韓國産業應用學會誌, 제5권, 제4호, 303-307, 2002.11
J. Korean Society of Industrial Application, Vol.5, No.4, 303-307, November, 2002

비국소 경계 조건들을 가진 상미분 방정식들의 근의 존재성에 음함수 정리들의 응용 II

Application of Implicit Function Theorem to Existence of Solutions to Ordinary Differential Equations with Nonlocal Boundary Conditions, II

도태석*
Tae-Sug Do*

<Abstract>

We consider the problem

$$y''=a(x,\ y)(y-b),\ (0<x<1)$$
$$y(0)=0,\ y'(1)=g(y(\xi),\ y'(\xi)),\ \xi \text{ fixed in } (0,\ 1).$$

This is a model of steady-state heat conduction in a rod when the heat flux at the end x=1 is determined by observation of the temperature and heat flux at some interior point ξ. We establish conditions sufficient for existence, uniqueness, and positivity of solutions.

Key Words : Nonlocal Boundary Conditions, Ordinary Differential Equation, A Priori Bounds.

Introduction

The problem formulated and proposed in [5] has the form

$$y''(x) = a(x, y(x))(y(x) - b) \quad (0 < x < 1)$$
$$y(0) = 0, \quad y'(1) = g(y(\xi), y'(\xi)) \quad (1)$$

for some fixed $\xi \in (0, 1)$. Existence and uniqueness for g decreasing in both arguments was studied in [5].

As an example of what may be expected, we consider the simple special case

$$y'' = a^2 y, \quad y(0) = 0,$$

$$y'(1) = \alpha y(\xi) + \beta y'(\xi) + \gamma \quad (2)$$

where $a, \alpha, \beta, \gamma \neq 0$ are constants. We are interested in conditions sufficient to guarantee existence for all $a>0$ and all $\xi \in (0, 1)$. Since solutions of the differential equation and the first boundary condition have the form $y(x) = k\sinh ax$ for some constant k, this y will be a solution of the boundary value problem if and only if k satisfies.

$$k(a\cosh a - \alpha\sinh(a\xi) - \beta a\cosh(a\xi)) = \gamma. \quad (3)$$

Considering briefly the case $\alpha=0$, we see that for $\beta > 1$ there is value of ξ for which no

* 관동대학교 사범대학 수학교육학과 부교수
* Dept. of Mathematics Education, College of Education, Kwandong University.

solution exists. Thus we expect that $\beta\leq 1$ is necessary. Returning to the case of general linear g in (2), i. e., to (3), we may take the limit as $\xi \rightarrow 1-$ to get that

$$\frac{1-\beta}{a} \neq \frac{1}{a}\tanh a$$

must hold. Since$(tanha)/a$ has range (0, 1) for $a>0$, it is sufficient to require that $\alpha+\beta<1$. On the other hand, if $\hat{\alpha}>0$ and , $\hat{\alpha}+\hat{\beta}>1$ then $1>\frac{1-\hat{\beta}}{\hat{\alpha}}=\frac{1}{a}\tanh a$ for some a, so $\alpha+\beta\leq 1$ is also necessary if a solution is to exist for each $a>0$.

In the following we first establish one specialized implicit function theorem which requires only continuity; this theorem will be utilized in establishing certain a priori estimates in the following section. We treat first existence and uniqueness questions for g increasing in both arguments. We then establish conditions under which the solution is positive.

Implicit Function Theorem

The following lemma is form of the implicit function theorem that does not require - or guarantee - differentiability, only continuity. it was used without detailed proof in [4]. this lemma can be strengthened in various ways, but the present form is adequate for our purposes.

Lemma 1. *Let* $\mu\in R^n$, *let* $x\in R$, *and let* $g(x,\mu)$ *have the following properties*

a. *for each* $x\in R$, $g(x,\cdot)$ *is continuous*
b. $z_2>z_1$
$\Rightarrow 0\leq g(z_2,\mu)-g(z_1,\mu)\leq\gamma(z_2-z_1)$

where the constant γ *satisfies* $0<\gamma<1$. *Then the equation* $x=g(x,\mu)$ *has a unique solution* $x(\mu)$, *which is moreover continuous in* μ.

Remark. The second hypothesis on g is a combination of the requirement that g is nondecreasing and that g satisfies a one-sided Lipschitz condition with Lipschitz constant γ less than one. Of course it guarantees that $g(\cdot,\mu)$ is continuous for each fixed μ.

