J. KSIAM **Vol.6**, No.2, 1-8, 2002

ON DISTRIBUTIONS IN GENERALIZED CONTINUED FRACTIONS

YOUNG-HO AHN

Abstract. Let T_ϕ be a generalized Gauss transformation and $[a_1, a_2, \cdots]_{T_\phi}$ be a symbolic representation of $x \in [0,1)$ induced by T_ϕ, i.e., generalized continued fraction expansion induced by T_ϕ. It is shown that the distribution of relative frequency of $[k_1, \cdots, k_n]$ in $[a_1, a_2, \cdots]_{T_p}$ satisfies Central Limit Theorem where $k_i \in \mathbb{N}$ for $1 \le i \le n$.

1. Introduction

Let $(X, \mathcal{B}, \mu)$ be a probability space. A transformation $T : X \to X$ is said to be μ-preserving transformation if $\mu(T^{-1}E) = \mu(E)$. Sometimes we say that μ is T-invariant measure. A transformations T is called ergodic if constant function is the only T-invariant function and it is called weakly mixing if constant function is the only eigenfunction. Ergodic Theorem says that if T is ergodic then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_E(T^i x) = \mu(E)$$

where E is a measurable subset of X and $\mathbf{1}_E$ is an indicator function of E. Now consider the case $X = [0,1]$. If μ is given by $d\mu = \rho dx$ for $\rho(x) \ge 0$ with $\int_X \rho dx = 1$ then $\mu(E) = \int_E \rho\, dx$ and $\int_X f d\mu = \int_X f \rho\, dx$ for every E and $f \in L^1(X, dx)$ where dx is the Lebesgue measure on $[0,1]$. In this case μ is said to be absolutely continuous and ρ is called a density function. For more information on ergodic theory, see [5, 6].

In [3], Choe introduced the generalized Gauss transformations as follows. Let $\{x\}$ be the fractional part of x. Recall that a piecewise differentiable transformation $T : [0,1] \to [0,1]$ is said to be *eventually expansive* if some iterate of T has its derivative bounded away from 1 in modulus, i.e., $|(T^n)'| > 1$ everywhere for some n.

Definition 1. A transformation T_ϕ on the unit interval defined by $T_\phi(x) = \{\phi(x)\}$ is called a generalized Gauss transformation if $\phi(x)$ satisfies the following conditions.

(i) $\phi(x)$ is twice continuous differentiable,

1991 *Mathematics Subject Classification.* 28D05, 47A35.

Key words and phrases. coboundary, invariant measure, Gauss transformation, eventually expansive.

(ii) $\phi'(x) < 0$ for every $0 < x < 1$,
(iii) $\phi(0+) = +\infty$, and $\phi(1) = 1$ and
(iv) T is eventually expansive.

For the notational convenience, if $\phi(x) = 1/x^p$, we denote T_ϕ as T_p.

Let $x = .d_1 d_2 \cdots$ be the decimal expansion of $0 < x < 1$. Since $10x = d_1.d_2 d_3 \cdots$, $d_1 = [10x]$ where $[y]$ is the integral part of y. Hence the decimal expansion is closely related to the transformation on the unit interval defined by $Sx = \{10x\}$, i.e., $d_i = [10(S^{i-1}x)]$. By using the properties of this transformation, it can be easily shown that the relative frequency of $0 \le k \le 9$ satisfies Central Limit Theorem [1].

The classical Gauss transformation on the unit interval is defined by $T_1(x) = \{1/x\}$ with its invariant measure $d\mu = \dfrac{1}{\ln 2} \cdot \dfrac{1}{1+x}\, dx$. Since $x = 1/([1/x] + T_1(x))$ and $T_1(x) = 1/([1/T_1(x)] + T_1^2(x))$, we have

$$x = \cfrac{1}{[1/x] + \cfrac{1}{(1/[T_1(x)] + T_1^2(x))}}.$$

Similarly as in the case of decimal expansion of x, we obtain

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}},$$

where $a_1 = [1/x]$, $a_2 = [1/T_1(x)]$, and so on. Hence $a_n(x) = a_1(T_1^{n-1}(x))$. We write $x = [a_0(x), a_1(x), \cdots]_{T_1}$ and say that $[a_0(x), a_1(x), \cdots]_{T_1}$ is a symbolic representation of x induced by classical Gauss transformation and in general it is called the classical continued fraction expansion of x. For $k \in \mathbb{N}$, $a_n(x) = k$ if and only if $T_1^{n-1}x \in (\frac{1}{k+1}, \frac{1}{k}]$. Thus the relative frequency of k in $[a_0, a_1, \cdots]_{T_1}$, i.e.,

