \subset \delta$, then $\vee\{\mu_\alpha \mid \alpha \in \Lambda\} \in \delta$.

Several articles on the subject all involve this definition. Amongst these the most important ones are [3, 6]. It is concept of fuzzy topology that will be used throughout the sequel. Chang's definition we will refer to as quasi fuzzy topology.

The fuzzy sets in δ are called open fuzzy sets. A fuzzy set $A \in I^X$ is called closed iff A^c is open .

DEFINITION 3. A prefilter $\mathcal{F}$ on X is a nonempty collection of subsets of I^X with the properties:
(1) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \wedge F_2 \in \mathcal{F}$
(2) If $F_1 \in \mathcal{F}$ and $F_2 \geq F_1$, then $F_2 \in \mathcal{F}$
(3) $0 \notin \mathcal{F}$

DEFINITION 4. A collection $\mathcal{B}$ of subsets of I^X is a prefilter base iff $\mathcal{B} \neq \emptyset$ and
(1) If $B_1, B_2 \in \mathcal{B}$ then $B_3 \leq B_1 \wedge B_2$ for some $B_3 \in \mathcal{B}$;
(2) $0 \notin \mathcal{B}$

The collection $\mathcal{F} = \{F \in I^X | \exists B \in \mathcal{B} \text{ s.t } F \geq B\}$ is prefilter. $\mathcal{F}$ is said to be generated by $\mathcal{B}$ and denoted $< \mathcal{B} >$.

3. Converges and Cluster Point

A directed set $(D, \prec)$ is a set with partial order $\prec$ such that for each pair a, b of elements of D, there exists an element c of D having the property that $a \prec c$ and $b \prec c$.

Let $(D, \prec)$ be a directed set. The terminal set T_a determined by an $a \in D$ is $\{b \in D|a \prec b\}$

Let $(D, \prec)$ be a directed set. A fuzzy net in X is a map $\varphi : D \to B_F(X)$.

If $\varphi(b) = p_b(x_b, t_b)$, we also denote φ by $\{\varphi(b)|b \in D\}$ or $\{p_b|b \in D\}$. From now on, x_b and t_b will be the support and the value of the fuzzy point p_b. If $a \in D$, the fuzzy set $\varphi(T_a) = \vee\{\varphi(b) \mid a \prec b\}$ is called a F-tail of φ.

Definition 3.1. Let (X, δ) be an f.t.s. Let φ be fuzzy nets on X and $p(x,t) \in B_F(X)$. We say that ;

(1) φ converges to p (written $\varphi \to p$), if for all $N \in N_p^\delta$ $\exists a \in D$, $\forall b \succ a$ s.t $\varphi(b) \in N$.
(2) p is a cluster point of φ (written $\varphi \propto p$), if $\forall N \in N_p^\delta$, $\forall a \in D$, $\exists b \succ a$ s.t $\varphi(b) \in N$.

Let $\varphi : D \to B_F(X)$ be a fuzzy net. Then the family $\mathcal{B}(\varphi) = \{\varphi(T_a)|a \in D\}$ is a prefilter base in X. For, given $\varphi(T_a)$ and $\varphi(T_b)$, first find a $c \in D$ such that $a \prec c, b \prec c$, and then observe that $T_c \leq T_a \wedge T_b$ because $\prec$ is transitive. $\mathcal{B}(\varphi)$ is called the prefilter base determined by the fuzzy net φ.

Definition 3.2. Let $(D, \prec)$ be a directed set and T_a be a terminal set. Then, for a fuzzy net $\varphi : D \to B_F(X)$, we have:

(1) $\varphi \to p$ if $\forall N \in N_p^\delta,\ \exists T_a$ s.t $\varphi(T_a) \leq N$.
(2) $\varphi \propto p$ if $\forall N \in N_p^\delta,\ \forall T_a,\ \varphi(T_a) \wedge N \neq 0$.

Definition 3.3.

(1) A prefilter $\mathcal{F}$ is said to converge to the fuzzy point p (written $\mathcal{F} \to p$) iff $N_p^\delta \subset \mathcal{F}$, that is, $\mathcal{F}$ is finer than the nhood prefilter at p.
(2) We say $\mathcal{F}$ has p as a cluster point (written $\mathcal{F} \propto p$) iff $\forall N \in N_p^\delta$, then $N \wedge F \neq 0,\ \forall F \in \mathcal{F}$.

We can express these notions in terms of prefilter base as follows:

(1) A prefilter base converges to a fuzzy point p ($\mathcal{B} \to p$) iff each $N \in N_p^\delta$ contains some $B \in \mathcal{B}$.
(2) A prefilter base has p as a cluster point ($\mathcal{B} \propto p$) iff each $N \in N_p^\delta$ meets each $B \in \mathcal{B}$.