Proof. Set $h(x,\mu)=x-g(x,\mu)$ and let $z_2>z_1$; then

$$\begin{aligned}&h(z_2,\mu)-h(z_1,\mu)\\&=(z_2-z_1)-(g(z_2,\mu)-g(z_1.\mu))\\&\geq(1-\gamma)(z_2-z_1)\end{aligned}$$

Fix z_1 and let $z_2\rightarrow\infty$ to get that $\lim_{z\rightarrow\infty}h(z,\mu)=\infty$; fix z_2 and let $z_1\rightarrow-\infty$ to get that $\lim_{z\rightarrow-\infty}h(z,\mu)=-\infty$. Existence of a value $x(\mu)$ satisfying $x(\mu)=g(x(\mu),\mu)$ follows from the intermediate value theorem. Suppose for some μ that $z_2>z_1$ are two solutions of $x=g(x,\mu)$; then

$$\begin{aligned}&0<z_2-z_1=g(z_2,\mu)-g(z_1,\mu)\\&\leq\gamma(z_2-z_1),\end{aligned}$$

contradicting $\gamma<1$. Thus the function $x(\mu)$ is well defined.

To establish continuity, let $\varepsilon>0$ and suppose first that δ is such that $x(\mu+\delta)\geq x(\mu)$. Then

$$\begin{aligned}&0\leq x(\mu+\delta)-x(\mu)\\&=g(x(\mu+\delta),\mu+\delta)-g(x(\mu),\mu)\\&=g(x(\mu+\delta),\mu+\delta)-g(x(\mu),\mu+\delta)\\&\quad+g(x(\mu),\mu+\delta)-g(x(\mu),\mu)\\&\leq\gamma[x(\mu+\delta)-x(\mu)]+g(x(\mu),\mu+\delta)\\&\quad-g(x(\mu),\mu)\end{aligned}$$

from which it follows that

$$\begin{aligned}&0\leq x(\mu+\delta)-x(\mu)\\&\leq\frac{g(x(\mu),\mu+\delta)-g(x(\mu),\mu)}{1-\gamma}\end{aligned}$$

The right-hand side can be made small by choosing the vector δ small, by the continuity of $g(x,\cdot)$.

Suppose now that $x(\mu+\delta)<x(\mu)$ for some δ. Then, much as before,

$$\begin{aligned}&0\geq x(\mu+\delta)-x(\mu)\\&=g(x(\mu+\delta),\mu+\delta)-g(x(\mu),\mu+\delta)\\&\quad+g(x(\mu),\mu+\delta)-g(x(\mu),\mu)\\&\geq\gamma[x(\mu+\delta)-x(\mu)]+g(x(\mu),\mu+\delta)\end{aligned}$$

$$- g(x(\mu), \mu)$$

or

$$0 \ge x(\mu+\delta) - x(\mu) \ge \frac{g(x(\mu), \mu+\delta) - g(x(\mu), \mu)}{1-\gamma}$$

which can again be made small by choosing $||\delta||$ adequately small.

Existence and Uniqueness of Solutions

Theorem 1. *Let $\xi \in (0,1)$ and let a be continuous and positive on $[0,1]\times(-\infty,\infty)$. Let g be continuous on $R\times R$ and let there exist γ_1, $\gamma_2 > 0$ such that $\gamma_1+\gamma_2<1$ and for all z_i and x_i with $z_i \ge x_i$, $i=1,2$, the inequalities*

$$0 \le g(z_1, z_2) - g(x_1, x_2) \le \gamma_1(z_1 - x_1) + \gamma_2(z_2 - x_2)$$

hold. Then (1) has a solution.

Proof. We consider the one-parameter family of problems

$$y_\lambda''(x) = \lambda a(x, y_\lambda(x))(y_\lambda(x) - b) \quad (0 < x < 1), \quad y_\lambda(0) = 0,$$
$$y_\lambda'(1) = g(y_\lambda(\xi), y_\lambda'(\xi)). \qquad (2)$$

We start with the case $\lambda = 0$. Solutions then are of the form $y_0(x) = \alpha x$, where α must satisfy $\alpha = g(\alpha\xi, \alpha)$. Set $G(x,\xi) = g(\xi x, x)$; then $z_2 > z_1$ implies that

$$0 \le G(z_2, \xi) - G(z_1, \xi) = g(\xi z_2, z_2) - g(\xi z_1, z_1) \le \gamma_1(\xi z_2, \xi z_1) + \gamma_2(z_2 - z_1) < (\gamma_1 + \gamma_2)(z_2 - z_1)$$

Thus $G(x,\xi)$ satisfies the hypotheses of Lemma 1 with $n=1$ and $\gamma = \gamma_1 + \gamma_2$. It follows that there is a unique α satisfying $\alpha = g(\alpha\xi, \alpha)$ and thus a unique solution of (2) for $\lambda=0$.