$$\lim_{n\to\infty} \frac{\mathrm{card}\{i; a_i(x) = k\}}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{(\frac{1}{k+1}, \frac{1}{k}]}(T_1^i x) = \mu\left(\frac{1}{k+1}, \frac{1}{k}\right]$$

by applying Ergodic Theorem to Gauss transformation and an indicator function $\mathbf{1}_{(\frac{1}{k+1}, \frac{1}{k}]}$. Similarly we have a symbolic representation $[a_0(x), a_1(x), \cdots]_{T_\phi}$ of x induced by a generalized Gauss transformation T_ϕ and they are called the generalized continued fraction expansion of x and we obtain the similar result as above. For example, $a_n(x) = k$ if and only if $T_p^{n-1}x \in (\sqrt[p]{\frac{1}{k+1}}, \sqrt[p]{\frac{1}{k}}]$. More precisely, consider the square-root continued fractions which is induced by T_2. Recall that

$$x = \frac{1}{\sqrt{[1/x^2] + T_2 x}}.$$

Since $T_2 x = \dfrac{1}{\sqrt{[1/(T_2x)^2] + T_2^2 x}}$, we have from the previous equation that

$$x = \cfrac{1}{\sqrt{[1/x^2] + \cfrac{1}{\sqrt{[1/(T_2x)^2] + T_2^2 x}}}}$$

and so on. Continuing indefinitely, we obtain

$$x = \cfrac{1}{\sqrt{a_1 + \cfrac{1}{\sqrt{a_2 + \cdots}}}} = [a_1, a_2, \cdots]_{T_2}$$

where $a_1(x) = [1/x^2]$ and $a_n(x) = [1/(T_2^{n-1}x)^2] = a_1(T_2^{n-1}x)$ for $n \geq 2$.

In this article, we will consider the distribution of relative frequency of $[k_1, \cdots, k_n]$ in the generalized continued fraction expansion $[a_1, a_2, \cdots]_{T_\phi}$ where $k_i \in \mathbb{N}$ for $1 \leq i \leq n$ and show that it satisfies Central Limit Theorem.

2. Densities of generalized Gauss transformations and Central Limit Theorem

For piecewise differentiable maps on the unit interval the existence of absolutely continuous invariant measure is proved under various similar conditions. In this article we consider piecewise differentiable maps with infinitely many discontinuities. The following folklore theorem guarantees the existence of absolutely continuous ergodic invariant measures on the unit interval for some transformations with countably infinite discontinuities [3].

Lemma 1 (Folklore Theorem)**.** *Let $\{\Delta_i\}$ be a countable partition of the unit interval by subintervals Δ_i with the property that the closure of the set of endpoints of Δ_i has zero Lebesgue measure. Suppose that an eventually expansive map T on the interval* $[0, 1]$ *satisfies*

(i) $T|_{\Delta_i}$ *has a C^2-extension to the closure of Δ_i,*
(ii) $T|_{\Delta_i}$ *is strictly monotone,*
(iii) $\overline{T(\Delta_i)} = [0, 1]$*, and*
(iv) $\sup_i \left\{ \sup_{x_1 \in \Delta_i} |T''(x_1)| / \inf_{x_2 \in \Delta_i} |T'(x_2)|^2 \right\} < \infty$.

Then there exists a measure μ which is (a) *T-invariant,* (b) *ergodic, and* (c) *finite and of the form $d\mu = \rho(x)dx$ where ρ is continuous and $1/C < \rho < C$ for some $C > 0$.*

As indicated in the introduction the relative frequency of $k \in \mathbb{N}$ in $[a_0(x), a_1(x), \cdots]_{T_p}$ converges to $\int \mathbf{1}_{P_k}\, \rho_p(x)\, dx$ where $P_k = \left(\sqrt[p]{\frac{1}{k+1}}, \sqrt[p]{\frac{1}{k}} \right]$ and $\rho_p(x)$ is the T_p-invariant density function. It is a consequence of applying Ergodic Theorem to an indicator function $f(x) = \mathbf{1}_{P_k}(x)$.

To investigate the distribution of relative frequency of $[k_1, \cdots, k_n]$ in $[a_1, a_2, \cdots]_{T_\phi}$ where $k_i \in \mathbb{N}$ for $1 \le i \le n$ we need the following Lemma.