These definitions are still valid if we use nhood bases at $p, \mathcal{B}_p^\delta$, instead of nhood systems as p, N_p^δ. Clearly, if $\mathcal{F} \to p$, then $\mathcal{F} \propto p$.

LEMMA 3.4. *Let $\mathcal{B}(\varphi)$ be the prefilter base determined by the fuzzy net $\varphi : D \to B_F(X)$: Then*
(1) $\varphi \to p$ iff $\mathcal{B}(\varphi) \to p$.
(2) $\varphi \propto p$ iff $\mathcal{B}(\varphi) \propto p$.

THEOREM 3.5. *Let (X, δ) be a f.t.s., and $\mathcal{F}$ a prefilter on X and $\mathcal{B}$ a prefilter base of $\mathcal{F}$. Then $\mathcal{B}$ converges to a fuzzy point p iff $\mathcal{F}$ converges to the fuzzy point p.*

Proof. Let $\mathcal{B}$ converges to a fuzzy point p. Since there exists $B_\alpha \in \mathcal{B}$ such that $B_\alpha \leq N$ for all N in N_p^δ. Then $N \in \mathcal{F}$ and $N_p^\delta \subset \mathcal{F}$. Hence $\mathcal{F} \to p$. Conversely, if $\mathcal{F} \to p$, then $N_p^\delta \subset \mathcal{F}$ and since $\mathcal{B}$ is prefilter base $\mathcal{F}$. There exists $B_\alpha \in \mathcal{B}$ such that $B_\alpha \leq N \in \mathcal{F}$ for all $N \in N_p^\delta \subset \mathcal{F}$. Hence $\mathcal{B} \to p$. □

DEFINITION 3.6. Let $\mathcal{U} = \{A_\alpha | \alpha \in \Lambda\}$ and $\mathcal{B} = \{B_\beta | \beta \in \Gamma\}$ be two prefilter bases on X. $\mathcal{B}$ is subordinate to $\mathcal{U}$, written $\mathcal{B} \vdash \mathcal{U}$, if there exist B_β in $\mathcal{B}$ such that $B_\beta \leq A_\alpha$ for all $A_\alpha \in \mathcal{U}$.

THEOREM 3.7. *Let $\mathcal{U} = \{A_\alpha | \alpha \in \Lambda\}$ and $\mathcal{B} = \{B_\beta | \beta \in \Gamma\}$ be two prefilter bases on X.*
(1) If $\mathcal{U} \subset \mathcal{B}$ then $\mathcal{B} \vdash \mathcal{U}$.
(2) If $\mathcal{B} \vdash \mathcal{U}$, then each member of $\mathcal{B}$ meets every member of $\mathcal{U}$.

Proof. (1) is obvious. (2). Assume there exist B_β, A_α such that $A_\alpha \wedge B_\beta = 0$; since $\mathcal{B} \vdash \mathcal{U}$, for this A_α we can find a $B_\gamma \leq A_\alpha$, and then $B_\gamma \wedge B_\beta = 0$ contradicts that $\mathcal{B}$ is a prefilter base. □

THEOREM 3.8. *An f.t.s. (X, δ) is Hausdorff iff each convergent prefilter base in X converges to exactly one fuzzy point.*

Proof. Assume that X is fuzzy Hausdorff and $\mathcal{B}$ is a prefilter base and $\mathcal{B} \to p$. For any pair of distinct fuzzy points p, q in X. Then there exists fuzzy open set μ, ν in I^X such that $p \in \mu,\ q \in \nu$ and $\mu \wedge \nu = 0$. Since by hypothesis there is some $B_1 \leq \mu$ and and since any two B_1, B_2 have nonempty intersection, there can be no $B_2 \leq \nu$, thus, $\mathcal{B}$ cannot converge to $q \neq p$.

Conversely, assume that X is not Hausdorff. Then there must exist p, q such that $N \wedge M \neq 0$ for all N in $N_p^\delta$and M in N_q^δ.

$\mathcal{B} = N_p^\delta \cap N_q^\delta$ is therefore a prefilter base , and evidently $\mathcal{B} \to p$, $\mathcal{B} \to q$. □

THEOREM 3.9. *Let $\mathcal{U} = \{A_\alpha | \alpha \in \Lambda\}$ and $\mathcal{B} = \{B_\beta | \beta \in \Gamma\}$ be two prefilter bases on X.*
(1) $(\mathcal{U} \to p) \Rightarrow (\mathcal{U} \propto p)$ and, in Hausdorff spaces, at no point other than p.
(2) Let $\mathcal{B} \vdash \mathcal{U}$. Then;
(a) $(\mathcal{U} \to p) \Rightarrow (\mathcal{B} \to p)$
(b) $(\mathcal{B} \propto p) \Rightarrow (\mathcal{U} \propto p)$

Proof. (1). Given $N \in N_p^\delta$, there is some $A_\alpha \in \mathcal{U}$ such that $A_\alpha \leq N$; since each A_β must intersect A_α, it follows that $A_\beta \wedge N \neq 0$ for all A_β, so $\mathcal{U} \propto p$.