We now establish a priori bounds on y_λ for $\lambda \in (0, 1]$. As in the proof of Theorem 1 [5], a solution y_λ of (2) cannot have an interior maximum exceeding b nor an interior minimum less than b. Suppose that the maximum of y_λ occurs at 1 ; then $y_\lambda(1)>0$. If $y_\lambda(1) \le b$ we have our bound, so suppose further that $y_\lambda(1)>b$. We divide the analysis into three cases:

Case 1. $b>0$ and $y_\lambda(\xi) \le b$. By the mean value theorem there is a $\zeta \in (0,\xi)$ such that

$$y_\lambda'(\xi) \le y_\lambda'(\zeta) = \frac{y_\lambda(\xi)}{\xi} \le \frac{b}{\xi},$$

the first inequality coming from the convexity of y_λ. It follows from the monotonicity of g that $y_\lambda'(1) \le g(b, b/\xi)$ and therefore that

$$y_\lambda(x) \le y_\lambda(1) \le b + y_\lambda'(1)(1-\xi) \le b + g(b, b/\xi).$$

Case 2. $y_\lambda(\xi) \ge b$, implying that $y_\lambda'(\xi) \le y_\lambda'(1)$ and that $y_\lambda(1) \le b + y_\lambda'(1)$. Therefore $y_\lambda'(1) \le g(b + y_\lambda'(1), y_\lambda'(1))$.

Suppose, by way of contradiction, that the inequality $z \le g(b+z, z)$ were to hold for arbitrarily large z. Let $z \ge 0$ be any value for which this inequality holds. Then

$$z - g(b,0) \le g(b+z,z) - g(b,0) \le (\gamma_1 + \gamma_2)z,$$

or

$$z \le \frac{g(b,0)}{1-\gamma_1-\gamma_2}.$$

Thus there must exist a number η (independent of $\lambda \in (0, 1]$ such that $z \le g(b+z, z)$ implies that $z \le \eta$. It follows that $y_\lambda'(1) \le \eta$ and that

$$y_\lambda(x) \le y_\lambda(1) \le b + \eta.$$

Case 3. $y_\lambda(\xi) \le b \le 0$. Since $y_\lambda(1)>0$, we must have $y_\lambda(x) \ge b$ for all $x \in [0, 1]$, and y_λ is concave up. Thus the maximum of y_λ' occurs at $x=1$, and $y_\lambda(x) \le y_\lambda'(1)$ for all x; in particular, $y_\lambda(\xi) \le y_\lambda'(1)$. Since we also have $y_\lambda'(\xi) \le y_\lambda'(1)$, we get that $y_\lambda'(1) \le g(y_\lambda'(1), y_\lambda'(1))$. The existence of an upper bound for this case now follows from the argument presented in Case 2, but with b set equal to zero.

We conclude that there exists a constant $\mathcal{C}_0$ such that any solution y_λ of (2) for any $\lambda\in[0,1]$ satisfies $y_\lambda(x)\le \mathcal{C}_0$ on $(0,1]$. Employing the invariance of the form of (2) and the properties of g stated in the theorem under the change of variables $y\to -y$, $b\to -b$, $g(s,t)\to -g(-s,-t)$, we conclude that there is a constant C_0 such that $|y_\lambda(x)|\le C_0$ holds for $x\in[0,1]$ and $\lambda\in[0,1]$. A priori bounds $|y_\lambda'|\le C_1$ and $|y_\lambda''|\le C_2$ now follow as before.