Lemma 2 (Central Limit Theorem)**.** *Let T be a transformation satisfying the condition of Lemma 1, μ be the T-invariant absolutely continuous measure and $f(x)$ be a bounded variation function. Assume that the equation*

$$f = C + \varphi \circ T - \varphi$$

does not have a solution $C \in \mathbb{R}$. Then

$$\sigma^2 = \lim_{n\to\infty} \int \left(\frac{S_n f - n\mu(f)}{\sqrt{n}} \right)^2 d\mu > 0$$

and, for any $z \in \mathbb{R}$,

$$\lim_{n\to\infty} \mu \left\{ x : \frac{S_n f(x) - n\mu(f)}{\sigma\sqrt{n}} \le z \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-t^2/2) dt,$$

where $S_n f(x) = \sum_{i=0}^{n-1} f(T^i x)$ and $\mu(f) = \int_X f d\mu$

Proof. For a proof, see [2]. □

Recall that a function $g(x)$ is called a *coboundary* if the equation $g(x)h(Tx) = h(x)$, $|h(x)| = 1$ has a solution. If the equation $f = C + \varphi \circ T - \varphi$ has a solution, then

$$\exp(2\pi i f(x)) = \exp(2\pi i C) \exp(2\pi i \varphi \circ T(x)) \overline{\exp(2\pi i \varphi(x))} \qquad (*)$$

has a solution and $g(x) = \overline{\exp(2\pi i C)} \exp(2\pi i f(x))$ is a coboundary with cobounding function $h(x) = \overline{\exp(2\pi i \varphi(x))}$. Hence if f does not have a solution of $(*)$ then f does not have a solution of the equation $f = C + \varphi \circ T - \varphi$. Hence we can apply Central Limit Theorem.

3. Distributions in generalized continued fractions

To investigate the applicability of Central Limit Theorem, we will study the solvability of coboundary equation.

Let $(Y, \mathcal{C}, \mu)$ be a probability space, $f \in L^1(Y, \mathcal{C}, \mu)$ and $\mathcal{B} \subset \mathcal{C}$ a sub σ-algebra. Put $\nu(B) = \int_B f\, d\mu$ for $B \in \mathcal{B}$. Radon-Nikodym Theorem implies that there is a function $g \in L^1(Y, \mathcal{B}, \mu)$ such that $\nu(B) = \int_B g\, d\mu$ for $B \in \mathcal{B}$. We use the notation $E(f|\mathcal{B})$ for g, and call it the *conditional expectation* of f with respect to $\mathcal{B}$. Let S be a transformation defined on Y and $\mathcal{B}$ be *exhaustive* i.e., $S^{-1}\mathcal{B} \subset \mathcal{B}$ and $S^n\mathcal{B} \uparrow \mathcal{C}$ as $n \to +\infty$.

Let $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots$ an increasing sequence of sub σ-algebra of $\mathcal{C}$. A sequence $f_1, f_2, \cdots$ of functions in $L^1(Y)$ such that f_n is measurable with respect to $\mathcal{B}_n$ for $n = 1, 2, \cdots$ is called a *martingale* if $E(f_{n+1}|\mathcal{B}_n) = f_n$ a.e. for $n = 1, 2, \cdots$. Martingale Theorem says that every L^1-bounded martingale (i.e., $\sup_n \int_Y |f_n|\, d\mu < \infty$) converges a.e. and in L^1. Let $\mathcal{B}_n = S^n\mathcal{B}$ and $f_n = E(f|S^n\mathcal{B})$. Then by the properties of the conditional expectation operator, $E(f_{n+1}|\mathcal{B}_n) = f_n$ a.e. for $n = 1, 2, \cdots$, i.e., $E(f|S^n\mathcal{B})$ is martingale with respect to the sequence of sub σ-algebra $\{S^n\mathcal{B}\}$. Since

$\mathcal{B}$ is exhaustive, Martingale Theorem says that $E(f|S^n\mathcal{B})$ converges to f a.e. and in $L^1(Y,\mathcal{C},\mu)$ for $f \in L^1(Y,\mathcal{C},\mu)$

From now on, let $\mathbb{T} = \{z \in \mathcal{C} : |z| = 1\}$.