Now let X be fuzzy Hausdorff and let $p \neq q$; choosing disjoint fuzzy nbds $N \in N_p^\delta, M \in N_q^\delta$, there must be some $A_\alpha \in \mathcal{U}$ contained in N; then $A_\alpha \wedge M = 0$, and so $\mathcal{U}$ cannot cluster point at q .

(2a). There is some $A_\alpha \in \mathcal{U}$ such that $A_\alpha \leq N$ for all $N \in N_p^\delta$; since $\mathcal{B} \vdash \mathcal{U}$, there is a $B_\beta \leq A_\alpha$, so $\mathcal{B} \to p$ also.

(2b) . Given $N \in N_p^\delta$ and A_α, there is some $B_\beta \leq A_\alpha$, and since $B_\beta \wedge N \neq 0$ for all B_β, we can find $A_\alpha \wedge N \neq 0$ for all A_α, which proves $\mathcal{U} \propto p$. □

As a immediately, we have the follows .

COROLLARY 3.10.
(1) $\mathcal{U} \to p$ iff $\forall \mathcal{B} \vdash \mathcal{U}, \exists \mathcal{C} \vdash \mathcal{B}$ s.t $\mathcal{C} \to p$
(2) $\mathcal{U} \propto p$ iff $\exists \mathcal{B} \vdash \mathcal{U}$ s.t $\mathcal{B} \to p$

Let $\mathcal{B}$ be a prefilter base on X. We say that a subset Y of X contains $\mathcal{B}$ if every member of $\mathcal{B}$ is an element of fuzzy set with support Y.

DEFINITION 3.11. Let $\mathcal{M}$ be a prefilter base in X is called fuzzy maximal if it has no property subordinated prefilter base, that is, if for all $\mathcal{U}$,$\mathcal{U} \vdash \mathcal{M} \Rightarrow \mathcal{M} \vdash \mathcal{U}$.

THEOREM 3.12. *A prefilter base $\mathcal{M}$ is fuzzy maximal iff for each $Y \subset X$, either Y or Y^c contains a member of $\mathcal{M}$.*

Proof. Assume $\mathcal{M} = \{M_\beta | \beta \in \Gamma\}$ is a fuzzy maximal prefilter base. Let $Y \subset X$ and A be a fuzzy set with support Y and B is a fuzzy set with support Y^c. Then we cannot have an Y contains M_β and Y^c contains M_γ, since $M_\beta \wedge M_\gamma = 0$. Assume now that Y not contains M_β for all M_β. Then $A \wedge M_\beta \neq 0$ and also then all $M_\beta \wedge B \neq 0$, so $\mathcal{U} = \{M_\beta \wedge B | M_\beta \in \mathcal{M}\}$ is a prefilter base. Since $\mathcal{U} \vdash \mathcal{M}$, also $\mathcal{M} \vdash \mathcal{U}$. Hence we have $M_\gamma \leq M_\beta \wedge B \leq B$. Therefore $M_\gamma \leq B$, that is Y^c contains M_γ.

Conversely, assume that for each $Y \subset X$, Y or Y^c contains a member of $\mathcal{M}$ and that $\mathcal{U} \vdash \mathcal{M}$. Given any $A \in \mathcal{U}$, the condition assures that either there is an $M_\beta \leq A$ or $M_\beta \leq B$ the latter possibility is excluded, since the assumption $\mathcal{U} \vdash \mathcal{M}$ implies [Theorem 3.7 (2)] that all $M_\beta \wedge A \neq 0$. Thus $\mathcal{M} \vdash \mathcal{U}$ and $\mathcal{M}$ is fuzzy maximal. □

Corollary 3.13. *Let $\mathcal{M}$ be a fuzzy maximal prefilter base in X. Then $\mathcal{M} \propto p$ iff $\mathcal{M} \to p$.*

Proof. Only the implication $(\mathcal{M} \propto p) \Rightarrow (\mathcal{M} \to p)$ need be proved. Given $N \in N_p^\delta$, there is an $M_\alpha \leq N$ or an $M_\alpha \leq N^c$; since $\mathcal{M} \propto p$, so that $M_\alpha \wedge N \neq 0$ for each M_α, the latter possibility is excluded, and therefore $\mathcal{M} \to p$. □

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