We define spaces C^0, C^1, C^2 and operators j, F, and L_λ exactly as before. We must again show that (3) has a solution $C_\lambda(u)$, and that $C_\lambda(u)$ depends continuously on λ and $u\in C^0$. Let us set $w=C_\lambda(u)$ for convenience and abbreviate

$$C_\lambda(u)=g\Big(C_\lambda(u)\xi-\lambda\int_0^\xi su(s)ds -\lambda\xi\int_\xi^1 u(s)ds,\ C_\lambda(u)-\lambda\int_\xi^1 u(s)ds\Big). \quad (3)$$

as $w=g(\xi w-p, w-q)$, where

$$p=\lambda\int_0^\xi su(s)ds+\lambda\xi\int_\xi^1 u(s)ds,$$

$$q=\lambda\int_\xi^1 u(s)ds$$

obviously depend continuously on λ and u. Let $G(x,p,q)\equiv g(\xi x-p, x-q)$; then for $z_2>z_1$ we have

$$\begin{aligned}0&\le G(z_2,p,q)-G(z_1,p,q)\\&=g(\xi z_2-p,\ z_2-q)-g(\xi z_1-p,\ z_1-q)\\&\le(\gamma_1\xi+\gamma_2)(z_2-z_1).\end{aligned}$$

That is, the hypotheses of Lemma 1 are satisfied with $n=2$ and $\gamma=\gamma_1\xi+\gamma_2$. Existence and continuity of $C_\lambda(u)$ now follow from that Lemma, so L_λ is well-defined and a continuous map into C^2, as in [5].

We define the homotopy H as in the proof of Theorem 1 [5], observing that fixed points of H are precisely the solutions of (2). The remainder of the proof is identical with that of Theorem 1 [5], except that Lemma 1 must be cited in lieu of Lemma 1 [5].

Theorem 1. *Let $\xi\in(0,1)$; let $a(x,y)$ be continuous on $[0,1]\times(-\infty,\infty)$, let $a(x,y)(y-b)$ be nondecreasing in y, and let $a(x,y)(y-b)$ be locally Lipschitz in y on $[0,1]\times(-\infty,\infty)$. Let g be continuous on $R\times R$ and let there exist $\gamma_1,\gamma_2>0$ such that $\gamma_1+\gamma_2<1$ and for all z_i and x_i with $z_i\ge x_i, i=1,2$, the inequalities*

$$\begin{aligned}0&\le g(z_1,z_2)-g(x_1,x_2)\\&\le\gamma_1(z_1-x_1)+\gamma_2(z_2-x_2)\end{aligned}$$

hold. Then the solution of (1) *is unique.*

Proof. Suppose to the contrary that u and v are two distinct solutions of (1). Since $v(0)=u(0)=0$, we cannot have $u'(0)=v'(0)$ else $u\equiv v$ by uniqueness for the initial value problem. Let $v'(0)>u'(0)$. Then on some interval $(0,\delta)$ we have $v(x)>u(x)$ whence

$$\begin{aligned}v''(x)&=a(x,v(x))(v(x)-b)\\&\ge a(x,u(x))(u(x)-b)=u''(x)\end{aligned}$$

also holds on $(0,\delta]$ and therefore $v'(x)>u'(x)$ on $(0,\delta]$. Thus $v(x)>u(x)$, $v'(x)>u'(x)$, and $v''(x)>u''(x)$ must hold throughout $(0,1]$. We have from this that

$$\begin{aligned}0&<v'(1)-u'(1)\\&=g(v(\xi),v'(\xi))-g(u(\xi),u'(\xi))\\&\le\gamma_1(v(\xi)-u(\xi))+\gamma_2(v'(\xi)-u'(\xi))\\&\le\gamma_1(v(1)-u(1))+\gamma_2(v'(1)-u'(1))\\&\le(\gamma_1+\gamma_2)(v'(1)-u'(1)),\end{aligned}$$

contradicting the hypothesis that

$$\gamma_1+\gamma_2<1.$$

Positivity of Solutions

Theorem 2. *Let $a>0$ on $[0,1]\times(-\infty,\infty)$ and $b\ge0$. Then any solution of (1) is nonnegative on $[0,1]$ if any of the following holds;*

i. $g: R\times R\to[0,\infty)$; or

ii. g is nondecreasing on R^2, and there exists $\eta>0$ such that $g(z_1,z_2)\le z_2$ only for $z_2\ge\eta$; or *iii. g is nonincreasing on R^2 with $g(b, b/\xi)>0$.*

Proof : Suppose that (i) holds. If a solution y is anywhere negative, then y has a negative minimum at some $\hat{x}\in(0,1]$. From the differential equation, we have the contradiction $y''(\hat{x})<0$ if $\hat{x}\in(0,1)$. Since $y'(1)=g(y(\xi),y'(\xi))\geq 0$, the nonnegativity is established. For the case (ii) suppose that y has its minimum at 1, so that $y(1)\leq y(\xi)$. If $y(\xi)>b$, then y has a maximum exceeding b, contradicting the differential equation. Since $y(\xi)\leq b$, $y''(x)\leq 0$ must hold on $[\xi,1]$, so that $y'(1)\leq y'(\xi)$. Then

$g(y(1),y'(1))\leq g(y(\xi),y'(\xi))=y'(1)$, which implies $y'(1)\geq\eta>0$.