Lemma 3. *Let S be a measure preserving transformation on $(Y,\mathcal{C},\mu)$, and $\mathcal{B}$ be an exhaustive σ-algebra $\mathcal{B} \subset \mathcal{C}$, and let $f : Y \to \mathbb{T}$ be a $\mathcal{B}$-measurable map to the circle group $\mathbb{T}$. If $q : Y \to \mathbb{T}$ is a $\mathcal{C}$-measurable solution to the equation $f \cdot q \circ S = q$, then q is $\mathcal{B}$-measurable.*

Proof. We follow the idea of Parry in [4]. Applying the conditional expectation operator $E(\cdot|\mathcal{B})$ to the equation

$$f \cdot q \circ S = q \qquad (**)$$

then

$$f \cdot E(q \circ S|\mathcal{B}) = E(q|\mathcal{B})$$

or

$$f \cdot E(q|S\mathcal{B}) \circ S = E(q|\mathcal{B}).$$

Multiplying this with $(**)$ inverted we have

$$\overline{q(y)} \cdot E(q|\mathcal{B})(y) = \overline{q(Sy)} \cdot E(q|S\mathcal{B}) \circ S(y)$$

a.e. so that

$$\int_Y \overline{q} \cdot E(q|\mathcal{B})\, d\mu = \int_Y \overline{q} \cdot E(q|S\mathcal{B})\, d\mu.$$

By exactly the same argument, using $S^n\mathcal{B}$ in place of $\mathcal{B}$, we have

$$\int_Y \overline{q} \cdot E(q|S^n\mathcal{B})\, d\mu = \int_Y \overline{q} \cdot E(q|S^{n+1}\mathcal{B})\, d\mu$$

so that

$$\int_Y \overline{q} \cdot E(q|\mathcal{B})\, d\mu = \int_Y \overline{q} \cdot E(q|S^n\mathcal{B})\, d\mu.$$

Taking limits, we get

$$\int_Y \overline{q} \cdot E(q|\mathcal{B})\, d\mu = \int_Y |q|^2\, d\mu.$$

Thus $E(q|\mathcal{B}) = q$ a.e., and q is $\mathcal{B}$-measurable. ◻

Proposition 1. *Let $Y = \prod_{-\infty}^{\infty}\{0,1,2,\cdots\}$ where σ be the shift map on Y with σ-invariant measure μ. Let's denote $\mathcal{P} = \{P_k : P_k = \{x : x_0 = k\} \quad for k = 0,1,2,\cdots\}$, i.e., $\mathcal{P}$ is a state partition and $\mathcal{B}_l^m = \bigvee_{i=l}^m \sigma^{-i}\mathcal{P}$ for $l \le m$. Assume that $\phi(x)$ is $\mathbb{T}$-valued $\mathcal{B}_l^m$ measurable function. If $g(x)$ is a $\mathbb{T}$-valued solution of the equation, $\phi(x)g(\sigma x) = g(x)$ then $g(x)$ is also $\mathcal{B}_l^m$ measurable function.*

Proof. Let $\mathcal{B} = \bigvee_{i=l}^{\infty} \sigma^{-i}\mathcal{P}$. Then $\phi(x)$ is $\mathcal{B}$ measurable and $\mathcal{B}$ is exhaustive with respect to σ. Since $\phi(x)g(\sigma x) = g(x)$, $g(x)$ is also $\mathcal{B}$-measurable by the above Lemma. Now let $\mathcal{A} = \bigvee_{i=-m}^{\infty} \sigma^{i}\mathcal{P}$. Then $\phi(\sigma^{-1}x)$ is $\mathcal{A}$ measurable and $\mathcal{A}$ is exhaustive with respect to σ^{-1}. Since $\phi(x)g(\sigma x) = g(x)$ can be rewritten as $\phi(\sigma^{-1}x)g(x) = g(\sigma^{-1}x)$, i.e., $\phi(\sigma^{-1}x)\overline{g(\sigma^{-1}x)} = \overline{g(x)}$, $g(x)$ is also $\mathcal{A}$ measurable by applying the above Lemma to the map σ^{-1}. Hence the conclusion follows. □

For fixed generalized Gauss transformation T_ϕ, we have a collection of intervals $\{\Delta_i\}$ satisfying the condition of Lemma 1, i.e., $\Delta_i = (b_{i+1}, b_i]$ where $\{b_i\}$ is a decreasing sequence of T_ϕ preimages of 1 with $b_1 = 1$. Let $\{\Delta_i^n\}$ be a collection of intervals such that $A \in \{\Delta_i^n\}$ if and only if $A = P_{i_0} \cap T_\phi^{-1}P_{i_1} \cap \cdots \cap T_\phi^{-n+1}P_{i_{n-1}}$ where P_{i_k} is taken from $\{\Delta_i\}$ for all $0 \le k \le n-1$.

Given a generalized Gauss transformation T_ϕ, construct an one-sided shift space on countable symbols as follows: To each $x \in (0,1]$ there corresponds a one-sided infinite sequence $[a_0, a_1, ..., a_n, ...]$ such that $T_\phi^n(x) \in \Delta_{a_n}$ if $\{T_\phi^{n-1}x\} \neq 0$ and $a_i = 0$ for all $i \ge n$ if $\{T_\phi^{n-1}x\} = 0$ for $n \ge 1$. For some $t \in (0,1]$, we can find N such that its representation $t = [a_0, a_1, ..., a_n, ...]$ satisfies the condition that $a_n = 0$ for all $n \ge N$. We call such a t as a *generalized rational point*. Let X be the set of all such sequences and ψ be the assignment of a sequence to a point. Since T_ϕ has a finite Lebesgue equivalent ergodic measure $\rho(x)\,dx$, we define a shift invariant measure ν on a cylinder set $C \subset \prod_0^\infty \{0,1,2,\cdots,L\}$ by $\nu(C) = \int_{\psi^{-1}(C)} \rho(x)\,dx$. Note that $\psi^{-1}(C)$ is a union of intervals with generalized rational endpoints. Kolmogorov Extension Theorem guarantees the existence and uniqueness of such a measure ν. We call the shift space X the *symbolic system obtained from* T_ϕ. Recall that two measure preserving transformations T_1 and T_2 on X_1 and X_2 are said to be *isomorphic* if there exists a measure preserving transformation $\psi : X_1 \to X_2$ which is one-to-one and $\psi \circ T_1 = T_2 \circ \psi$ on X_1 modulo measure zero sets. Let σ be the one-sided shift transformation on X defined by $\sigma : [a_0, a_1, a_2, ...] \mapsto [a_1, a_2, a_3, ...]$.

In the commutative diagram

$$\begin{array}{ccc} ([0,1), \rho\,dx) & \xrightarrow{T_\phi} & ([0,1), \rho\,dx) \\ \psi \big\downarrow & & \psi \big\downarrow \\ (X, d\nu) & \xrightarrow{\sigma} & (X, d\nu) \end{array}$$

T_p and σ are isomorphic.

Let $T : (X, \mathcal{B}) \to (X, \mathcal{B})$ be a measurable transformation. We define a natural extension S_T of T as follows: Let

$$X_T = \{(x_0, x_1, x_2, \cdots) : x_n = T(x_{n+1}), x_n \in X, n = 0, 1, 2, \cdots\},$$

and let $S_T : X_T \to X_T$ be defined by

$$S_T((x_0, x_1, x_2, \cdots)) = (T(x_0), x_0, x_1, x_2, \cdots).$$

S_T is one to one on X_T. If T preserves a measure μ, then we can define a measure $\overline{\mu}$ on X_T by defining $\overline{\mu}$ on the cylinder sets

$$C(A_0, A_1, \cdots, A_k) = \{(x_0, x_1, x_2, \cdots) : x_0 \in A_0, x_1 \in A_1, \cdots, x_k \in A_k\}$$

where $A_i \in \mathcal{B}$ for $0 \le i \le k$ as follows:

$$\overline{\mu}(C(A_0, A_1, \cdots, A_k)) = \mu(T^{-k}A_0 \cap T^{-k+1}A_1 \cap \cdots \cap A_k).$$

Recall that if T preserves the measure μ, then S_T preserves the measure $\overline{\mu}$. Furthermore (T, μ) is ergodic if and only if $(S_T, \overline{\mu})$ is ergodic and (T, μ) is weakly mixing if and only if $(S_T, \overline{\mu})$ is weakly mixing.