Hence, $y\geq 0$ on $[0,1]$ and case (ii) is established. In the case where (iii) holds, we continue to have that no solution has a negative local minimum on (0, 1), so it will again suffice to show that $y(1)\geq 0$. By way of contradiction, assume that $y(1)<0$. Then $y'(1)\leq 0$. In addition, assume that $y(\xi)\leq 0$; if $y'(\xi)>0$, then y would have a negative minimum somewhere on $(0, \xi)$, so $y'(\xi)\leq 0$ must hold. Then

$$y'(1)=g(y(\xi),y'(\xi))\geq g(0,0)\geq g(b,b/\xi)>0,$$

a contradiction. Now suppose that $y(\xi)\geq 0$. Since y cannot have a local maximum greater than b, $y(\xi)\leq b$. Were $y'(\xi)<0$, we would have

$$y'(1)=g(y(\xi),y'(\xi))\geq g(b,0)\geq g(b,b/\xi)>0,$$

a contradiction. Were $y'(\xi)>0$, we would have $y'(\xi)\leq y(\xi)/\xi\leq b/\xi$ from the concavity of y. Hence, $y'(1)\geq g(b,b/\xi)>0$. This contradiction to our assumption that $y(1)<0$ completes the proof.

References

1) R. P. Agarwal, D. O'Regan, & P. J. Y. Wong, Positive solutions of differential, difference and integral equations, Kluwer Academic Publishers, Dordrecht, 1999.
2) K. Al-Khaled, On the existence of solutions for a class of first order differential equations, Internat. J. Math. Math. Sci. 25 (2001), 1-10.
3) H. A. Antosiewicz, Boundary value problems for nonlinear ordinary differential equations, Pacific J. Math. 17 (1966), 191-197.
4) L. E. Bobisud, Three-point boundary value problems for some nonlinear second-order differential equations, Dynamic Systems Appl. 10 (2001) , 489-516.
5) T.S.Do, Application of Implicit Function Theorem to Existence of Solutions to Ordinary differential Equations with nonlocal Boundary Conditions, I, to appear.
6) A. Granas, R. B. Guenther, & J. W. Lee, Nonlinear boundary value problems for ordinary differential equations, Dissertationes Math. (Rozprawy Mat.), 1985
7) A. Granas, R. B. Guenther, & J. W. Lee, Some general existence principles in the Caratheodory theory of nonlinear differential systems, J. Math. pures et appl. 70 (1991) , 153-196.
8) M. Gregus, F. Neuman, & F. M. Arscott, Three-point boundary value problems in differential equations, J. London Math. Soc. 2 (1971), 429-436.
9) G. L. Karakostas and P. Ch. Tsamatos, Existence results for some n-dimensional nonlocal boundary value problems, J. Math. Anal. Applic. 259 (2001) , 429-438.
10) A. G. Lomtatidze, A singular three-point boundary value problem, Tbiliss. Gos. Univ. Inst. Prikl. Mat, Trudy 17 (1986), 122-134.
11) A. G. Lomtatidze, On the problem of the solvability of singular boundary value problemw for second-order ordinary differential equations, Tbiliss. Gos. Univ. Inst. Prikl. Mat. Yrudy 22 (1987) 135-149, 235-236.
12) A. Lomtatidze, On a nonlocal boundary value problem for second order linear ordinary differential equations, J. Math. Anal. Applic. 193 (1995), 889-908.
13) S. Stanek, Three-point boundary value problems for nonlinear second-order differential equations with parameter, Czech. Math. Jour. 42(117) (1992), 241-256.
14) S. Stanek, On certain three-point regular boundary value problems for nonlinear second -order differential equations depending on the parameter, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. 33 (1994), 125-132.

(2002년 7월 3일 접수, 2002년 10월10일 채택)