Now consider the case when $X = \prod_{k=0}^{\infty}\{0, 1, 2, \cdots, L\}$ where $L \le \infty$ and T is a shift map, i.e.,

$$T((x_0, x_1, x_2, \cdots)) = (x_1, x_2, x_3, \cdots).$$

Then T is noninvertible. We will construct a natural extension of T. By the definition of natural extension, we define

$$X_T = \{\overline{x} = (y_0, y_1, y_2, \cdots) : y_i = (x_0^i, x_1^i, x_2^i, \cdots), T(y_{i+1}) = y_i, y_i \in X, i = 0, 1, \cdots\}.$$

By virtue of the condition $T(y_{i+1}) = y_i$, $i = 0, 1, \cdots$, the sequence y_i are of the form

$$\begin{aligned}
y_0 &= (x_0, x_1, x_2, \cdots),\\
y_1 &= (x_{-1}, x_0, x_1, x_2, \cdots),\\
y_2 &= (x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots),\\
&\vdots\\
y_n &= (x_{-n}, x_{-n+1}, \cdots, x_{-1}, x_0, x_1, x_2, \cdots).
\end{aligned}$$

It is natural then to write the double sequence $\overline{x} = (y_0, y_1, y_2, \cdots)$ as one two-sided sequence:

$$\overline{x} = (\cdots, x_{-n}, x_{-n+1}, \cdots, x_{-1}, x_0, x_1, x_2, \cdots).$$

Hence then natural extension of one-sided shift map is just a two-sided shift map.

Proposition 2. *For fixed generalized Gauss transformation T_ϕ, let $\Delta_i = (b_{i+1}, b_i]$ where $\{b_i\}$ is a decreasing sequence of T_ϕ preimages of 1 with $b_1 = 1$. Let $\{\Delta_i^n\}$ be a collection of intervals such that $A \in \{\Delta_i^n\}$ if and only if $A = P_{i_0} \cap T_\phi^{-1}P_{i_1} \cap \cdots \cap T_\phi^{-n+1}P_{i_{n-1}}$ where P_{i_k} is taken from $\{\Delta_i\}$ for all $0 \le k \le n-1$. Assume that B_k be a finite sequence of intervals taken from $\{\Delta_i^n\}$. Then for generalized Gauss transformation T_ϕ, a nonconstant function $f(x) = \exp(2\pi i \sum_{k=1}^n b_k \mathbf{1}_{B_k}(x))$ is not coboundary. Hence $h(x) = \sum_{k=1}^n b_k \mathbf{1}_{B_k}(x)$ satisfies Central Limit Theorem.*

Proof. Let ρ_ϕ be the T_ϕ-invariant density function. Let $Y = \prod_{-\infty}^{\infty}\{0, 1, 2, \cdots\}$ and $Y^+ = \prod_0^{\infty}\{0, 1, 2, \cdots\}$. For notational convenience, let $T = T_\phi$. Consider the following commutative diagram

$$\begin{array}{ccc} [0,1) & \xrightarrow{T} & [0,1) \\ \psi \big\downarrow & & \big\downarrow \psi \\ Y^+ & \xrightarrow{\sigma^+} & Y^+ \end{array}$$

where $(\psi(x))_i = j$ if $T^i x \in \Delta_j$ for $i = 0, 1, 2, \cdots$. Then ψ is a measure theoretically isomorphism between $([0,1), T_\phi, \rho_\phi dx)$ and (Y^+, σ^+, ν^+) where ν^+ is a induced measure by ψ and σ^+ is the one-sided shift map on Y^+. And (Y, σ, ν) is the natural extension of (Y^+, σ^+, ν^+) where σ is the two-sided shift map on Y. Hence if $f(x)g(Tx) = g(x)$ then $g(x)$ is also step function and there exist an interval I taken from $\{\Delta_i^m\}$ for some m such that $g(x)$ is constant on I by Proposition 1. Since $T^m I = [0,1)$ by the property of T, $f(x)$ is a function with finite discontinuity points, and $f(x)g(Tx) = g(x)$, $g(x)$ is also a function with finite discontinuity points. Since $f(x)g(Tx) = g(x)$ can be rewritten as $f(x) = g(x)\overline{g(Tx)}$, $f(x)$ must be a function with infinite discontinuity points. It is a contradiction. □

References

[1] Y. Ahn, *On compact group extension of Bernoulli shifts,* Bull. of Austral. Math. Soc. **61** (2000), 277–288.
[2] A. Boyarsky and P. Góra, Laws of Chaos, Birkhäuser, 1997.
[3] G. H. Choe, *Generalized continued fractions,* Appl. Math. and Comput. **109** (2000), 287–299.
[4] W. Parry, *A Cocycle equation for shift,* Contemp. Math. **135** (1992), 327–333.
[5] K. Petersen, Ergodic Theory, Cambridge Univ. Press (1983).
[6] P. Walters, An Introduction to Ergodic Theory, Springer-Verlag New York, 1982.

School of Mathematical Sciences
Seoul National University,
Seoul 151-747, Korea
email: ahn@euclid.kaist.ac.kr